Solution of nonlinear integral equations via fixed point of generalized contractive condition in G-metric space

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Abstract

The main aim of this paper is to prove that the existence and uniquencess of solutions of systems of simultaneous Volterra Hammerstein and Urysohn nonlinear integral equations in G-metric spaces setting by using common fixed point theorems satisfying generalized contractive conditions.

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1 Introduction and preliminaries

The fixed point theory is one of the most rapidly growing topic of nonlinear functional analysis. It is a vast and interdisciplinary subject whose study belongs to several mathematical domains such as: classical analysis, functional analysis, operator theory, topology and algebraic topology, etc. This topic has grown very rapidly perhaps due to its interesting applications in various fields within and out side the mathematics such as: integral equations, initial and boundary value problems for ordinary and partial differential equations, game theory, optimal control, nonlinear oscillations, variational inequalities, complementary problems, economics and others.

Existence of fixed points in ordered metric spaces has been initiated by Ran and Reurings [1] and further studied by Nieto and López [2]. They studied existence and uniqueness of fixed points on partially ordered metric spaces and applied their results to boundary value problems for ordinary differential equations. Subsequently, several interesting and valuable results have appeared in this direction see for examples [3]-[9].

Integral equation methods are very useful for solving many problems in several applied fields like mathematical economics and optimal control theory, because these problems are often reduced to integral equations. Since these equations usually cannot be solved explicitly, so it is necessary to get different numerical techniques. There are numerous advanced and efficient methods, which have been focusing on the solution of integral equations. Many papers have been appeared on the problem of existence and uniqueness of solutions of nonlinear integral equations and the results are established by applying various fixed point techniques, see e.g., [10]-[17].

In 2006, Mustafa and Sims [18, 19], introduced the concept of a G-metric and G-metric space, which is a generalization of metric space. After this pioneering work, G-metric spaces and particularly fixed points of various maps on G-metric spaces have been intensively studied, see, e.g., [20]-[23].

Definition 1.1. *[19] Let X be a nonempty set and let $G : X^3 \to [0,\infty)$ be a function satisfying:*

(G_1) $G(x,y,z) = 0$ *if* $x = y = z$,

(G_2) $0 < G(x,x,y)$, *for all* $x,y \in X$, *with* $x \neq y$,

(G_3) $G(x,x,y) \leq G(x,y,z), \forall x,y,z \in X$, *with* $z \neq y$,

(G_4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \ldots$, *(symmetry in all three variables)*,

(G_5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z), \forall x,y,z,a \in X$, *(rectangle inequality).*

Then the function G is called a G-metric on X, and the pair (X,G) is called a G-metric space.

Definition 1.2. *[19] Let (X,G) be a G-metric space, a sequence (x_n) is said to be*

(i) G-convergent if for every $\varepsilon > 0$, there exists an $x \in X$, and $k \in \mathbf{N}$ such that for all $m,n \geq k, G(x,x_n,x_m) < \varepsilon$.

(ii) G-Cauchy if for every $\varepsilon > 0$, there exists an $k \in \mathbf{N}$ such that for all $m,n,p \geq k, G(x_m,x_n,x_p) < \varepsilon$, that is $G(x_m,x_n,x_p) \to 0$ as $m,n,p \to \infty$.

(iii) A space (X,G) is said to be G-complete if every G-Cauchy sequence in (X,G) is G-convergent.

Proposition 1.3. *[19] Every G-metric space (X,G) will define a metric space (X,d_G) by*

$$d_G(x,y) = G(x,y,y) + G(y,x,x), \qquad \forall x,y \in X.$$

Lemma 1.4. *[19] Let (X,G) be a G-metric space. Then the following are equivalent:*

(i) $\{x_n\}$ is G-convergent to x.

(ii) $d_G(x_n,x) \to 0$, as $n \to \infty$ (that is, $\{x_n\}$ converges to x relative to the metric d_G).

(iii) $G(x_n,x_n,x) \to 0$ as $n \to \infty$,

(iv) $G(x_n,x,x) \to 0$ as $n \to \infty$,

(v) $G(x_n,x_m,x) \to 0$ as $n,m \to \infty$,

Lemma 1.5. *[19] Let (X,G) be a G-metric space. Then the following are equivalent:*

(i) The sequence (x_n) is G-Cauchy,

(ii) for every $\varepsilon > 0$, there exists $k \in \mathbf{N}$ such that $G(x_n,x_m,x_m) < \varepsilon$ for $m,n \geq k$.

