International Journal of Applied Mathematical Research, 1 (4) (2012) 593-603

www.sciencepubco.com/index.php/IJAMR

Right Circulant Matrices With Geometric Progression

Aldous Cesar F. Bueno

Department of Mathematics and Physics, Central Luzon State Univeristy
Science City of Muñoz 3120, Nueva Ecija, Philippines
E-mail: aldouz_cezar@yahoo.com

Abstract

In this paper, the right circulant matrix $RCIRC_n(\vec{g}) \in M_{nxn}(\mathbb{R})$ with circulant vector $\vec{g} = (a, ar, ar^2, \dots, ar^{n-1})$, where $a \neq 0$ and $r \neq 0,1$, was investigated and its inverse $RCIRC_n{}^{-1}(g)$ was obtained. The eigenvalues, determinant, Euclidean norm, and spectral norm of both $RCIRC_n(\vec{g})$ and $RCIRC_n{}^{-1}(\vec{g})$ were determined. Some examples were provided to illustrate the obtained results.

Keywords: *Determinant, Eigenvalue, Euclidean norm, Right circulant matrix with geometric progression, Spectral norm*

1 Introduction

A matrix $C \in M_{nxn}(\mathbb{R})$ is said to be a right circulant matrix if it is of the form

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & & c_{n-3} & c_{n-2} \\ & \vdots & & \ddots & \vdots & \\ c_2 & c_3 & c_4 & \cdots & c_0 & c_1 \\ c_1 & c_2 & c_3 & & c_{n-1} & c_0 \end{pmatrix}$$

The matrix C has the following structure:

1. Each row is a right cyclic shift of the row above it. Thus, C is determined by the first row

$$(c_0, c_1, c_2, \dots, c_{n-1})$$

2. $c_k = c_{i,j}$ whenever i – j = k (mod n)

Definition 1.1

1. Let $\vec{c} = (c_0, c_1, c_2, \dots, c_{n-1})$ then the right circulant matrix $C \in M_{nxn}(\mathbb{R})$ is denoted by $\boldsymbol{RCIRC_n(\vec{c})}$
2. The vector $\boldsymbol{\vec{c}} = (\boldsymbol{c_0}, \boldsymbol{c_1}, \boldsymbol{c_2}, \dots, \boldsymbol{c_{n-1}})$ is called the **circulant vector**
3. $\boldsymbol{RCIRC_n}(\mathbb{R}) = \{\boldsymbol{RCIRC_n(\vec{c})} | \boldsymbol{\vec{c}} \in \mathbb{R}^{\boldsymbol{n}}\}$

Properties of $\boldsymbol{RCIRC_n(\vec{c})}$

1. The eigenvalues of $\boldsymbol{RCIRC_n(\vec{c})}$ are just the **Discrete Fourier Transform** of the circulant vector $\vec{c}$. That is

$$\lambda_m = \sum_{k=0}^{n-1} c_k \omega^{-mk}$$

where $\omega = e^{2\pi i/n}$ and m=0,1, …, n-1

2. The eigenvectors of $\boldsymbol{RCIRC_n(\vec{c})}$ are the columns of the Fourier matrix F. That is

$$v_m = \frac{1}{\sqrt{n}}\left(1, \omega^m, \omega^{2m}, \dots, \omega^{(n-1)m}\right)$$

3. The Fourier matrix F is a simultaneous, unitary, diagonalizing matrix for $RCIRC_n(\vec{c})$. That is, for any $RCIRC_n(\vec{c})$

