# Implicit stomer cowell k-step hybrid block methods for solution of first order ordinary differential equations (ODEs)

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## Abstract

This paper focuses on derivation of two implicit hybrid Block methods at $k = 3$ and one method at $k = 4$. Each block method produced at $k = 3$ yielded uniform order 5 and 7 respectively, while the Block implicit method at k = 4 produced uniform order 9. The block analysis of all the methods derived shows that they are zero stable, consistent and convergent. The superiority of these methods over some existing one were demonstrated numerically.

*Keywords*: *Consistent And Convergent, Implicit Block Methods, Uniform Order and Zero Stable.*

## 1. Introduction

Many life and physical problems can be modeled into differential equation of the form
$$y' = f(x, y) \tag{0.1}$$
The equation (1.0) occurs in field of sciences and engineering when we come across physical and natural phenomena which, when represented by mathematical models, happen to be differential equations. Some of these differential equations do not possess the closed form solution hence the numerical computation which is the area of mathematics and computer science that creates analyses and implements algorithms for numerical or approximate solutions is adopted to obtain the solution of (1.0).

Many Researchers have worked extensively in this area such as Awoyemi [1], Badmus and Mshelia ([2],[3]), Badmus and Adegboye [4],Badmus et al [5], Odekunle et al ([7],[8], Yahaya and Adegboye [10], to mention but a few.

The aim of this research paper is to develop a high order, zero stable and consistent block method at various values of k and use it to solve some existing known problems to ascertain the level of convergence.

**Definition 1.0:** One-Step Method

The construct an approximate solution $x_{(k+1)} = x(t)_k$, using only one previous approximation $x_k$. The approach in this method enjoys the virtue that the step size (h) can be changed at every iteration, if desired, thus providing a mechanism for error control.

A general expression of one-step method is $y_{n+1} = y_n + hf(x_n, y_n)$ where $f(x_n, y_n, h) = f_n = f(x_n, y_n)$
(Lambert [6])

**Definition 1.1:** Stomer Cowell Method

A linear Multistep Method of (1.2) is said to be Stomer Cowell if and only if
$$\sum_{j=0}^{t+m-1} \alpha_j = 0 \tag{1.1}$$

**Definition 1.2:** Linear Multistep Method (LMM)

If a computational method for determining a sequence between $[y_n]$ takes the form of a linear relationship between $y_{n+j}, f_{n+j}, \ j = 0,1,2, \dots, k$, then we call it a LMM of step number K or a linear k-step method.

A linear k-step method is mathematically defined as

$a_k y_{n+k} + a_{k-1} y_{n+k-1} \ldots\ldots\ldots + a_1 y_{n+1} + a_o y_n = h(\beta_k f_{n+k} + \ldots\ldots + \beta_1 f_{n+1} + \beta_o f_n)$ Which can be written in compartment form as?

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \tag{1.2}$$

Where $|\alpha_k| + |\beta_k| \neq 0$ and $\beta_k = 1$ when $\beta_k \neq 0$, the scheme becomes an implicit scheme, otherwise explicit scheme. Subair [9]

**Definition 1.3:** Zero Stability
The linear multistep method (1.2) is said to satisfy the root conditions if all the roots of the first characteristics polynomial have modulus less than or equal to unity and those of modulus unity are simple. The method (1.2) is said to be zero stable if it satisfies the root condition Lambert [6].

**Theorem 1.0:** *Fundamental Theorem of Dahlquist.*
The necessary and sufficient conditions for a Linear Multistep Method (LMM) to be convergent are that it must be Consistent and Zero stable.

**Theorem 1.1:** *Dalhquist Order Barrier for LMM.*
a)  A Zero-stable, K-step LMM is maximum order P with $p = \begin{pmatrix} k+1, \text{for k odd} \\ k+2, \text{for k even} \end{pmatrix}$
b)  An explicit LMM cannot attain A-stability if the step number, k is suck that
c)  $k > 2$
d)  The Order P of an A-stable LMM cannot exceed two. In fact, the Trapezoidal rule which is of Order $P = 2$ with step number, $k = 1$ known for its A- stability has the smallest Error constant of $C^* = \frac{1}{12}$ Lambert [6].

## 2. Methodology

Given a power series of the form
$p(x) = \sum_{j=0}^{\infty} \alpha_j x^j$
Which is used as our basis to produce an approximate solution to (1.0) as?

$$y(x) = \sum_{j=0}^{s+t-1} \alpha_j x^j \tag{2.1}$$

and

$$y'(x) = \sum_{j=0}^{s+t-1} j\alpha_j x^j = f(x, y) \tag{2.2}$$

Where $\alpha_j$'s are the parameters to be determined, s and t are the points of collocation and interpolation respectively. This process leads to $(s + t - 1)$ non-linear system of equations with $(s + t - 1)$ unknown coefficients, which are to be determined by the use of maple 17 mathematical software.

### 2.1. Hybrid Block methods derived at $k = 3$

a)  Using equations (2.1) and (2.2), $s = 3, t = 3$. our choice of degree of polynomial is $(s + t - 1)$. Equations (2.1) and (2.2) are interpolated and collocated respectively at the points $x = \{x_{n+\frac{1}{2}}, x_{n+\frac{3}{2}}, x_{n+\frac{5}{2}}\}$ which gives the following non-linear system of equations of the form

$$\sum_{j=0}^{s+t-1} \alpha_j x_{n+i}^j = y_{n+i} \tag{2.3}$$

$$\sum_{j=1}^{s+t-1} j\alpha_j x_{n+i}^{j-1} = f_{n+i} \quad \text{where } i = \left(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}\right) \tag{2.4}$$

With the mathematical software, we obtain the continuous formulation of equations (2.3) and (2.4) as follows