Lemma 1.6. *[19] Let (X,G) be a G-metric space. Then the function $G(x,y,z)$ is jointly continuous in all three of its variables.*

Definition 1.7. *[19] A G metric space X is symmetric if $G(x,y,y) = G(y,x,x)$ for all $x,y \in X$.*

Proposition 1.8. *[19] Let (X,G) be a G-metric space. Then for any x,y,z, and $a \in X$, it follows that*

(i) if $G(x,y,z) = 0$ then $x = y = z$,

(ii) $G(x,y,z) \leq G(x,x,y) + G(x,x,z)$,

(iii) $G(x,y,y) \leq 2G(x,x,y)$,

(iv) $G(x,y,z) \leq G(x,a,z) + G(a,y,z)$,

(v) $G(x,y,z) \leq \frac{2}{3}(G(x,y,a) + G(x,a,z) + G(a,y,z))$,

(vi) $G(x,y,z) \leq G(x,a,a) + G(y,a,a) + G(z,a,a)$,

Proposition 1.9. *[19] A G-metric space (X,G) is G-complete if and only if (X,d_G) is a complete metric space.*

Corollary 1.10. *[19] If Y is a non-empty subset of a G-complete metric space (X,G), then $(Y,G|_Y)$ is complete if and only if Y is G-closed in (X,G).*

Definition 1.11. *[25] A function $\psi : [0,\infty) \to [0,\infty)$ is called altering distance function if*

(i) ψ is increasing and continuous,

(ii) $\psi(t) = 0$ if and only if $t = 0$.

Definition 1.12. *[21] Let X be a nonempty set. Then $(X,\preceq,G)$ is called an ordered G-metric space if (X,G) is a G- metric space and $(X,\preceq)$ is a partial order set.*

Definition 1.13. *Let $(X,\preceq)$ be a partial ordered set. Then two points $x,y \in X$ are said to be comparable if $x \preceq y$ or $y \preceq x$.*

Definition 1.14. *[26] Let $(X,\preceq)$ be a partially ordered set. A mapping f is called weak annihilator of g if $fgx \preceq x$ for all $x \in X$.*

Definition 1.15. *[26] Let $(X,\preceq)$ be a partially ordered set. A mapping f on X is called dominating if $x \preceq fx$ for all $x \in X$.*

For examples illustrating the above definitions are given in [26].

Definition 1.16. *A subset W of a partially ordered set X is said to be well ordered if every two elements of W are comparable.*

In this paper, we discuss two applications for the solutions of nonlinear Volterra-Hammerstein integral equations in partially ordered G-metric space and Urysohn integral equations in G-metric space.

2 Solutions of nonlinear integral equations

The following two theorems 2.1 and 2.2 were proved by Rashwan et.al [8], [23] respectively.

Theorem 2.1. *[8] Let $(X, \preceq, G)$ be an ordered G-metric space and let f, g, h, S, T and R be self-maps on X satisfying the following condition*

$$G(fx, gy, hz) \leq kM(x, y, z), \tag{2.1}$$

where $k \in [0, \frac{1}{2})$ and

$$\begin{aligned} M(x,y,z) = \max\{&G(Sx, Ty, Rz), G(fx, fx, Sx), G(gy, gy, Ty), G(hz, hz, Rz), \\ &(gy, gy, Sx), G(Ty, hz, hz), G(Rz, fx, fx)\}, \end{aligned}$$

for all comparable elements $x, y, z \in X$. Suppose that

(i) $f(X) \subseteq T(X)$, $g(X) \subseteq R(X)$, $h(X) \subseteq S(X)$,

(ii) dominating maps f, g, h are weak annihilators of T, R, S respectively,

(iii) one of $S(X)$, $T(X)$ or $R(X)$ is a G-complete subspace of X.

If, for a non-decreasing sequence $\{x_n\}$ with $x_n \preceq y_n$ for all n and $y_n \to q$ implies that $x_n \preceq q$, then f, g, h, S, T and R have a common fixed point. Moreover, the set of common fixed points of f, g, h, S, T and R is well ordered if and only if f, g, h, S, T and R have one and only one common fixed point.