$$RCIRC_n(\vec{c}) = F\mathrm{DF}^{-1},$$

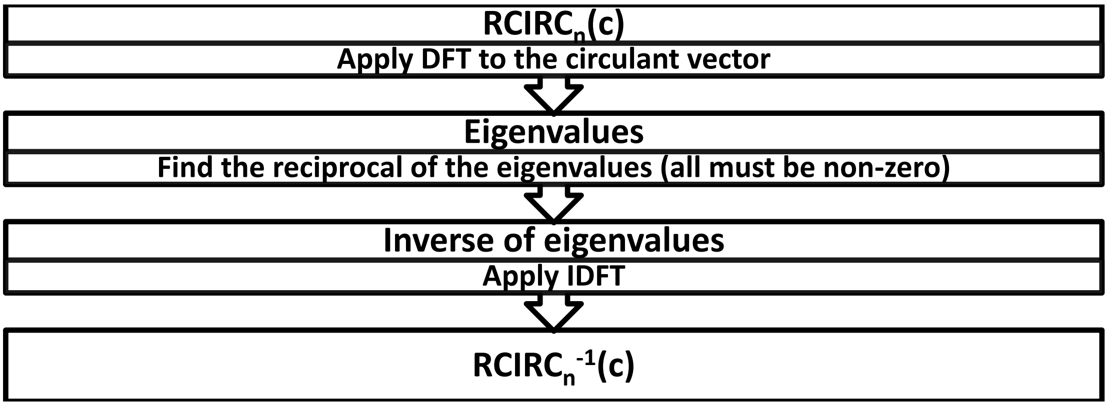
where $\mathrm{D} = \mathrm{diag}(\lambda_0, \lambda_1, \dots, \lambda_{n-1})$ and $FF^* = I$.

4. The determinant of $RCIRC_n(\vec{c})$ denoted by $|RCIRC_n(\vec{c})|$ is given by

$$|RCIRC_n(\vec{c})| = \prod_{m=0}^{n-1} \lambda_m = \prod_{m=0}^{n-1} \left(\sum_{k=0}^{n-1} c_k \omega^{-mk} \right)$$

where $\omega = e^{2\pi i/n}$ and m=0,1, …, n-1

Inversion of $RCIRC_n(\vec{c})$
From the circulant vector we can obtain the eigenvalues of $RCIRC_n(\vec{c})$ through Discrete Fourier Transform (DFT). Furthermore, through the Inverse Discrete Fourier Transform (IDFT) of the eigenvalues of $RCIRC_n(\vec{c})$ the circulant vector can be obtained and hence $RCIRC_n(\vec{c})$ itself.
Note that if $\lambda \neq 0$ is an eigenvalue of an invertible matrix A, then λ^{-1} is also an eigenavlue of the matrix A^{-1}. With the help of this concept we can derive the matrix ${RCIRC_n}^{-1}(\vec{c})$, the inverse of $RCIRC_n(\vec{c})$.
The flowchart of finding ${RCIRC_n}^{-1}(c)$ is as given below.



Given $RCIRC_n(\vec{c})$ with eigenvalues $\lambda_m = \sum_{k=0}^{n-1} c_k \omega^{-mk} \neq 0$ where m=0,1, … , n-1, then ${RCIRC_n}^{-1}(\vec{c})$ is given by

$$
{RCIRC_n}^{-1}(\vec{c}) = \begin{pmatrix} \mathcal{C}_0 & \mathcal{C}_1 & \mathcal{C}_2 & \cdots & \mathcal{C}_{n-2} & \mathcal{C}_{n-1} \\ \mathcal{C}_{n-1} & \mathcal{C}_0 & \mathcal{C}_1 & & \mathcal{C}_{n-3} & \mathcal{C}_{n-2} \\ \vdots & & & \ddots & & \vdots \\ \mathcal{C}_2 & \mathcal{C}_3 & \mathcal{C}_4 & \cdots & \mathcal{C}_0 & \mathcal{C}_1 \\ \mathcal{C}_1 & \mathcal{C}_2 & \mathcal{C}_3 & & \mathcal{C}_{n-1} & \mathcal{C}_0 \end{pmatrix}
$$

where $\mathcal{C}_k = \frac{1}{n}\sum_{m=0}^{n-1} {\lambda_m}^{-1}\omega^{mk}$ and $(\mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_{n-1})$ is the circulant vector.

As an example, consider the matrix

$$
\begin{pmatrix} 1 & 2 & 4 & 8 \\ 8 & 1 & 2 & 4 \\ 4 & 8 & 1 & 2 \\ 2 & 4 & 8 & 1 \end{pmatrix}
$$

Its eigenvalues are $\lambda_0 = 15, \lambda_1 = -3 + 6i, \lambda_2 = -3 - 6i, \lambda_3 = -5$ with inverses $\frac{1}{15}, \frac{-1}{15} \pm \frac{2}{15}i, \frac{-1}{2}$ respectively. Performing the IDFT to each we will obtain the following: $\mathcal{C}_0 = -\frac{1}{15}, \mathcal{C}_1 = \frac{2}{15}, \mathcal{C}_2 = \mathcal{C}_3 = 0$.