$y(x) = \left[-\frac{225}{128} + \frac{915}{64h}(x - x_n) - \frac{407}{16h^2}(x - x_n)^2 + \frac{149}{8h^3}(x - x_n)^3 - \frac{49}{8h^4}(x - x_n)^4 + \frac{3}{4h^5}(x - x_n)^5\right] y_{n+\frac{1}{2}} \quad . \quad + \left[\frac{25}{16} - \right.$
$152hx-xn+232h2x-xn2-6h3x-xn3+1h4x-xn4yn+32$

$$+ \left[\frac{153}{128} - \frac{435}{64h}(x - x_n) + \frac{223}{16h^2}(x - x_n)^2 - \frac{101}{8h^3}(x - x_n)^3 + \frac{41}{8h^4}(x - x_n)^4 - \frac{3}{4h^5}(x - x_n)^5\right] y_{n+\frac{5}{2}} \tag{2.5}$$

$$+ \left[-\frac{225h}{128} + \frac{465}{64}(x - x_n) - \frac{167}{16h}(x - x_n)^2 + \frac{55}{8h^2}(x - x_n)^3 - \frac{17}{8h^3}(x - x_n)^4 + \frac{1}{4h^4}(x - x_n)^5\right] f_{n+\frac{1}{2}}$$

$$+ \left[-\frac{75h}{32} + \frac{205}{16}(x - x_n) - \frac{99}{4h}(x - x_n)^2 + \frac{41}{2h^2}(x - x_n)^3 - \frac{15}{2h^3}(x - x_n)^4 + \frac{1}{h^4}(x - x_n)^5\right] f_{n+\frac{3}{2}}$$

$$+ \left[-\frac{45h}{128} + \frac{129}{64}(x - x_n) - \frac{67}{16h}(x - x_n)^2 + \frac{31}{8h^2}(x - x_n)^3 - \frac{13}{8h^3}(x - x_n)^4 + \frac{1}{4h^4}(x - x_n)^5\right] f_{n+\frac{5}{2}}$$

When equation (2.5) evaluated at $x = x_{n+j}$ where $j = 0, 1, 2$ and its first derivative also evaluated at $x = x_{n+j}, j = 0, 2$ gives the following set of discrete schemes to form the first hybrid block method at $k = 3$.

$$\frac{153}{128} y_{n+\frac{5}{2}} + \frac{25}{16} y_{n+\frac{3}{2}} - \frac{225}{128} y_{n+\frac{1}{2}} - y_n = \frac{45}{128} hf_{n+\frac{5}{2}} + \frac{75}{32} hf_{n+\frac{1}{2}} + \frac{225}{128} hf_{n+\frac{3}{2}}$$

$$\frac{11}{128}y_{n+\frac{5}{2}} + \frac{9}{16}y_{n+\frac{3}{2}} - y_{n+1} + \frac{45}{128}y_{n+\frac{1}{2}} = \frac{3}{128}hf_{n+\frac{5}{2}} + \frac{9}{32}hf_{n+\frac{3}{2}} - \frac{9}{128}hf_{n+\frac{1}{2}}$$

$$\frac{45}{128}y_{n+\frac{5}{2}} - y_{n+2} + \frac{9}{16}y_{n+\frac{3}{2}} + \frac{11}{128}y_{n+\frac{1}{2}} = \frac{9}{128}hf_{n+\frac{5}{2}} - \frac{9}{32}hf_{n+\frac{3}{2}} - \frac{3}{128}hf_{n+\frac{1}{2}}$$

$$435y_{n+\frac{5}{2}} + 480y_{n+\frac{3}{2}} - 915y_{n+\frac{1}{2}} = 129hf_{n+\frac{5}{2}} + 820hf_{n+\frac{3}{2}} + 465hf_{n+\frac{1}{2}} - 64hf_n$$

$$93y_{n+\frac{5}{2}} - 96y_{n+\frac{3}{2}} + 3y_{n+\frac{1}{2}} = 15hf_{n+\frac{5}{2}} + 64hf_{n+2} + 12hf_{n+\frac{3}{2}} - hf_{n+\frac{1}{2}}$$

(2.6)

Equations (2.6) are of uniform order 5, with error constant as follows

$[0.00488281250, \ 0.00019531250, \ 0.00019531250, \ 0.02994791667, \ 0.01666666667]^T$

b)     The second block scheme from k = 3 derived as follows

Equation (2.1) is interpolated at x = $x_{n+j}$, j = 0, 1 and equation (2.2) is collocated at **x = $x_{n+j}$**, j = $0, \frac{1}{2}, \frac{3}{2}, 2, \frac{5}{2}, 3$ which gives the system of non-linear equations of the form

$$\sum_{j=0}^{s+t-1} \alpha_j x_{n+u}^j = y_{n+u} \quad , u = 0,1$$ 

(2.7)

$$\sum_{j=1}^{s+t-1} j\alpha_j x_{n+w}^{j-1} = f_{n+w} \quad , w = 0, \frac{1}{2}, \frac{3}{2}, 2, \frac{5}{2}, 3$$ 

(2.8)

With the use of maple 17 mathematical software, we obtain the continuous formula for the method as

$$y(x) = \left[1 + \frac{210(90)}{11h^2}(x-x_n)^2 - \frac{140(351)}{11h^3}(x-x_n)^3 + \frac{105(461)}{11h^4}(x-x_n)^4 - \frac{168(137)}{11h^5}(x-x_n)^5 + \frac{280(19)}{11h^6}(x-x_n)^6 - 48011h7 x-xn7 \right] yn$$

$$+ \left[-\frac{210(90)}{11h^2}(x-x_n)^2 + \frac{140(351)}{11h^3}(x-x_n)^3 - \frac{105(461)}{11h^4}(x-x_n)^4 + \frac{168(137)}{11h^5}(x-x_n)^5 - \frac{280(19)}{11h^6}(x-x_n)^6 + 48011h7 x-xn7 \right] yn+1$$

$$+ \left[\frac{2(106245)}{165h}(x-x_n)^2 - \frac{2(275907)}{165h^2}(x-x_n)^3 + \frac{2(54327)}{33h^3}(x-x_n)^4 - \frac{2(129121)}{165h^4}(x-x_n)^5 + \frac{8(7460)}{165h^5}(x-x_n)^6 - 5384165h6 x-xn7 \right] fn+12$$

$$+ \left[(x-x_n) + \frac{507699}{1980h}(x-x_n)^2 - \frac{1990013}{2970h^2}(x-x_n)^3 + \frac{261731}{396h^3}(x-x_n)^4 - \frac{2(77829)}{495h^4}(x-x_n)^5 + \frac{2(53987)}{1485h^5}(x-x_n)^6 - 3248495h6 x-xn7 \right] fn$$

$$+ \left[\frac{4(7635)}{99h}(x-x_n)^2 - \frac{4(59663)}{297h^2}(x-x_n)^3 + \frac{4(19643)}{99h^3}(x-x_n)^4 - \frac{4(9363)}{99h^4}(x-x_n)^5 + \frac{8(3253)}{297h^5}(x-x_n)^6 - \frac{784}{99h^6}(x-xn7 \right] fn+32$$

$$+ \left[-\frac{24705}{132h}(x-x_n)^2 + \frac{32199}{66h^2}(x-x_n)^3 - \frac{63693}{132h^3}(x-x_n)^4 + \frac{2(3803)}{33h^4}(x-x_n)^5 - \frac{2(883)}{33h^5}(x-x_n)^6 + \frac{160}{33h^6}(x-xn7 \right] fn+2$$