Theorem 2.2. *[23] Let (X, G) be a complete G-metric space and f, g and h be self maps on X satisfying inequality*

$$\psi(G(fx, gy, hz)) \leq \psi(M(x, y, z)) - \varphi(M(x, y, z)), \tag{2.2}$$

where

$$\begin{aligned} M(x,y,z) = \max\{&G(x,y,z), G(x,y,gy), G(y,z,hz), G(z,x,fx), \\ &\alpha G(fx,x,gy) + (1-\alpha)G(y,gy,hz)\}, \end{aligned}$$

for all $x, y, z \in X$, where $0 < \alpha < 1$, ψ is an altering distance function, and $\varphi : [0, \infty) \to [0, \infty)$ is a continuous function with $\varphi(t) = 0$ if and only if $t = 0$.. Then f, g, and h have a unique common fixed point in X.

In this section, we present an application of Theorem 2.1 to study the existence and uniqueness of solution for a system of nonlinear Volterra-Hammerstein integral equation in G-metric spaces. Let $X = (L[0, \infty), R)$ be the space of real valued functions that are measurable on $[0, \infty)$ and let $d : X \times X \to R^+$ be defined by

$$d(x, y) = \int_0^\infty |x(t) - y(t)| dt,$$

for all $x,y \in X$. Equip X with the G-metric given by

$$G(x,y,z) = \max\{d(x,y), d(y,z), d(z,x)\}$$

for all $x,y,z \in X$. Clearly, (X,G) is a complete G-metric space. We endow X with the partial ordered $\preceq$ given by

$$x \preceq y \iff x(t) \leq y(t) \tag{2.3}$$

for all $t \in [0,\infty)$. Motivated by the work in [16], we apply Theorem 2.1 to prove the existence of solution in partially ordered G-metric space (X,G) of the following simultaneous Volterra-Hammerstein nonlinear integral equations:

$$\begin{aligned}
x(t) &= p_1(t) - p_2(t) + \lambda \int_0^t m(t,s) g_1(s,x(s))ds + \mu \int_0^\infty k(t,s) h_2(s,x(s))ds, \\
x(t) &= p_1(t) - p_2(t) + \lambda \int_0^t m(t,s) g_2(s,x(s))ds + \mu \int_0^\infty k(t,s) h_3(s,x(s))ds, \qquad (2.4) \\
x(t) &= p_1(t) - p_2(t) + \lambda \int_0^t m(t,s) g_3(s,x(s))ds + \mu \int_0^\infty k(t,s) h_1(s,x(s))ds,
\end{aligned}$$

for all $t \in [0,\infty)$, where $p_1, p_2 \in X$ are known with $p_1(t) \geq p_2(t)$, $m(t,s)$, $k(t,s)$, $g_i(s,x(s))$ and $h_i(s,x(s))$, $i = 1,2,3$ are real valued functions that are measurable both in t and s on $[0,\infty)$ and λ, μ are real numbers. These functions satisfy the following conditions:

(C_0) $\int_0^\infty \sup_{s\in[0,\infty)} |m(t,s)|dt = M_1 < +\infty$,

(C_1) $\int_0^\infty \sup_{s\in[0,\infty)} |k(t,s)|dt = M_2 < +\infty$,

(C_2) $g_i(s,x(s)) \in X$, $i = 1,2,3$ for all $x \in X$ and there exists $K_1 > 0$ such that for all $s \in [0,\infty)$

$$|g_i(s,x(s)) - g_j(s,y(s))| \leq K_1 |x(s) - y(s)|, \qquad \forall x,y \in X,\ i = 1,2,3$$

(C_3) $h_i(s,x(s)) \in X$, $i = 1,2,3$ for all $x \in X$ and there exists $K_2 > 0$ such that for all $s \in [0,\infty)$

$$|h_i(s,x(s)) - h_j(s,y(s))| \leq K_2 |x(s) - y(s)|, \qquad \forall x,y \in X,\ i = 1,2,3.$$

The existence and uniqueness theorem can be formulated as follows:

Theorem 2.3. *Under the assumptions (C_0)–(C_3), if the following conditions are also satisfied:*