Thus the inverse is given by $\begin{pmatrix} -\frac{1}{15} & \frac{2}{15} & 0 & 0 \\ 0 & -\frac{1}{15} & \frac{2}{15} & 0 \\ 0 & 0 & -\frac{1}{15} & \frac{2}{15} \\ \frac{2}{15} & 0 & 0 & -\frac{1}{15} \end{pmatrix}$.

Furthermore

$$\begin{pmatrix} 1 & 2 & 4 & 8 \\ 8 & 1 & 2 & 4 \\ 4 & 8 & 1 & 2 \\ 2 & 4 & 8 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{15} & \frac{2}{15} & 0 & 0 \\ 0 & -\frac{1}{15} & \frac{2}{15} & 0 \\ 0 & 0 & -\frac{1}{15} & \frac{2}{15} \\ \frac{2}{15} & 0 & 0 & -\frac{1}{15} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2 The Right Circulant Matrix $\mathbf{RCIRC_n(\vec{g})}$

Let $\vec{g} = (a\ ar\ ...\ ar^{n-1})$ be the circulant vector of a right circulant matrix where $a \neq 0, r \neq 0,1$. Then the right circulant matrix with geometric progression denoted by $RCIRC_n(\vec{g})$ is the matrix of the form

$$\boldsymbol{RCIRC_n(\vec{g})} = \begin{pmatrix} a & ar & ar^2 & \cdots & ar^{n-2} & ar^{n-1} \\ ar^{n-1} & a & ar & & ar^{n-3} & ar^{n-2} \\ \vdots & & & \ddots & & \vdots \\ ar^2 & ar^3 & ar^4 & \cdots & a & ar \\ ar & ar^2 & ar^3 & & ar^{n-1} & a \end{pmatrix}$$

Now, for the rest of the paper the following notations will be used:

$\vec{\boldsymbol{g}} = (a\ ar\ ...\ ar^{n-1})$: circulant vector with geometric sequence whose first term is a≠0 and whose common ratio is r≠0,1

$\boldsymbol{RCIRC_n(\vec{g})}$: right cicrulant matrix with geometric sequence with dimension nxn

$|\boldsymbol{RCIRC_n(\vec{g})}|$: determinant of $RCIRC_n(\vec{g})$

$||\boldsymbol{RCIRC_n(\vec{g})}||_{\boldsymbol{E}}$: Euclidean norm of $RCIRC_n(\vec{g})$

$\|\boldsymbol{RCIRC_n(\vec{g})}\|_2$: Spectral norm of $RCIRC_n(\vec{g})$
$\boldsymbol{RCIRC_n}^{-1}(\vec{\boldsymbol{g}})$: inverse of $RCIRC_n(\vec{g})$

3 Main Results

Theorem 3.1

$$|\boldsymbol{RCIRC_n(\vec{g})}| = \boldsymbol{a^n(1-r^n)^{n-1}}$$

Proof:

$$\boldsymbol{RCIRC_n(\vec{g})} = \begin{pmatrix} a & ar & ar^2 & \cdots & ar^{n-2} & ar^{n-1} \\ ar^{n-1} & a & ar & & ar^{n-3} & ar^{n-2} \\ & \vdots & & \ddots & & \vdots \\ ar^2 & ar^3 & ar^4 & \cdots & a & ar \\ ar & ar^2 & ar^3 & & ar^{n-1} & a \end{pmatrix}$$
$$= a\begin{pmatrix} 1 & r & r^2 & \cdots & r^{n-2} & r^{n-1} \\ r^{n-1} & 1 & r & & r^{n-3} & r^{n-2} \\ & \vdots & & \ddots & & \vdots \\ r^2 & r^3 & r^4 & \cdots & 1 & r \\ r & r^2 & r^3 & & r^{n-1} & 1 \end{pmatrix}$$

Now let $K = \begin{pmatrix} 1 & r & r^2 & \cdots & r^{n-2} & r^{n-1} \\ r^{n-1} & 1 & r & & r^{n-3} & r^{n-2} \\ & \vdots & & \ddots & & \vdots \\ r^2 & r^3 & r^4 & \cdots & 1 & r \\ r & r^2 & r^3 & & r^{n-1} & 1 \end{pmatrix}$