When equation (2.9) is evaluated at $x = x_{n+j}$, j = $\frac{1}{2}, \frac{3}{2}, 2, \frac{5}{2}, 3$ and its derivative is evaluated at x = $x_{n+1}$ gives the following second hybrid block method at k = 3 as follows

$$+ \left[\frac{6(51)}{5h}(x-x_n)^2 - \frac{2(399)}{5h^2}(x-x_n)^3 + \frac{2(79)}{h^3}(x-x_n)^4 - \frac{6(63)}{5h^4}(x-x_n)^5 + \frac{8(11)}{5h^5}(x-x_n)^6 - \frac{8}{5h^6}(x-x_n)^7 \right] f_{n+\frac{5}{2}}$$

$$\frac{15487}{176}y_{n+\frac{1}{2}} (t-y_{n+1}) + \frac{4245}{495h} \frac{15663}{1485h^2} (x = \frac{5201}{11880}hf_{n+3} \frac{2(1097)}{99h^3} \frac{25}{8}hf_{n+\frac{5}{2}} \frac{204857}{2495h^4} hf_{n+2}-x_{n} \frac{37463}{2376} \frac{16(230)}{1485h^5}(x \frac{21841}{330}-x_{n}) hf_{n+\frac{3}{2}} \frac{1254995}{2495h9008} hf_n$$

$$y_{n+2} - \frac{13449}{176}y_{n+1} + \frac{10279}{176}(2.9)y_n = \frac{25}{88}hf_{n+3} - \frac{81}{40}hf_{n+\frac{5}{2}} + \frac{4347}{704}hf_{n+2} - \frac{875}{88}hf_{n+\frac{3}{2}} - \frac{1917}{44}hf_{n+\frac{1}{2}} - \frac{30947}{3520}hf_n$$

$$y_{n+2} - \frac{512}{11}y_{n+1} + \frac{501}{11}y_n = \frac{332}{1485}hf_{n+3} - \frac{8}{5}hf_{n+\frac{5}{2}} + \frac{167}{33}hf_{n+2} - \frac{2192}{297}hf_{n+\frac{3}{2}} - \frac{5608}{165}hf_{n+\frac{1}{2}} - \frac{10189}{1485}hf_n$$

$$y_{n+\frac{5}{2}} - \frac{10625}{176}y_{n+1} + \frac{10449}{176}y_n = \frac{25}{88}hf_{n+3} - \frac{15}{8}hf_{n+\frac{5}{2}} + \frac{4875}{704}hf_{n+2} - \frac{875}{88}hf_{n+\frac{3}{2}} - \frac{975}{22}hf_{n+\frac{1}{2}} - \frac{6295}{704}hf_n$$

$$y_{n+3} - \frac{243}{11}y_{n+1} + \frac{232}{11}y_n = \frac{14}{55}hf_{n+3} + \frac{27}{11}hf_{n+2} - \frac{32}{11}hf_{n+\frac{3}{2}} - \frac{864}{55}hf_{n+\frac{1}{2}} - \frac{35}{11}hf_n$$

$$7560y_{n+1} - 7560y_n = -37hf_{n+3} + 264hf_{n+\frac{5}{2}} - 807hf_{n+2} + 1328hf_{n+\frac{3}{2}} + 33hf_{n+1}$$

$$+5640hf_{n+\frac{1}{2}} + 1139hf_n$$

(2.10)

Equations (2.10) are of uniform order 7, with error constant as follows.

$[0.002964268903, -0.001924034212, -0.001506132756, -0.001951936003, -0.007305194805, -0.250000000000]^T$

c)     The third block method at k = 4 derived as follows

Equation (2.1) is interpolated at $x = x_{n+j}$ , j = $0, \frac{1}{2}$ and equation (2.2) is collocated at $x = x_{n+j}$    , j = 0, 1, $\frac{3}{2}$, 2, $\frac{5}{2}$, 3, $\frac{7}{2}$, 4 which gives the system of non-linear equations of the form

$$\sum_{j=0}^{s+t-1} \alpha_j x_{n+i}^j = y_{n+i} \quad , i = 0, \frac{1}{2}$$ 

(2.11)

$$\sum_{j=1}^{s+t-1} j\alpha_j x_{n+r}^{j-1} = f_{n+r} \quad , r = 0, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4$$ 

(2.12)

By applying the mathematical software to evaluate $\alpha_j$ in equations (2.11) and (2.12), we obtain the continuous formulation of the method as follows

$$y(x) = \left[1 - \frac{11520(2520)}{2233547h^2}(x-x_n)^2 + \frac{7680(8658)}{2233547h^3}(x-x_n)^3 - \frac{40320(1745)}{2233547h^4}(x-x_n)^4 + \frac{32256(1316)}{2233547h^5}(x-x_n)^5 - \frac{26880(575)}{2233547h^6}(x-x_n)^6 + \frac{46080(73)}{2233547h^7}(x-x_n)^7 - \frac{11520(35)}{2233547h^8}(x-x_n)^8 + \frac{20480}{2233547h^9}(x-x_n)^9 \right] y_n + \left[\frac{11520(2520)}{2233547h^2}(x-x_n)^2 - \frac{7860(8658)}{2233547h^3}(x-x_n)^3 + \frac{40320(1745)}{2233547h^4}(x-x_n)^4 - \frac{32256(1316)}{2233547h^5}(x-x_n)^5 + \frac{26880(575)}{2233547h^6}(x-x_n)^6 - \frac{46080(73)}{2233547h^7}(x-x_n)^7 + \right.$$

$$\frac{11520(35)}{2233547h^8}(x-x_n)^8 - \frac{20480}{2233547h^9}(x-x_n)^9\Big]y_{n+\frac{1}{2}} \quad + \quad \Big[(x-x_n) - \frac{26083334763}{5628538440h}(x-x_n)^2 + \frac{140072533945}{16885615320h^2}(x-x_n)^3 -$$