(a)

$$\mu \int_0^\infty k(t,s)h_2(s,\lambda \int_0^s m(s,r)g_1(r,x(r))dr + p_1(s) - p_2(s))ds = 0,$$
$$\mu \int_0^\infty k(t,s)h_3(s,\lambda \int_0^s m(s,r)g_2(r,x(r))dr + p_1(s) - p_2(s))ds = 0,$$
$$\mu \int_0^\infty k(t,s)h_1(s,\lambda \int_0^s m(s,r)g_3(r,x(r))dr + p_1(s) - p_2(s))ds = 0,$$

(b) for all $x \in X$,

$$x(t) \leq \lambda \int_0^t m(t,s)g_i(s,x(s))ds - p_2(t), \qquad i = 1,2,3,$$

(c) for all $x \in X$,

$$\lambda \int_0^t m(t,s)g_1(s,x(s) - p_1(t) - \mu \int_0^\infty k(t,r)h_2(r,x(r))dr)ds - p_2(t) \leq x(t),$$
$$\lambda \int_0^t m(t,s)g_2(s,x(s) - p_1(t) - \mu \int_0^\infty k(t,r)h_3(r,x(r))dr)ds - p_2(t) \leq x(t),$$
$$\lambda \int_0^t m(t,s)g_3(s,x(s) - p_1(t) - \mu \int_0^\infty k(t,r)h_1(r,x(r))dr)ds - p_2(t) \leq x(t).$$

Then the system of simultaneous Volterra-Hammerstein nonlinear integral equation (2.4) *has a unique solution in X for each pair of real numbers λ, μ with $|\mu|K_2M_2 < 1$ and $\frac{|\lambda|K_1M_1}{1-|\mu|K_2M_2} < \frac{1}{2}$.*

Proof. We define, for every $x \in X$

$$\begin{aligned} fx(t) &= -p_2(t) + \lambda \int_0^t m(t,s)g_1(s,x(s))ds, \\ gx(t) &= -p_2(t) + \lambda \int_0^t m(t,s)g_2(s,x(s))ds, \\ hx(t) &= -p_2(t) + \lambda \int_0^t m(t,s)g_3(s,x(s))ds, \end{aligned}$$

$$Sx(t) = (I-A)x(t), \quad Tx(t) = (I-B)x(t), \quad Rx(t) = (I-C)x(t),$$

where I is the identity operator on X and

$$\begin{aligned}
Ax(t) &= p_1(t)+\mu\int_0^\infty k(t,s)h_1(s,x(s))ds,\\
Bx(t) &= p_1(t)+\mu\int_0^\infty k(t,s)h_2(s,x(s))ds,\\
Cx(t) &= p_1(t)+\mu\int_0^\infty k(t,s)h_3(s,x(s))ds.
\end{aligned}$$

We see that x is a solution to (2.4) if and only if x is a common fixed point of f, g, h, S, T, and R. To prove the existence of such a point, we shall use Theorem 2.1. So, we have to check that all the hypotheses of Theorem 2.1 are satisfied.

We shall show that each f, g, h, S, T, R, A, B, and C are operators from X into itself.

$$|fx(t)|\le|\lambda|\int_0^\infty|m(t,s)g_1(s,x(s))|ds+|p_2(t)|\le|\lambda|\sup_{0\le s<\infty}|m(t,s)|\int_0^\infty|g_1(s,x(s))|ds+|p_2(t)|,$$

applying conditions (C_0) and (C_2), we have

$$\int_0^\infty|fx(t)|dt\le|\lambda|\int_0^\infty\sup_{0\le s<\infty}|m(t,s)|dt\int_0^\infty|g_1(s,x(s))|ds+\int_0^\infty|p_2(t)|dt<+\infty.$$

Hence $f\in X$. Similarly g, $h\in X$.
For mapping A we apply conditions (C_1) and (C_3) as following:

$$\int_0^\infty|Ax(t)|dt\le|\mu|\int_0^\infty\sup_{0\le s<\infty}|k(t,s)|dt\int_0^\infty|h_1(s,x(s))|ds+\int_0^\infty|p_1(t)|dt<+\infty.$$

Thus $A\in X$, similarly $B, C\in X$. we conclude that S, T, $R\in X$.