Note that $|cA| = c^n|A|$ so $|\boldsymbol{RCIRC_n(\vec{g})}| = a^n|K|$

$$K = \begin{pmatrix} 1 & r & r^2 & \cdots & r^{n-2} & r^{n-1} \\ r^{n-1} & 1 & r & & r^{n-3} & r^{n-2} \\ & \vdots & & \ddots & & \vdots \\ r^2 & r^3 & r^4 & \cdots & 1 & r \\ r & r^2 & r^3 & & r^{n-1} & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & r & r^2 & \cdots & r^{n-2} & r^{n-1} \\ 0 & -(r^n-1) & -r(r^n-1) & & -r^{n-3}(r^n-1) & -r^{n-2}(r^n-1) \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & -(r^n-1) & -r \\ 0 & 0 & 0 & & 0 & -(r^n-1) \end{pmatrix}$$

by applying the row operation $-r^{n-k}R_1 + R_{k+1} \rightarrow R_{k+1}$ where k=1,2,3,…,n-1
From the equivalent lower diagonal matrix we get $|K| = (1-r^n)^{n-1}$.
Thus $|\boldsymbol{RCIRC_n(\vec{g})}| = a^n(1-r^n)^{n-1}$

Theorem 3.2

The eigenvalues of $RCIRC_n(\vec{g})$ are $\lambda_0 = S_n \; and \; \lambda_m = \frac{a(r^n-1)}{re^{-2\pi im/n}-1}$ where m=1,2,..,n-1

Proof:
Note that $\lambda_m = \sum_{k=0}^{n-1} c_k e^{-2\pi imk/n}$

For m=0, we have

$$\lambda_0 = \sum_{k=0}^{n-1} c_k = \sum_{k=0}^{n-1} ar^k = \frac{a(r^n-1)}{r-1} = S_n$$

For m≠0, we have

$$\lambda_m = \sum_{k=0}^{n-1} c_k e^{-2\pi imk/n}$$

$$= \sum_{k=0}^{n-1} ar^k e^{-\frac{2\pi imk}{n}} = a\sum_{k=0}^{n-1} r^k e^{-2\pi imk/n} = \frac{a\left(r^n e^{-2\pi im} - 1\right)}{re^{-2\pi im/n} - 1}$$

$$= \frac{a(r^n-1)}{re^{-2\pi im/n} - 1}$$

Theorem 3.3

$$\|\boldsymbol{RCIRC_n}(\vec{\boldsymbol{g}})\|_{\boldsymbol{E}} = |a|\sqrt{\frac{n(1-r^{2n})}{1-r^2}}$$

Proof:

$$\|\boldsymbol{RCIRC_n}(\vec{\boldsymbol{g}})\|_E = \sqrt{\sum_{i,j=0}^{n-1} {a_{ij}}^2} = \sqrt{\sum_{k=0}^{n-1} n(ar^k)^2} = \sqrt{a^2 n\sum_{k=0}^{n-1} r^{2k}}$$

$$= |a|\sqrt{\frac{n(1-r^{2n})}{1-r^2}}$$

Theorem 3.4

$$\|\boldsymbol{RCIRC_n}(\vec{\boldsymbol{g}})\|_{\boldsymbol{2}} = max\left\{|S_n|, \frac{|a(r^n-1)|}{\sqrt{r^2 - 2rcos\frac{2\pi m}{n} + 1}}\right\}$$

Proof:
For m=0, $|\lambda_0| = |S_n|$

For m≠0, we have

$$|\lambda_m| = \left|\frac{a(r^n-1)}{re^{\frac{2\pi im}{n}}-1}\right| = \frac{|a(r^n-1)|}{\left|r\left(cos\frac{2\pi im}{n}+isin\frac{2\pi im}{n}\right)-1\right|}$$

$$= \frac{|a(r^n-1)|}{\sqrt{\left(rcos\frac{2\pi im}{n}-1\right)^2+r^2sin^2\frac{2\pi im}{n}}}$$

$$= \frac{|a(r^n-1)|}{\sqrt{r^2-2rcos\frac{2\pi m}{n}+1}}$$

Corollary 3.5

$$\left|\boldsymbol{RCIRC_n}^{-\mathbf{1}}(\vec{\boldsymbol{g}})\right| = \frac{1}{a^n(1-r^n)^{n-1}}$$