$$\frac{12835932101}{1608153840h^3}(x-x_n)^4 + \frac{3685624841}{804076920h^4}(x-x_n)^5 - \frac{977094389}{603057690h^5}(x-x_n)^6 + \frac{243330815}{703567305h^6}(x-x_n)^7 - \frac{28776259}{703567305h^7}(x-$$

$$x_n)^8 + \frac{32166}{15634829h^8}(x-x_n)^9\Big]f_n \quad + \quad \Big[-\frac{161926615320}{2814269220h}(x-x_n)^2 + \frac{95032968858}{4221403830h^2}(x-x_n)^3 - \frac{24922255031}{804076920h^3}(x-x_n)^4 +$$

$$\frac{4405881735}{201019230h^4}(x-x_n)^5 - \frac{2674228355}{301528845h^5}(x-x_n)^6 + \frac{16(91437513)}{703567305h^6}(x-x_n)^7 - \frac{2(92605849)}{703567305h^7}(x-x_n)^8 + \frac{5882792}{422140383h^8}(x-$$

$$x_n)^9\Big]f_{n+1} \quad + \quad \Big[\frac{8(760388160)}{703567305h}(x-x_n)^2 - \frac{8(9602705048)}{2110701915h^2}(x-x_n)^3 + \frac{4(1392410347)}{100509615h^3}(x-x_n)^4 - \frac{8(534221437)}{100509615h^4}(x-x_n)^5 +$$

$$\frac{8(690939655)}{301528845h^5}(x-x_n)^6 - \frac{8(397082471)}{703567305h^6}(x-x_n)^7 + \frac{32(13063456)}{703567305h^7}(x-x_n)^8 -$$

$$\frac{1521440}{46904487h^8}(x-x_n)^9\Big]f_{n+\frac{3}{2}} + \Big[-\frac{1592185770}{187617948h}(x-x_n)^2 + \frac{6930177687}{187617948h^2}(x-x_n)^3 - \frac{1583999026}{26802564h^3}(x-x_n)^4 + \frac{1283634229}{26802564h^4}(x-$$

$$x_n)^5 - \frac{2(72767514)}{6700641h^5}(x-x_n)^6 + \frac{2(131109257)}{46904487h^6}(x-x_n)^7 - \frac{4(8955226)}{46904487h^7}(x-x_n)^8 + \frac{2018012}{46904487h^8}(x-x_n)^9\Big]f_{n+2} \quad +$$

$$\Big[\frac{8(489445992)}{703567305h}(x-x_n)^2 - \frac{16(3257587350)}{2110701915h^2}(x-x_n)^3 + \frac{4(1022387309)}{100509615h^3}(x-x_n)^4 - \frac{16(214704897)}{100509615h^4}(x-x_n)^5 +$$

$$\frac{8(607526579)}{301528845h^5}(x-x_n)^6 - \frac{16(189618885)}{703567305h^6}(x-x_n)^7 + \frac{32(13425587)}{703567305h^7}(x-x_n)^8 - \frac{14999104}{422140383h^8}(x-x_n)^9\Big]f_{n+\frac{5}{2}} \quad +$$

$$\Big[-\frac{1325107560}{562853844h}(x-x_n)^2 + \frac{8930443094}{844280766h^2}(x-x_n)^3 - \frac{2857792229}{160815384h^3}(x-x_n)^4 + \frac{615336859}{40203846h^4}(x-x_n)^5 - \frac{448405193}{60305769h^5}(x-$$

$$x_n)^6 + \frac{8(36160979)}{140713461h^6}(x-x_n)^7 - \frac{2(21192619)}{140713461h^7}(x-x_n)^8 + \frac{849992}{46904487h^8}(x-x_n)^9\Big]f_{n+3} \quad + \quad \Big[\frac{8(17077320)}{234522435}(x-x_n)^2 -$$

$$\frac{8(11057796)}{33503205h^2}(x-x_n)^3 + \frac{8(18835348)}{33503205h^3}(x-x_n)^4 - \frac{8(16521835)}{33503205h^4}(x-x_n)^5 + \frac{8(8212230)}{33503205h^5}(x-x_n)^6 - \frac{8(16337317)}{234522435h^6}(x-x_n)^7 +$$

$$\frac{32(617683)}{234522435h^7}(x-x_n)^8 - \frac{35168}{6700641h^8}(x-x_n)^9\Big]f_{n+\frac{7}{2}} \quad + \quad \Big[-\frac{361321065}{5628538440h}(x-x_n)^2 + \frac{4945277457}{16885615320h^2}(x-x_n)^3 -$$

$$\frac{810541603}{1608153840h^3}(x-x_n)^4 + \frac{360476553}{804076920h^4}(x-x_n)^5 - \frac{136825315}{603057690h^5}(x-x_n)^6 + \frac{46414443}{703567305h^6}(x-x_n)^7 - \frac{7223237}{703567305h^7}(x-$$

$$x_n)^8 + \frac{279698}{422140383h^8}(x-x_n)^9\Big]f_{n+4} \tag{2.13}$$

When equation (2.13) is evaluated at $x = \{x_{n+1}, x_{n+\frac{3}{2}}, x_{n+2}, x_{n+\frac{5}{2}}, x_{n+3}, x_{n+\frac{7}{2}}, x_{n+4}\}$ and its derivative is evaluated at $x = \{x_{n+\frac{1}{2}}\}$, we have the following discrete block schemes at $k = 4$ as follows.