Now, we show the condition (i) of Theorem 2.1 is hold. We prove that $f(X)\subseteq T(X)$. For all $x\in X$, by using hypothesis (a), we get

$$\begin{aligned}
T(fx(t)+p_1(t)) &= (I-B)(fx(t)+p_1(t))\\
&= fx(t)-\mu\int_0^\infty k(t,s)h_2(s,fx(s)+p_1(s))ds\\
&= fx(t)-\mu\int_0^\infty k(t,s)h_2(s,-p_2(s)+\lambda\int_0^s h(s,r)g_1(r,x(r))dr+p_1(s))ds\\
&= fx(t).
\end{aligned}$$

Hence $f(X)\subseteq T(X)$. Similarly $g(X)\subseteq R(X)$, $h(X)\subseteq S(X)$.

From condition (b) we conclude that $x(t)\le fx(t)$ and hence $x\preceq fx$. Similarly we get $x\preceq gx$ and $x\preceq hx$. That is f, g, h are dominating operators. Also from condition (c) we obtain

$$fTx(t) = \lambda \int_0^t m(t,s)g_1(s,x(s) - p_1(t) - \mu \int_0^\infty k(t,r)h_2(r,x(r))dr)ds - p_2(t) \leq x(t).$$

Thus $fTx \preceq x$ Similarly we have $gRx \preceq x$ and $hSx \preceq x$. Hence the ordered pairs (f,T), (g,R) and (h,S) are weak annihilators. Therefore the condition (ii) of Theorem 2.1 is valid.

Further, we show the condition (iii) of Theorem 2.1. Suppose $\{x_n\} \subseteq X$ such that $x_n \to x$ as $n \to \infty$ and $Sx_n \to y$ as $n \to \infty$ we want to prove that $y \in S(X)$ and $y = S(x)$. From (C_3) we conclude that

$$\begin{aligned}
d(Ax_n,Ax) &= \int_0^\infty |Ax_n(t) - Ax(t)|dt \\
&= \int_0^\infty |\mu \int_0^t k(t,s)[h_1(s,x_n(s)) - h_1(s,x(s))]ds|dt \\
&\leq \int_0^\infty |\mu| \sup_{0\leq s<\infty} |k(t,s)|dt \int_0^\infty |h_1(s,x_n(s)) - h_1(s,x(s))|ds \\
&\leq |\mu|M_2K_2 \int_0^\infty |x_n(s) - x(s)|ds \\
&= |\mu|M_2K_2 d(x_n,x) \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Therefore

$$\begin{aligned}
d(Sx_n,Sx) &= d((I-A)x_n,(I-A)x) \\
&= \int_0^\infty |x_n(t) - x(t) - Ax_n(t) + Ax(t)|dt \\
&\leq \int_0^\infty |x_n(t) - x(t)|dt + \int_0^\infty |Ax_n(t) - Ax(t)|dt \\
&= d(x_n,x) + d(Ax_n,Ax) \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Therefore

$$G(Sx_n,Sx,Sx) = d(Sx_n,Sx) \to 0 \quad \text{as } n \to \infty.$$

This implies that $Sx_n \to Sx$ as $n \to \infty$. Hence $S(X)$ is G-closed. Thus from Corollary 1.10 $S(X)$ is complete subspace of (X,G). Now, we check the condition (2.1). Let $x,y, \in X$ such that $x \preceq y$ then

$$\begin{aligned}
d(fx,gy) &= \int_0^\infty |fx(t)-gy(t)|dt \\
&= \int_0^\infty |\lambda \int_0^t m(t,s)g_1(s,x(s))ds - \lambda \int_0^t m(t,s)g_2(s,y(s))ds|dt \\
&= \int_0^\infty |\lambda \int_0^t m(t,s)[g_1(s,x(s)) - g_2(s,y(s))]ds|dt \\
&\leq \int_0^\infty |\lambda| \sup_{0\leq s<\infty} |m(t,s)|dt \int_0^\infty |g_1(s,x(s)) - g_2(s,y(s))|ds \\
&\leq |\lambda|M_1K_1 \int_0^\infty |x(s)-y(s)|ds \\
&= |\lambda|M_1K_1 d(x,y) \\
&\leq |\lambda|M_1K_1 G(x,y,z).
\end{aligned}$$

Hence

$$d(fx,gy) \leq |\lambda|M_1K_1G(x,y,z). \tag{2.5}$$

Similarly, we can show that

$$d(gy,hz) \leq |\lambda|M_1K_1G(x,y,z) \quad \text{and} \quad d(hz,fx) \leq |\lambda|M_1K_1G(x,y,z). \tag{2.6}$$

