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Research Paper

An *ABS* approach for interior point methods for some special QCPs

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Abstract

We make use of the properties of the class of extended *ABS* (*EABS*) algorithms to present an efficient class of algorithms for computing the search directions of the primal–dual infeasible interior point methods (*IIPMs*) for solving convex quadratic programming problems (CQPs), when when the number of variables and constraints are equal. We show that, in this case, the parameters of the *EABS* algorithms for computing a search direction can always be chosen so that a part of the search vectors of the corresponding member of the class of *EABS* algorithms does not change in various iterations of the *IIPMs*.

Keywords: *Quadratic programming, Infeasible interior point methods, Primal–dual algorithms,* ABS *algorithms, Search direction.*

1. Introduction

Consider the CQP,

$$Min \qquad \frac{1}{2}x^TQx + c^Tx \qquad s.t. \quad Ax = b, \qquad x \geq 0,$$

where $c, x \in R^n$, $Q \in R^{n\times n}$ is symmetric positive semidefinite, $A \in R^{m\times n}$ and $b \in R^m$. Here, we assume that $rank(A) = m$ and $m \leq n$. In the kth iteration of an *IIPM* for solving CQPs, the search direction is computed by solving the $(2n + m) \times (2n + m)$ system of linear equations [3],

$$\begin{pmatrix} -Q & A^T & I_n \\ A & 0 & 0 \\ S^k & 0 & X^k \end{pmatrix} \begin{pmatrix} \Delta x^k \\ \Delta y^k \\ \Delta s^k \end{pmatrix} = \begin{pmatrix} -r_c^k \\ -r_b^k \\ -r_{xs}^k \end{pmatrix}, \tag{1}$$

where r_c^k, r_b^k and r_{xs}^k are given by

$$r_b^k = Ax^k - b, \qquad r_c^k = A^Ty^k + s^k - c - Qx^k, \qquad r_{xs}^k = X^kS^ke - \sigma_k\mu_ke,$$

with (x^k, y^k, s^k) the kth iterate of the *IIPM*, X^k and S^k the diagonal matrices with its diagonal elements being the components of the vectors x^k and s^k, respectively, and $e = (1, \cdots, 1)^T \in R^n$. Moreover, $\sigma_k \in (0, 1)$ and $\mu_k = (x^k)^Ts^k/n$ are termed as *centering parameter* and *duality gap*, respectively.