$$y_{n+1} - \frac{2921344}{2233547}y_{n+\frac{1}{2}} + \frac{687797}{2233547}y_n = \frac{16521235}{6754246128}hf_{n+4} - \frac{5339968}{234522435}hf_{n+\frac{7}{2}} + \frac{321957235}{3377123064}hf_{n+3} - \frac{501362452}{2110701915}hf_{n+\frac{5}{2}} +$$

$$\frac{37060021}{93808974}hf_{n+2} - \frac{202157692}{422140383}hf_{n+\frac{3}{2}} + \frac{10848341641}{16885615320}hf_{n+1} - \frac{1691229383}{33771230640}hf_n$$

$$y_{n+\frac{3}{2}} - \frac{2871531}{2233547}y_{n+\frac{1}{2}} + \frac{637984}{2233547}y_n = \frac{1185903}{625393160}hf_{n+4} - \frac{5458077}{312696580}hf_{n+\frac{7}{2}} + \frac{8979181}{125078632}hf_{n+3} - \frac{13527684}{78174145}hf_{n+\frac{5}{2}} +$$

$$\frac{33120711}{125078632}hf_{n+2} - \frac{309760791}{312696580}hf_{n+\frac{3}{2}} + \frac{534113739}{625393160}hf_{n+1} - \frac{14316959}{312696580}hf_n$$

$$y_{n+2} - \frac{2889728}{2233547}y_{n+\frac{1}{2}} + \frac{656181}{2233547}y_n = \frac{356227}{156348290}hf_{n+4} - \frac{1668864}{78174145}hf_{n+\frac{7}{2}} + \frac{1417114}{15634829}hf_{n+3} - \frac{18389824}{78174145}hf_{n+\frac{5}{2}} +$$

$$\frac{9020556}{15634829}hf_{n+2} - \frac{12336704}{78174145}hf_{n+\frac{3}{2}} + \frac{64841534}{78174145}hf_{n+1} - \frac{7415927}{156348290}hf_n y_{n+\frac{5}{2}} - \frac{2876875}{2233547}y_{n+\frac{1}{2}} + \frac{643328}{2233547}y_n = \frac{762400}{422140383}hf_{n+4} -$$

$$\frac{754475}{46904487}hf_{n+\frac{7}{2}} + \frac{25265375}{422140383}hf_{n+3} + \frac{12835510}{422140383}hf_{n+\frac{5}{2}} + \frac{41250875}{46904487}hf_{n+2} \quad\quad - \frac{43726475}{422140383}hf_{n+\frac{3}{2}} + \frac{355948525}{422140383}hf_{n+1} -$$

$$\frac{19519615}{422140383}hf_n$$

$$y_{n+3} - \frac{2892672}{2233547}y_{n+\frac{1}{2}} + \frac{659125}{2233547}y_n = \frac{101655}{35736752}hf_{n+4} - \frac{470880}{15634829}hf_{n+\frac{7}{2}} + \frac{5125415}{17868376}hf_{n+3} + \frac{871020}{2233547}hf_{n+\frac{5}{2}} + \frac{3437775}{4467094}hf_{n+2}$$

$$+ \frac{341620}{2233547}hf_{n+\frac{3}{2}} + \frac{14796945}{1786376}hf_{n+1} - \frac{11939465}{250157264}hf_n$$

$$y_{n+\frac{7}{2}} - \frac{2858219}{2233547}y_{n+\frac{1}{2}} + \frac{624672}{2233547}y_n = -\frac{34055}{7868376}hf_{n+4} + \frac{7151193}{44670940}hf_{n+\frac{7}{2}} + \frac{13065115}{17868376}hf_{n+3} + \frac{1733032}{11167735}hf_{n+\frac{5}{2}} +$$

$$\frac{16803129}{17868376}hf_{n+2} + \frac{543263}{8934188}hf_{n+\frac{3}{2}} + \frac{76733069}{89341880}hf_{n+1} - \frac{1985543}{44670940}hf_n$$

$$y_{n+4} - \frac{3014656}{2233547}y_{n+\frac{1}{2}} + \frac{781109}{2233547}y_n = \frac{43979726}{301528845}hf_{n+4} + \frac{25883648}{33503205}hf_{n+\frac{7}{2}} + \frac{6585152}{60305769}hf_{n+3} + \frac{270104576}{301528845}hf_{n+\frac{5}{2}} +$$

$$\frac{1979768}{6700641}hf_{n+2} + \frac{132772864}{301528845}hf_{n+\frac{3}{2}} + \frac{218727488}{301528845}hf_{n+1} - \frac{17926706}{301528845}hf_n$$

$$7257600y_{n+\frac{1}{2}} - 7257600y_n$$

$$= -33953hf_{n+4} + 312874f_{n+\frac{7}{2}} - 1291214hf_{n+3} + 3146338hf_{n+\frac{5}{2}} - 5033120hf_{n+2} + 5595358hf_{n+\frac{3}{2}} -$$

$$4604594hf_{n+1} + 4467094hf_{n+\frac{1}{2}} + 1070017hf_n \tag{2.14}$$

Equation (2.14) is of uniform order 9, with error constant as follows. [-0.000003803135250, -0.000003097181856, -0.000003495965842, -0.000003115623046, -0.000003704193292, -0.000002155631643, -0.00001040304010, -55.938476562500000]$^T$

## 3.   Block analysis of the methods

The method in (2.6) is arranged in matrix form as

$$
\begin{pmatrix}
\frac{225}{128} & 0 & -\frac{25}{16} & 0 & -\frac{153}{128} \\
-\frac{45}{128} & 1 & -\frac{9}{16} & 0 & -\frac{11}{128} \\
-\frac{11}{128} & 0 & -\frac{9}{16} & 1 & -\frac{45}{128} \\
915 & 0 & -480 & 0 & -435 \\
3 & 0 & -96 & 0 & -93
\end{pmatrix}
\begin{pmatrix}
y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \\ y_{n+\frac{5}{2}}
\end{pmatrix}
=
\begin{pmatrix}
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
y_{n-2} \\ y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n
\end{pmatrix}
$$

$$
+
\begin{pmatrix}
-\frac{225}{128} & 0 & -\frac{75}{32} & 0 & -\frac{45}{128} \\
\frac{9}{128} & 0 & -\frac{9}{32} & 0 & -\frac{3}{128} \\
\frac{3}{128} & 0 & \frac{9}{32} & 0 & -\frac{9}{128} \\
-465 & 0 & -820 & 0 & -129 \\
-1 & 0 & 12 & 64 & 15
\end{pmatrix}
\begin{pmatrix}
f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \\ f_{n+\frac{5}{2}}
\end{pmatrix}
+
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 64 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
f_{n-2} \\ f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n
\end{pmatrix}
\qquad (3.1)
$$