Therefore from (2.5) and (2.6) we conclude that

$$G(fx,gy,hz) \leq |\lambda|M_1K_1G(x,y,z). \tag{2.7}$$

Likewise, we obtain

$$d(Ax,By) \leq |\mu|M_2K_2G(x,y,z),\ d(Bx,Cy) \leq |\mu|M_2K_2G(x,y,z),\ d(Cx,Ay) \leq |\mu|M_2K_2G(x,y,z).$$

Therefore

$$\begin{aligned}
d(Sx,Ty) &= d((I-A)x,(I-B)y) \\
&= \int_0^\infty |x(t)-y(t)-Ax(t)+By(t)|dt \\
&\geq \int_0^\infty |x(t)-y(t)|dt - \int_0^\infty |Ax(t)-By(t)|dt \\
&\geq d(x,y) - d(Ax,By) \\
&\geq d(x,y) - |\mu|M_2K_2G(x,y,z).
\end{aligned}$$

Thus

$$d(Sx,Ty) \geq (1-|\mu|M_2K_2)G(x,y,z). \tag{2.8}$$

Similarly, we get

$$d(Ty,Rz) \geq (1-|\mu|M_2K_2)G(x,y,z) \quad \text{and} \quad d(Rz,Sx) \geq (1-|\mu|M_2K_2)G(x,y,z). \tag{2.9}$$

Hence from (2.8) and (2.9) we obtain

$$G(Sx,Ty,Rz) \geq (1-|\mu|M_2K_2)G(x,y,z).$$

That is

$$G(x,y,z) \leq \frac{1}{(1-|\mu|M_2K_2)}G(Sx,Ty,Rz). \tag{2.10}$$

From (2.7) and (2.10) we obtain

$$\begin{aligned} G(fx,gy,hz) &\leq \frac{|\lambda|M_1K_1}{(1-|\mu|M_2K_2)}G(Sx,Ty,Rz) \\ &\leq kM(x,y,z), \end{aligned}$$

where

$$\begin{aligned} M(x,y,z) &= \max\{G(Sx,Ty,Rz),G(fx,fx,Sx),G(gy,gy,Ty),G(hz,hz,Rz), \\ &\quad (gy,gy,Sx),G(Ty,hz,hz),G(Rz,fx,fx)\}, \end{aligned}$$

and $k = \frac{|\lambda|M_1K_1}{(1-|\mu|M_2K_2)}$ Hence the generalized contractive condition (2.1) of Theorem 2.1 is satisfied. If $\{x_n\}$, $\{y_n\}$ are sequences in (X,G) such that $\{x_n\}$ is a monotone nondecreasing and $x_n \preceq y_n$ for all n with $y_n \to q$ as $n \to \infty$. From (2.3) we have

$$x_{n-1}(t) \leq x_n(t) \leq y_n(t), \qquad \forall n \geq 0.$$

Since $y_n \to q$ as $n \to \infty$ hence $\{y_n\}$ is bounded, this implies that $\{x_n\}$ is bounded above and so $\{x_n(t)\}$ is bounded above and since $\{x_n(t)\}$ is a monotone nondecreasing then $x_n(t) \to \sup_n\{x_n(t)\}$ and $x_n(t) \leq \sup_n\{x_n(t)\} \leq q(t)$. Therefore $x_n \preceq q$.
Thus all the conditions of Theorem 2.1 are satisfied and the solution of equations (2.4) exists. Further, since $([0,\infty),\leq)$ is well ordered set then $(X,\preceq)$ is well ordered set. Therefore the solution is unique. □

As an application of the existence and uniqueness of Theorem 2.2, we consider the problem of existence and uniqueness of solutions defined by a system of Urysohn integral equations as follows:

$$\begin{aligned} x(t) &= \int_a^b L_1(t,s,x(s))ds + p(t), \\ x(t) &= \int_a^b L_2(t,s,x(s))ds + p(t), \\ x(t) &= \int_a^b L_3(t,s,x(s))ds + p(t), \end{aligned} \tag{2.11}$$

where $t \in [a,b] \subseteq R^+$. A solution of the equation (2.11) is a function $x \in X = C[a,b]$ (the set of real continuous functions on $[a,b]$) and $d : X \times X \to R^+$ be defined by

$$d(x,y) = \max_{t\in[a,b]} |x(t) - y(t)|, \qquad x,y \in X,$$

with the G-metric defined by

$$G(x,y,z) = \max\{d(x,y), d(y,z), d(z,x)\}, \qquad x,y,z \in X.$$

Clearly (X,G) is a complete G-metric space.