The major computational work in every iteration of an *IIPM* for solving CQPs, affecting stability and robustness of the method, is due to the computation of the search direction as the solution of the linear system (1). Here, we present an efficient approach for reducing the original system (1), using the class of extended *ABS* (*EABS*) algorithms. We first explain the class of *ABS* and *EABS* algorithms. *ABS* algorithms, introduced by Abaffy, Broyden and Spedicato [1], are a class of direct iteration type methods for solving a linear system where the point computed at the ith iteration solves the first i equations of the linear system. Therefore, a system of m equations is solved in at most m iterations. Let a_i^T denote the ith row of the coefficient matrix A in the linear system $Ax = b$ and π_i denote the rank of the first i rows of A. Chen et al. [2] introduced a generalization of the *ABS* algorithms, called extended *ABS* (*EABS*) class of algorithms, which differs from the *ABS* algorithms only in its updating of the Abaffian matrices H_i. In the *EABS* algorithms, the Abaffian matrices H_i, with rank j_i, are updated as $H_{i+1} = G_i H_i$, where $G_i \in R^{j_{i+1}\times j_i}$ is such that we have $G_i x = 0$ if and only if $x = \alpha H_i a_i$, for some $\alpha \in R$. Note that in the *EABS* algorithms we can set $G_i = I - H_i a_i w_i^T$, where $w_i^T H_i a_i = 1$. Therefore, the class of *EABS* algorithms contains the class of *ABS* algorithms. It is observed that, like the basic *ABS* algorithm, an *EABS* algorithm solves a new equation at each iteration. Thus, at most m iterations are needed to find a solution. Some properties of the *EABS* algorithms are noted next (see [1, 2] for the proofs). 1. $H_i a_i = s_i = 0$ if and only if a_i is linearly dependent on $a_1, a_2, \cdots, a_{i-1}$. 2. The rows of H_{i+1} generate the null space of the first i rows of A (note that this implies $AH_{m+1}^T = 0$). 3. If the first i equations has a solution, then the general solution for the first i equations is written as $x_{i+1} + H_{i+1}^T s$, for $s \in R^n$. 4. Let $x^* = x_{i+1} + H_{i+1}^T s^*$ be a special solution of the system $Ax = b$ and $Z = (H_{i+1}a_{i+1}, \cdots, H_{i+1}a_m)^T$, $r_{i+1} = b - Ax_{i+1}$. Then, s^* is the solution of the linear system $Zy = (r_{i+1})_{m-i}$, where $(r_{i+1})_{m-i}$ denotes the last $m-i$ components of the vector r_{i+1}. Moreover, we have $s^* = Z^T d$, where d is a solution of the linear system $ZZ^T d = (r_{i+1})_{m-i}$. If A has full row rank, then the rows of Z are nonzero and linearly independent (see [1]) and hence ZZ^T is symmetric and positive definite. 5. If $x_1 = 0$, then the solution of the system is $x_{m+1} = P\tau$, where $P = (p_1, \cdots, p_m)$ and $\tau = (\tau_1, \cdots, \tau_m)^T$. 6. If $rank(A) = n \le m$, then H_{m+1} is the zero matrix. Now, we explain our basic idea. Starting with x_1 as the zero vector, let p_i, $1 \le i \le 2n+m$, be the search vectors obtained by an application of an *EABS* algorithm to the coefficient matrix in (1). Then, by property 5, the solution of the system can be obtained by $\left(\Delta x^{k^T}, \Delta y^{k^T}, \Delta s^{k^T}\right)^T = P\tau^k$, where $P = (p_1, p_2, \cdots, p_{2n+m})$ and $\tau^k = (\tau_1^k, \tau_2^k, \cdots, \tau_{2n+m}^k)^T$ with τ_j^k, $1 \le j \le 2n+m$, as the step sizes. Note that if the search vectors p_i, $1 \le i \le 2n+m$, are independent of the iteration number k, then in every iteration of the *IIPM*, using this fact, the solution of the linear system (1), that is, the search direction, can efficiently be computed merely by determining the step sizes of the corresponding *EABS* algorithm. We name such search vectors as *iteration–free* search vectors. In Section 2, we discuss *iteration–free* search vectors of the *EABS* algorithm for solving (1). In Section 3, we show how we can use these *iteration–free* search vectors to characterize the *EABS* solutions of (1).

2. Iteration–free search vectors

In the kth iteration of an *IIPM* for solving CQPs, the search direction is computed by solving the linear system (1). We start the *EABS* algorithm with $x_1 = 0 \in R^{2n+m}$ and $H_1 = I_{2n+m}$, where I_{2n+m} is the identity matrix of dimension $2n+m$.

Theorem 2.1 *For $1 \le i \le n$, let $z_i = w_i = \left(0, 0, e_i^T\right)^T \in R^{2n+m}$, where the first two zeros are of dimensions n and m, respectively, and e_i is the ith column of the identity matrix I_n and the Abaffian matrices are updated as in Step 5 of a basic ABS algorithm. Then, in the ith iteration of the EABS algorithm for solving (1), we have*

$$p_i = \begin{pmatrix} 0 \\ 0 \\ e_i \end{pmatrix}, \qquad H_{i+1} = \begin{pmatrix} I_n & 0 & \sum_{j=1}^{i} q_j e_j^T \\ 0 & I_m & -\sum_{j=1}^{i} A e_j e_j^T \\ 0 & 0 & I_n - \sum_{j=1}^{i} e_j e_j^T \end{pmatrix}, \tag{2}$$

where q_j is the jth column of the matrix Q^T.

Proof: First of all, we note that for i, $1 \le i \le n$, the ith row of the coefficient matrix of system (1) is $\overline{a}_i^T = \left(-q_i^T, e_i^T A^T, e_i^T\right)$. We proceed by induction. Since $H_1 = I_{2n+m}$, for $i = 1$ we have $\overline{a}_1^T H_1^T \overline{e}_1 = 1 \ne 0$. Thus, we can choose $w_1 = z_1 = \overline{e}_1 = \left(0, 0, e_1^T\right)^T$, and the corresponding search vector p_1, and the Abaffian matrix H_2 are computed as:

$$p_1 = H_1^T z_1 = \begin{pmatrix} 0 \\ 0 \\ e_1 \end{pmatrix}, \quad H_2 = H_1 - \begin{pmatrix} -q_1 \\ Ae_1 \\ e_1 \end{pmatrix} \left(0, 0, e_1^T\right) = \begin{pmatrix} I_n & 0 & q_1 e_1^T \\ 0 & I_m & -Ae_1 e_1^T \\ 0 & 0 & I_n - e_1 e_1^T \end{pmatrix}.$$