Corollary 3.6

The eigenvalues of $\boldsymbol{RCIRC_n}^{-\mathbf{1}}(\vec{\boldsymbol{g}})$ are $\lambda_0{}^{-1} = \frac{1}{S_n}$ *and* $\lambda_m{}^{-1} = \frac{re^{-2\pi im/n}-1}{a(r^n-1)}$ where m=1,2,…,n-1

Corollary 3.7

$$\left\|\boldsymbol{RCIRC_n}^{-\mathbf{1}}(\vec{\boldsymbol{g}})\right\|_{\boldsymbol{E}} \geq \frac{1}{|a|\sqrt{\frac{n(1-r^{2n})}{1-r^2}}}$$

Theorem 3.8

For n≥3,

$$\boldsymbol{RCIRC_n}^{-\mathbf{1}}(\vec{\boldsymbol{g}}) = \begin{pmatrix} \mathcal{C}_0 & \mathcal{C}_1 & 0 & \cdots & 0 & 0 \\ 0 & \mathcal{C}_0 & \mathcal{C}_1 & & 0 & 0 \\ & \vdots & & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & \mathcal{C}_0 & \mathcal{C}_1 \\ \mathcal{C}_1 & 0 & 0 & & 0 & \mathcal{C}_0 \end{pmatrix}$$

$$= \frac{1}{a(r^n-1)}\begin{pmatrix} -1 & r & 0 & \cdots & 0 & 0 \\ 0 & -1 & r & & 0 & 0 \\ & \vdots & & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & -1 & r \\ r & 0 & 0 & & 0 & -1 \end{pmatrix}$$

where $C_0 = \frac{-1}{a(r^n-1)}$ and $C_1 = \frac{r}{a(r^n-1)}$

Proof:
Note that the first row entries of a circulant matrix which determines the matrix is just the Inverse Discrete Fourier Transform (IDFT) of its eigenvalues. From Corollary 3.6, the eigenvalues of $\boldsymbol{RCIRC_n}^{-1}(\vec{\boldsymbol{g}})$ are $\lambda_0{}^{-1} = \frac{1}{S_n}$ $and\ \lambda_m{}^{-1} = \frac{re^{-2\pi im/n}-1}{a(r^n-1)}$ where m=1,2,…,n-1. By performing IDFT to them, we will get the entries of $\boldsymbol{RCIRC_n}^{-1}(\vec{\boldsymbol{g}})$.

$$C_k = \frac{1}{n}\sum_{m=0}^{n-1}\frac{r\theta^{-m}-1}{a(r^n-1)}\theta^{km}\ where\ \theta = e^{\frac{2\pi i}{n}}$$

For k=0

$$C_0 = \frac{1}{n}\sum_{m=0}^{n-1}\frac{r\theta^{-m}-1}{a(r^n-1)} = \frac{1}{na(r^n-1)}\sum_{m=0}^{n-1}(r\theta^{-m}-1)$$
$$= \frac{1}{na(r^n-1)}\left[\frac{r(1-\theta^n)}{1-\theta} - n\right] = \frac{-1}{a(r^n-1)}$$

For k=1

$$C_1 = \frac{1}{n}\sum_{m=0}^{n-1}\frac{r\theta^{-m}-1}{a(r^n-1)}\theta^m$$
$$= \frac{1}{na(r^n-1)}\sum_{m=0}^{n-1}(r-\theta^m) = \frac{1}{na(r^n-1)}\left(rn - \frac{1-\theta^n}{1-\theta}\right)$$
$$= \frac{r}{a(r^n-1)}$$

For k=2, 3, … , n-1

$$C_k = \frac{1}{n}\sum_{m=0}^{n-1}\frac{r\theta^{-m}-1}{a(r^n-1)}\theta^{km}$$
$$= \frac{1}{na(r^n-1)}\sum_{m=0}^{n-1}\left(r\theta^{m(k-1)} - \theta^{km}\right)$$
$$= \frac{1}{na(r^n-1)}\left[\frac{r(1-\theta^{(k-1)n})}{1-\theta^{k-1}} - \frac{1-\theta^{kn}}{1-\theta^k}\right] = 0$$

which is as desired.