Now, we state the following existence and uniqueness theorem for the solution of (2.11).

Theorem 2.4. *Consider* (2.11) *and assume that:*

(i) $x, p \in C[a,b]$ *and* $L_i : [a,b] \times [a,b] \times R \to R$, $i = 1,2,3$ *are continuous functions,*

(ii) for all $s,t \in [a,b]$ *and all* $x,y \in C[a,b]$ *we have*

$$|L_i(s,x(s)) - L_j(s,y(s))| \le q(t,s)|x(s) - y(s)|,$$

where $q : [a,b] \times [a,b] \to [0,\infty)$ *is a continuous function satisfying*

$$\sup \int_a^b |q(t,s)|dt < \frac{1}{\lambda(b-a)}, \quad \textit{where } \lambda \ge 1.$$

Then the system (2.11) *has a solution* $x \in X$.

Proof. Let $f,g,h : X \to X$ be the mappings defined by

$$\begin{aligned} fx(t) &= \int_a^b L_1(t,s,x(s))ds + p(t), \\ gx(t) &= \int_a^b L_2(t,s,x(s))ds + p(t), \\ hx(t) &= \int_a^b L_3(t,s,x(s))ds + p(t), \end{aligned}$$

for all $x \in X$ and for all $t \in [a,b]$. Obviously, the existence of a solution for (2.11) is equivalent to the existence of a common fixed point of f, g and h. Now, let $x, y \in X$, from condition (2.2) for all $t \in [a,b]$ we obtain

$$\begin{aligned}
\lambda |fx(t) - gy(t)|^2 &\leq \lambda \left(\int_a^b |L_1(t,s,x(s)) - L_2(t,s,y(s))| ds \right)^2 \\
&\leq \lambda \left(\int_a^b 1^2 ds \right) \left(\int_a^b |L_1(t,s,x(s)) - L_2(t,s,y(s))|^2 ds \right) \\
&\leq \lambda (b-a) \int_a^b q(t,s)|x(s) - y(s)|^2 ds \\
&\leq \lambda (b-a) \int_a^b q(t,s) d(x,y)^2 ds \\
&\leq \lambda (b-a) \int_a^b q(t,s) G(x,y,z)^2 ds \\
&\leq \lambda (b-a) G(x,y,z)^2 \left(\int_a^b q(t,s) ds \right) \\
&\leq \lambda (b-a) M(x,y,z)^2 \left(\sup \int_a^b q(t,s) ds \right) \\
&< M(x,y,z)^2 \\
&= \lambda M(x,y,z)^2 - \lambda M(x,y,z)^2 + M(x,y,z)^2.
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda (d(fx,gy))^2 &= \max_{t \in [a,b]} |fx(t) - gy(t)|^2 \\
&< \lambda M(x,y,z)^2 - \lambda M(x,y,z)^2 + M(x,y,z)^2. \qquad (2.12)
\end{aligned}$$

Similarly, for x, y, $z \in X$ we can show that

$$\begin{aligned}
\lambda (d(gy,hz))^2 &= \max_{t \in [a,b]} |gy(t) - hz(t)|^2 \\
&< \lambda M(x,y,z)^2 - \lambda M(x,y,z)^2 + M(x,y,z)^2, \qquad (2.13)
\end{aligned}$$

and

$$\begin{aligned}
\lambda (d(hz,fx))^2 &= \max_{t \in [a,b]} |hz(t) - fx(t)|^2 \\
&< \lambda M(x,y,z)^2 - \lambda M(x,y,z)^2 + M(x,y,z)^2. \qquad (2.14)
\end{aligned}$$

Thus from (2.12), (2.13) and (2.14), we get

$$\begin{aligned}\lambda(G(fx,gy,hz))^2 &= \lambda\max\{d(fx,gy)^2, d(gy,hz)^2, d(hz,fx)^2\}\\ &< \lambda M(x,y,z)^2 - \lambda M(x,y,z)^2 + M(x,y,z)^2.\end{aligned}$$

Taking $\psi(t) = \lambda t^2$, $\varphi(t) = (\lambda-1)t^2$ in Theorem 2.2, there exists a unique common fixed point of f, g and h, which is a solution of (2.11). $\square$

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