Suppose that (2) is true for $i = 1, 2, \cdots, t-1$, $(t \le n)$. For $i = t$, we have

$$(0,0,e_t^T)\begin{pmatrix} I_n & 0 & \sum_{j=1}^{t-1} q_j e_j^T \\ 0 & I_m & -\sum_{j=1}^{t-1} Ae_j e_j^T \\ 0 & 0 & I_n - \sum_{j=1}^{t-1} e_j e_j^T \end{pmatrix}\begin{pmatrix} -q_t \\ Ae_t \\ e_t \end{pmatrix} = e_t^T e_t = 1 \neq 0.$$

Therefore, we can choose $z_t = w_t = \left(0,0,e_t^T\right)^T$, and by doing simple algebraic calculations, we obtain

$$p_t = H_t^T z_t = \begin{pmatrix} 0 \\ 0 \\ e_t \end{pmatrix},\ H_{t+1} = H_t - H_t \overline{a}_t w_t^T H_t = \begin{pmatrix} I_n & 0 & \sum_{j=1}^{t} q_j e_j^T \\ 0 & I_m & -\sum_{j=1}^{t} Ae_j e_j^T \\ 0 & 0 & I_n - \sum_{j=1}^{t} e_j e_j^T \end{pmatrix}.$$

So, by induction, (2) is true for $1 \le i \le n$ and the proof is complete.

Note that, by Theorem 2.1, after n iterations we have

$$H_{n+1} = \begin{pmatrix} I_n & 0 & \sum_{j=1}^{n} q_j e_j^T \\ 0 & I_m & -\sum_{j=1}^{n} \widehat{a}_j e_j^T \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} I_n & 0 & Q^T \\ 0 & I_m & -A \\ 0 & 0 & 0 \end{pmatrix}. \tag{3}$$

Now, let a_i^T, $1 \le i \le m$, denote rows of the matrix A, and consider the linear system $Ax = 0$. Applying the *EABS* algorithm with $j_{i+1} = n - i$ to this system, starting with $\overline{H}_1 = I_n$ and $\overline{x}_1 = 0 \in R^n$, in the ith iteration, choose $\overline{z}_i$ so that $\overline{z}_i^T \overline{H}_i a_i \neq 0$, and let $\overline{p}_i = \overline{H}_i^T \overline{z}_i$. After computing x_{i+1}, choose $\overline{G}_i \in R^{(n-i)\times(n-i+1)}$ so that $\overline{G}_i x = 0$ if and only if $x = \alpha \overline{H}_i a_i$, for some $\alpha \in R$, and let $\overline{H}_{i+1} = \overline{G}_i \overline{H}_i$. The following theorem establishes the *EABS* parameters for the $(n+i)$th, $1 \le i \le m$, iteration of the algorithm for solving (1).

Theorem 2.2 *For i, $1 \le i \le m$, let*

$$z_{n+i} = \begin{pmatrix} \overline{z}_i \\ 0 \\ 0 \end{pmatrix} \in R^{2n+m}, \qquad G_{n+i} = \begin{pmatrix} \overline{G}_i & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & I_n \end{pmatrix} \in R^{2n+m},$$

where the $\overline{z}_i$ and $\overline{G}_i$ are obtained by the application of an EABS algorithm to the linear system $Ax = 0$. Then, in the $(n+i)$th iteration of the EABS algorithm, with the same parameters for the first n iterations as in Theorem 2.1 applied to solve (1), we have

$$p_{n+i} = \begin{pmatrix} \overline{p}_i \\ 0 \\ Q\overline{p}_i \end{pmatrix}, \quad H_{n+i+1} = \begin{pmatrix} \overline{H}_{i+1} & 0 & \overline{H}_{i+1} Q^T \\ 0 & I_m & -A \\ 0 & 0 & 0 \end{pmatrix}, \quad 1 \le i \le m. \tag{4}$$

Proof: We observe that, for $1 \le i \le m$, the $(n+i)$th row of the coefficient matrix of system (1) is $\left(a_i^T, 0, 0\right)$. We proceed by induction. For $i = 1$, using (3), we have

$$\left(\overline{z}_1^T, 0, 0\right) H_{n+1} \begin{pmatrix} a_1 \\ 0 \\ 0 \end{pmatrix} = \left(\overline{z}_1^T, 0, 0\right) \begin{pmatrix} I_n & 0 & Q^T \\ 0 & I_m & -A \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ 0 \\ 0 \end{pmatrix} = \overline{z}_1^T \overline{H}_1 a_1 \neq 0.$$