Let

$$
A^{(o)} =
\begin{pmatrix}
\frac{225}{128} & 0 & -\frac{25}{16} & 0 & -\frac{153}{128} \\
-\frac{45}{128} & 1 & -\frac{9}{16} & 0 & -\frac{11}{128} \\
-\frac{11}{128} & 0 & -\frac{9}{16} & 1 & -\frac{45}{128} \\
915 & 0 & -480 & 0 & -435 \\
3 & 0 & -96 & 0 & -93
\end{pmatrix}
\qquad
A^{(1)} =
\begin{pmatrix}
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
B^{(o)} =
\begin{pmatrix}
-\frac{225}{128} & 0 & -\frac{75}{32} & 0 & -\frac{45}{128} \\
\frac{9}{128} & 0 & -\frac{9}{32} & 0 & -\frac{3}{128} \\
\frac{3}{128} & 0 & \frac{9}{32} & 0 & -\frac{9}{128} \\
-465 & 0 & -820 & 0 & -129 \\
-1 & 0 & 12 & 64 & 15
\end{pmatrix}
\qquad
B^{(1)} =
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 64 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

We shall normalize the block method (3.1) by multiplying matrices $A^{(o)}$, $A^{(1)}$, $B^{(o)}$, $B^{(1)}$ with inverse of $A^{(o)}$ to obtain $A'^{(o)}$. $A'^{(1)}$ , $B'^{(o)}$ and $B'^{(1)}$

Then, $\rho(R) = det\ [RA'^{(o)} - A'^{(1)}]$

$$
= det\left[ R
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
-
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\right] = 0
$$

$$
= det\left[
\begin{pmatrix}
R & 0 & 0 & 0 & -1 \\
0 & R & 0 & 0 & -1 \\
0 & 0 & R & 0 & -1 \\
0 & 0 & 0 & R & -1 \\
0 & 0 & 0 & 0 & R-1
\end{pmatrix}
\right] = R^5 - R^4 = 0
\qquad (3.2)
$$

Which implies that $R_1 = R_2 = R_3 = R_4 = 0$ and $R_5 = 1$. Hence from the definition (1.3) equation (3.2), the method (3.1) is zero stable and also consistent as its order is $[5,5,5,5,5]^T > 1$, thus convergent.

The same analysis holds for block methods (2.10) and (2.12), thus they are zero stable and convergent

## 4.   Numerical experiments

The two block methods derived at k = 3 and one block method at k = 4 are demonstrated with the following problems

Problem 1

$y' = -y \quad y(0) = 1, \quad 0 \le x \le 2, \quad h = 0.1$

Exact solution: $y(x) = e^{-x}$

Problem 2

$y' = xy \quad y(0) = 1, \ h = 0.1$

Exact solution: $y(x) = e^{\frac{x^2}{2}}$

Problem 3

$y' + 4y = 20 \qquad y(0) = 2 \qquad h = 0.01$

Exact solution: $y(x) = 5 - 3e^{-4x}$

**Table 1:** Approximate Solution to Problem 1 from Odekunle Et Al [7]

| x | Exact solution | EOAS (1) | EOAS (2) | EMY |
|---|---|---|---|---|
| 0.1000 | 0.9048374180359595 | 1.5490(-08) | 2.3231(-07) | 2.5292(-06) |
| 0.2000 | 0.8187307530779818 | 9.3743(-09) | 1.0067(-07) | 2.0937(-06) |
| 0.3000 | 0.7408182206817178 | 1.6932(-08) | 3.2505(-07) | 2.0079(-06) |
| 0.4000 | 0.6703200460356393 | 1.6406(-08) | 4.6622(-07) | 1.6198(-06) |
| 0.5000 | 0.6065306597126334 | 2.5228(-08) | 3.4071(-07) | 3.1608(-06) |
| 0.6000 | 0.5488116360940264 | 1.9716(-08) | 4.8161(-07) | 2.7294(-06) |
| 0.7000 | 0.4965853037914095 | 2.3504(-08) | 5.6328(-07) | 2.5457(-06) |
| 0.8000 | 0.4493289641172216 | 2.1995(-08) | 4.4956(-07) | 2.1713(-06) |
| 0.9000 | 0.4065696597405991 | 2.6862(-08) | 5.3518(-07) | 3.1008(-06) |
| 1.0000 | 0.3678794411714423 | 2.2220(-08) | 5.7870(-07) | 2.7182(-06) |

**Table 2:** Approximate Solution to Problem 1 with New Block Methods Derived

| x | Exact solution | Method A (k = 3) | Method B (k = 3) | Method C (k = 4) |
|---|---|---|---|---|
| 0.1000 | 0.9048374180359595 | 0.9048374180359595 | 0.904837418035702 | 0.904837418035966 |
| 0.2000 | 0.8187307530779818 | 0.8187307530779818 | 0.818730753077778 | 0.818730753077968 |
| 0.3000 | 0.7408182206817178 | 0.7408182206817178 | 0.740818220681792 | 0.740818220681725 |
| 0.4000 | 0.6703200460356393 | 0.6703200460356393 | 0.670320046041488 | 0.670320046035640 |
| 0.5000 | 0.6065306597126334 | 0.6065306597126334 | 0.606530659699514 | 0.606530659712627 |
| 0.6000 | 0.5488116360940264 | 0.5488116360940264 | 0.548811636184434 | 0.548811636094037 |
| 0.7000 | 0.4965853037914095 | 0.4965853037914095 | 0.496585303377540 | 0.496585303791423 |
| 0.8000 | 0.4493289641172216 | 0.4493289641172216 | 0.449328966219365 | 0.449328964117217 |
| 0.9000 | 0.4065696597405991 | 0.4065696597405991 | 0.406569649330592 | 0.406569659740614 |
| 1.0000 | 0.3678794411714423 | 0.3678794411714423 | 0.367879493036357 | 0.367879441171460 |