Moreover, since

$$\begin{pmatrix} \overline{G}_1 & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_n \end{pmatrix} \begin{pmatrix} a_1 \\ 0 \\ 0 \end{pmatrix} = \overline{G}_1 a_1$$

and $\overline{G}_1$ is chosen so that $\overline{G}_1 x = 0$ if and only if $x = \alpha \overline{H}_1 a_1$, for some $\alpha \in R$, therefore the choice of G_{n+1} is valid. Thus, by choosing G_{n+1} as above and $z_{n+1} = \left(\overline{z}_1^T, 0, 0\right)^T$, we will have

$$p_{n+1} = H_{n+1}^T \begin{pmatrix} \overline{z}_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \overline{H}_1^T & 0 & 0 \\ 0 & I_m & 0 \\ Q\overline{H}_1^T & -A^T & 0 \end{pmatrix} \begin{pmatrix} \overline{z}_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \overline{z}_1 \\ 0 \\ Q\overline{z}_1 \end{pmatrix} = \begin{pmatrix} \overline{p}_1 \\ 0 \\ Q\overline{p}_1 \end{pmatrix},$$

and, by doing simple algebraic calculations, we have

$$\begin{aligned} H_{n+2} &= G_{n+1}H_{n+1} \\ &= \begin{pmatrix} \overline{G}_1 & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_n \end{pmatrix} \begin{pmatrix} \overline{H}_1 & 0 & \overline{H}_1 Q^T \\ 0 & I_m & -A \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \overline{G}_1\overline{H}_1 & 0 & \overline{G}_1\overline{H}_1 Q^T \\ 0 & I_m & -A \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \overline{H}_2 & 0 & \overline{H}_2 Q^T \\ 0 & I_m & -A \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Note that we have assumed $\overline{H}_1 = I_n$. Suppose that (4) holds for $i = 1, 2, \cdots, t-1$, $(t \leq m)$ and consider the case $i = t$. We have

$$\left(\overline{z}_t^T, 0, 0\right) H_{n+t} \begin{pmatrix} a_t \\ 0 \\ 0 \end{pmatrix} = \left(\overline{z}_t^T, 0, 0\right) \begin{pmatrix} \overline{H}_t & 0 & \overline{H}_t Q^T \\ 0 & I_m & -A \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_t \\ 0 \\ 0 \end{pmatrix} = \overline{z}_t^T \overline{H}_t a_t \neq 0.$$

Therefore, by choosing $z_{n+t}^T = (\overline{z}_t^T, 0, 0)$, we will have

$$p_{n+t} = H_{n+t}^T z_{n+t} = \begin{pmatrix} \overline{H}_t^T & 0 & 0 \\ 0 & I_m & 0 \\ Q\overline{H}_t^T & -A^T & 0 \end{pmatrix} \begin{pmatrix} \overline{z}_t \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \overline{H}_t^T \overline{z}_t \\ 0 \\ Q\overline{H}_t^T \overline{z}_t \end{pmatrix} = \begin{pmatrix} \overline{p}_t \\ 0 \\ Q\overline{p}_t \end{pmatrix}.$$

From

$$\begin{pmatrix} \overline{G}_t & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_n \end{pmatrix} \begin{pmatrix} a_t \\ 0 \\ 0 \end{pmatrix} = \overline{G}_t a_t$$

and the fact that $\overline{G}_t$ is chosen so that $\overline{G}_r x = 0$ if and only if $x = \alpha \overline{H}_t a_t$, for some $\alpha \in R$, the choice of G_{n+t} is then valid. Moreover, we can write

$$H_{n+t+1} = \begin{pmatrix} \overline{G}_t\overline{H}_t & 0 & \overline{G}_t\overline{H}_t Q^T \\ 0 & I_m & -A \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \overline{H}_{t+1} & 0 & \overline{H}_{t+1} Q^T \\ 0 & I_m & -A \\ 0 & 0 & 0 \end{pmatrix}.$$

This completes the proof of the theorem.