**Table 3:** Absolute Error of Problem 1

| X | Error of Method A at k = 3 | Error of Method B at k = 3 | Error of Method C at k = 4 |
|---|---|---|---|
| 0.1000 | 2.5198(-10) | 2.5800(-13) | 6.0000(-15) |
| 0.2000 | 6.9206(-10) | 2.0400(-13) | 1.4000(-14) |
| 0.3000 | 7.3009(-11) | 7.4000(-14) | 7.0000(-15) |
| 0.4000 | 1.7709(-9) | 5.8490(-12) | 1.0000(-15) |
| 0.5000 | 4.2900(-9) | 1.3119(-11) | 6.0000(-15) |
| 0.6000 | 1.4894(-8) | 9.0408(-11) | 1.1000(-14) |
| 0.7000 | 4.6803(-8) | 4.1387(-10) | 1.3000(-14) |
| 0.8000 | 1.5088(-7) | 2.1021(-9) | 5.0000(-15) |
| 0.9000 | 4.8319(-7) | 1.0410(-8) | 1.5000(-14) |
| 1.0000 | 1.5500(-6) | 5.1865(-8) | 1.8000(-14) |
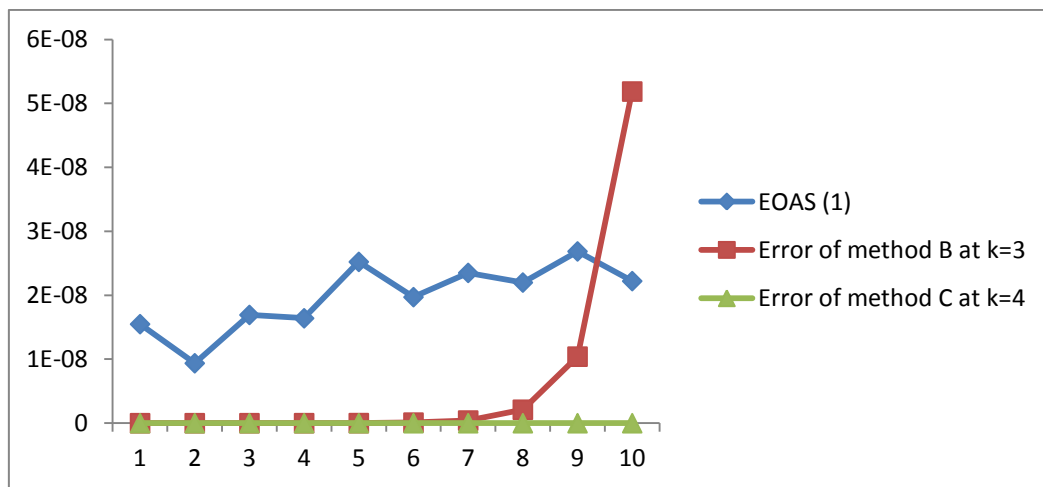


**Fig. 1:** Error Graph of Problem1

**Table 4:** Approximate Solution to Problem 2 from Odekunle Et Al [8]

| x | Exact solution | EOAS (1) | EOAS (2) | EBM |
|---|---|---|---|---|
| 0.1000 | 1.0050125208594010 | 3.1386(-07) | 5.2398(-07) | 5.29(-07) |
| 0.2000 | 1.0202013400267558 | 1.3364(-07) | 1.6913(-07) | 1.77(-07) |
| 0.3000 | 1.0460278599087169 | 3.1819(-07) | 8.7243(-07) | 8.99(-07) |
| 0.4000 | 1.0832870676749586 | 9.8972(-09) | 3.0098(-06) | 3.09(-06) |
| 0.5000 | 1.1331484530668263 | 6.9521(-07) | 1.7466(-06) | 1.91(-06) |
| 0.6000 | 1.1972173631218102 | 4.0794(-07) | 4.1710(-06) | 4.48(-06) |
| 0.7000 | 1.2776213132048868 | 7.7319(-07) | 9.6465(-06) | 1.02(-05) |
| 0.8000 | 1.3771277643359572 | 7.0821(-07) | 6.7989(-06) | 7.74(-05) |
| 0.9000 | 1.4993025000567668 | 2.8682(-06) | 1.2913(-05) | 1.44(-05) |
| 1.0000 | 1.6487212707001282 | 2.0664(-06) | 2.6575(-05) | 2.93(-05) |

**Table 5:** Approximate Solution to Problem 2 at H = 0.1with the New Block Methods Derived

| x | Exact solution | Method A (k = 3) | Method A (k = 3) | Method C  (k = 4) |
|---|---|---|---|---|
| 0.1000 | 1.0050125208594010 | 1.005012524786800 | 1.005012520821450 | 1.005012520858670 |
| 0.2000 | 1.0202013400267558 | 1.020201353563710 | 1.020201339988210 | 1.020201340026000 |
| 0.3000 | 1.0460278599087169 | 1.046027864154210 | 1.046027859908610 | 1.046027859907970 |
| 0.4000 | 1.0832870676749586 | 1.083287098480390 | 1.083287068619520 | 1.083287067674950 |
| 0.5000 | 1.1331484530668263 | 1.133148409900560 | 1.133148451587550 | 1.133148453059620 |
| 0.6000 | 1.1972173631218102 | 1.197217554200290 | 1.197217376385660 | 1.197217363191800 |
| 0.7000 | 1.2776213132048868 | 1.277620800696280 | 1.277621259685030 | 1.277621312469230 |
| 0.8000 | 1.3771277643359572 | 1.377129412581450 | 1.377128039667030 | 1.377127764031650 |
| 0.9000 | 1.4993025000567668 | 1.499297604095880 | 1.499301198635990 | 1.499302579518210 |
| 1.0000 | 1.6487212707001282 | 1.648736250960000 | 1.648727572198520 | 1.648719757723030 |