Note that by Theorem 2.2, after $m+n+1$ iterations, we have

$$H_{n+m+1} = \begin{pmatrix} \overline{H}_{m+1} & 0 & \overline{H}_{m+1} Q^T \\ 0 & I_m & -A \\ 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

Next, we construct the last n iterations of the *EABS* algorithm applied to the system (1). Consider the linear system $A^T y = 0$. In applying the *ABS* algorithm to this system, starting with $\widehat{H}_1 = I_m$ and $\widehat{x}_1 = 0 \in R^m$, in the ith iteration we choose $\widehat{z}_i$ so that $\widehat{z}_i^T \widehat{H}_i^T A e_i \neq 0$, and let $\widehat{p}_i = \widehat{H}_i^T \widehat{z}_i$. Then, we choose $\widehat{w}_i$ so that $\widehat{w}_i^T \widehat{H}_i^T A e_i = 1$, and let $\widehat{H}_{i+1} = \widehat{H}_i - \widehat{H}_i A e_i \widehat{w}_i^T \widehat{H}_i$. Let $\Pi^k = S^k (X^k)^{-1}$, $\overline{H} = \overline{H}_{m+1}$ and $B_k^0 = 0$. Define the matrices $B_k^j \in R^{n \times m}$ according to the following formula:

$$B_k^j = B_k^{j-1} - B_k^{j-1} A e_j \widehat{w}_j^T \widehat{H}_j + \overline{H}[Q^T + \Pi^k] e_j \widehat{w}_j^T \widehat{H}_j, \quad 1 \leq j \leq m. \tag{6}$$

Remark 2.3 *Note that, since by properties of* ABS *algorithms, $\widehat{H}_{m+1} a_i = 0$, $1 \leq i \leq m$, and a_i, $1 \leq i \leq m$, are linearly independent, then $\widehat{H}_{m+1}$ would be the zero matrix.*

3. Computing the search directions

Here, we provide the search directions of *IIPM* using the search vectors obtained in Section 2 for the system (1). The solution of the first $2m+n$ equations of (1) is:

$$\begin{pmatrix} \Delta x^k \\ \Delta y^k \\ \Delta s^k \end{pmatrix} = \begin{pmatrix} P\lambda^k \\ \widehat{P}\beta^k \\ -r_c^k + QP\lambda^k - A^T\widehat{P}\beta^k \end{pmatrix},$$

where $P = (\overline{p}_1, \overline{p}_2, \cdots, \overline{p}_m)$, $\widehat{P} = (\widehat{p}_1, \widehat{p}_2, \cdots, \widehat{p}_m)$, $\lambda^k = (\lambda_1^k, \lambda_2^k, \cdots, \lambda_m^k)$, $\beta^k = (\beta_1^k, \beta_2^k, \cdots, \beta_m^k)$, and the ith component of the vectors λ^k and β^k are defined, using the properties of the *EABS* algorithm as:

$$\lambda_1^k = \frac{-(r_b^k)^T e_1}{a_1^T \overline{p}_1}, \quad \lambda_i^k = \frac{-(r_b^k)^T e_i - \sum_{j=1}^{i-1} \lambda_j^k a_i^T \overline{p}_j}{a_i^T \overline{p}_i},$$

for $2 \leq i \leq m$, and

$$\begin{aligned} \beta_1^k &= \frac{s_1^k e_1^T P\lambda^k - x_1^k e_1^T r_c^k + (r_{xs}^k)^T e_1 + x_1^k e_1^T QP\lambda^k}{x_1^k e_1^T A^T \widehat{p}_1}, \\ \beta_i^k &= \frac{(r_{xs}^k)^T e_i + s_i^k e_i^T P\lambda^k - x_i^k e_i^T (\sum_{j=1}^{i-1} \beta_j^k A^T \widehat{p}_j) - x_j^k e_j^T r_c^k + x_j^k e_j^T QP\lambda^k}{x_j^k e_j^T A^T \widehat{p}_j}, \end{aligned}$$

for $2 \leq i \leq m$. Now, let A_m and A_{n-m} denote the first m columns and the last $n-m$ columns of A, respectively, $L = AP$ and $\widehat{L}_m = A_m^T \widehat{P}$. Then, we can also compute λ^k and β^k, by solving the linear systems $L\lambda^k = -r_b^k$ and $\widehat{L}_m \beta^k = \widehat{r}_m^k$, where $\widehat{r}_m^k$ denotes the first m components of $\widehat{r}^k = (X^k)^{-1} r_{xs}^k + \Pi^k P\lambda^k - r_c^k + QP\lambda^k$. Note that L and $\widehat{L}_m$ are nonsingular lower triangular matrices. If $m = n$, then the solution of (1) is at hand. Moreover, in this case all search vectors of the *EABS* algorithm for solving (1) are *iteration–free*.

4. Conclusion

We made use of some properties of the class of *EABS* algorithms to show that in the *EABS* approach, a portion of the search vectors of the *EABS* algorithm for computing the search direction of the Newton system does not change in every iteration of the *IIPM*.

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