**Table 6:** Absolute Error of Problem 2

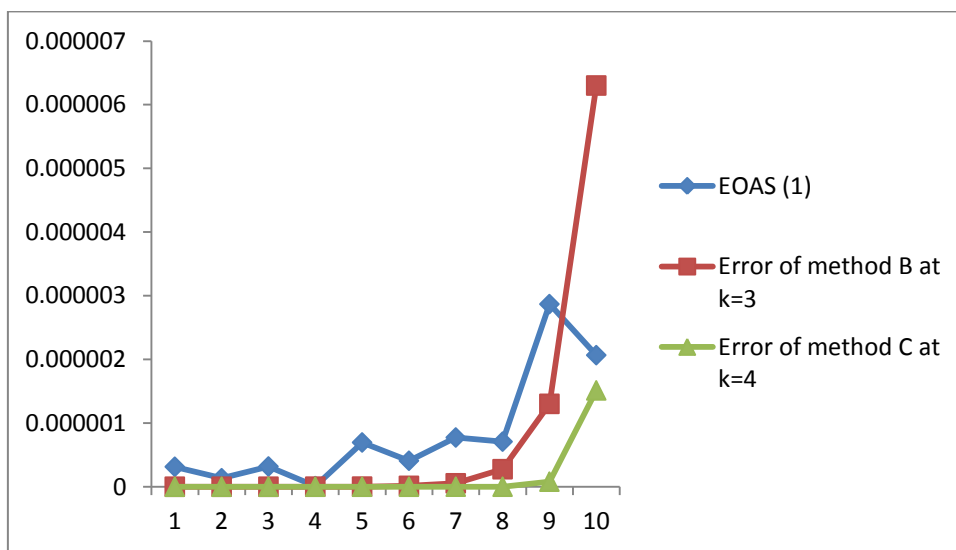| X | Error of Method A at k = 3 | Error of Method B at k = 3 | Error of Method C at k = 4 |
|---|---|---|---|
| 0.1000 | 3.9274(-9) | 3.7950(-11) | 7.3000(-13) |
| 0.2000 | 1.3537(-8) | 3.8550(-11) | 7.6000(-13) |
| 0.3000 | 4.2455(-9) | 1.1000(-13) | 7.5000(-13) |
| 0.4000 | 3.0805(-8) | 9.4456(-10) | 1.0000(-14) |
| 0.5000 | 4.3166(-8) | 1.4793(-9) | 7.2100(-12) |
| 0.6000 | 1.9108(-7) | 1.3264(-8) | 6.9990(-11) |
| 0.7000 | 5.1251(-7) | 5.3520(-8) | 7.3566(-10) |
| 0.8000 | 1.6482(-6) | 2.7533(-7) | 3.0431(-10) |
| 0.9000 | 4.8960(-6) | 1.3014(-6) | 7.9461(-8) |
| 1.0000 | 1.4980(-5) | 6.3015(-6) | 1.5130(-6) |



**Fig. 2:** Error Graph of Problem2

**Table 7:** Approximate Solution to Problem 3 at H = 0.01 with the New Block Methods Derived

| X | Exact solution | Method A k = 3 | Method B k = 3 | Method C k = 4 |
|---|---|---|---|---|
| 0.01 | 2.117631682543030 | 2.117631682539970 | 2.117631682543050 | 2.11763168254302 |
| 0.02 | 2.230650960840090 | 2.230650960830580 | 2.230650960840060 | 2.23065096084009 |
| 0.03 | 2.339238689848530 | 2.33923868984853 | 2.339238689848450 | 2.33923868984851 |
| 0.04 | 2.443568633101370 | 2.44356863310137 | 2.443568633100170 | 2.44356863310137 |
| 0.05 | 2.543807740766050 | 2.54380774076605 | 2.543807740765050 | 2.543807740766560 |
| 0.06 | 2.640116416800340 | 2.64011641680034 | 2.640116416773370 | 2.640116416796300 |
| 0.07 | 2.732648775632820 | 2.73264877563282 | 2.732648775627240 | 2.732648775665830 |
| 0.08 | 2.821552888778930 | 2.82155288877893 | 2.821552888157530 | 2.821552888218170 |
| 0.09 | 2.906971021786910 | 2.90697102178691 | 2.906971022057390 | 2.906971029253120 |
| 0.10 | 2.989039861893080 | 2.98903986189308 | 2.989039847321860 | 2.989039776916520 |

**Table 8:** Absolute Error of Problem 3

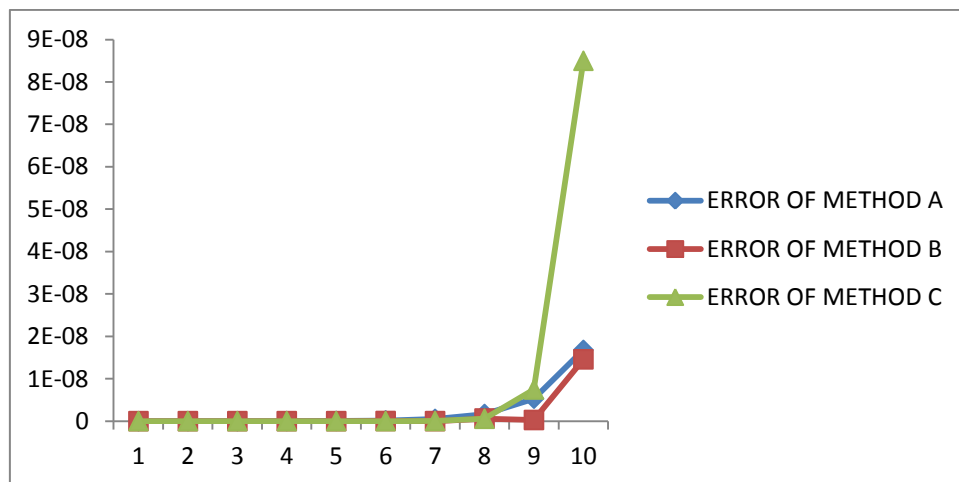| x | ERROR 0f Method A | ERROR 0f Method B | ERROR 0f Method C |
|---|---|---|---|
| 0.01 | 3.0600(-12) | 2.0000(-14) | 1.1000(-14) |
| 0.02 | 9.5100(-12) | 3.0000(-14) | - |
| 0.03 | 6.3000(-13) | 8.0000(-14) | 2.1000(-14) |
| 0.04 | 2.2760(-11) | 1.2000(-12) | - |
| 0.05 | 4.7450(-11) | 1.0000(-12) | 5.1000(-13) |
| 0.06 | 1.7270(-10) | 2.6970(-11) | 4.0400(-12) |
| 0.07 | 5.2519(-10) | 5.5800(-12) | 3.3010(-11) |
| 0.08 | 1.6816(-9) | 6.2140(-10) | 5.6076(-10) |
| 0.09 | 5.3022(-9) | 2.7048(-10) | 7.4662(-9) |
| 0.10 | 1.6794(-8) | 1.4571(-8) | 8.4977(-8) |



**Fig. 3:** Error Graph of Problem 3

## 5.   Discussion of results

We observed that for the first two problems tested with our new block methods of (2.6) which is method A, (2.10) which is method B while (2.12) which is method C, the block method (2.12) of k = 4 shows it superiority but for the third problem the block method (2.10) performed best. (See Tables 1, 2, 3 and fig 1, 2 and 3)

## 6.   Conclusion

We want to draw our conclusion that the block of (2.12) of uniform order 9 at k = 4  and the block of (2.10) of uniform order 7 at k = 3   are suitable for the solution of (1.0). Also each of the new block method display its superiority over Odekunle e tal [7], [8] see error graphs of 1, 2 and 3.

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