Remarks: For n=1, 2

(a) has inverse $(1/a)$

$\begin{pmatrix} a & ar \\ ar & a \end{pmatrix}$ has inverse $\frac{-1}{a^2(r^2-1)}\begin{pmatrix} -1 & r \\ r & -1 \end{pmatrix}$

Theorem 3.9

$$\left\|\boldsymbol{RCIRC_n}^{-1}(\vec{\boldsymbol{g}})\right\|_{\boldsymbol{E}} = \frac{\sqrt{n(r^2+1)}}{|a(r^n-1)|}$$

Proof:

$$\left\|\boldsymbol{RCIRC_n}^{-1}(\vec{\boldsymbol{g}})\right\|_{\boldsymbol{E}} = \sqrt{\sum_{k=0}^{n-1} n\mathcal{C}_k{}^2} = \sqrt{n\left(\frac{-1}{a(r^n-1)}\right)^2 + n\left(\frac{r}{a(r^n-1)}\right)^2}$$

$$= \sqrt{\frac{n(r^2+1)}{[a(r^n-1)]^2}} = \frac{\sqrt{n(r^2+1)}}{|a(r^n-1)|}$$

Corollary 3.10

$$\left\|\boldsymbol{RCIRC_n}^{-1}(\vec{\boldsymbol{g}})\right\|_{\boldsymbol{2}} = max\left\{\frac{1}{|S_n|}, \frac{\sqrt{r^2 - 2rcos\frac{2\pi m}{n} + 1}}{|a(r^n-1)|}\right\}$$

4 Examples

Consider the right circulant matrix with the circulant vector $\vec{g} = (4, 12, 36, 108)$

Hence $RCIRC_4(\vec{g}) = \begin{pmatrix} 4 & 12 & 36 & 108 \\ 108 & 4 & 12 & 36 \\ 36 & 108 & 4 & 12 \\ 12 & 36 & 108 & 4 \end{pmatrix} = 4\begin{pmatrix} 1 & 3 & 9 & 27 \\ 27 & 1 & 3 & 9 \\ 9 & 27 & 1 & 3 \\ 3 & 9 & 27 & 1 \end{pmatrix}$

$$|\boldsymbol{RCIRC_4}(\vec{\boldsymbol{g}})| = 4^4(1-4^4)^3 = -4244832000$$

Eigenvalues of $\boldsymbol{RCIRC_4}(\vec{\boldsymbol{g}})$:

$$\lambda_0 = 160$$
$$\lambda_1 = -32 + 96i$$
$$\lambda_2 = -32 - 96i$$
$$\lambda_3 = -80$$

$$\|\boldsymbol{RCIRC_4}(\vec{\boldsymbol{g}})\|_{\boldsymbol{E}} = |4|\sqrt{\frac{4(1-3^8)}{1-3^2}} = 4\sqrt{\frac{4(-6560)}{-8}} = 8\sqrt{820} = 16\sqrt{205}$$

$$\|\boldsymbol{RCIRC_4}(\vec{\boldsymbol{g}})\|_{\boldsymbol{2}} = \max\{\lambda_m\} = \lambda_0 = 160$$

$$\boldsymbol{RCIRC_4}^{-\boldsymbol{1}}(\vec{\boldsymbol{g}}) = \frac{1}{4(3^4-1)}\begin{pmatrix} -1 & 3 & 0 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & -1 & 3 \\ 3 & 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-1}{320} & \frac{3}{320} & 0 & 0 \\ 0 & \frac{-1}{320} & \frac{3}{320} & 0 \\ 0 & 0 & \frac{-1}{320} & \frac{3}{320} \\ \frac{3}{320} & 0 & 0 & \frac{-1}{320} \end{pmatrix}$$

$$\left|\boldsymbol{RCIRC_4}^{-\boldsymbol{1}}(\vec{\boldsymbol{g}})\right| = \frac{-1}{4244832000}$$

Eigenvalues of $\boldsymbol{RCIRC_4}^{-\boldsymbol{1}}(\vec{\boldsymbol{g}})$

$$\lambda_0^{-1} = \frac{1}{160}$$

$$\lambda_1^{-1} = \frac{1}{-32+96i} = \frac{-32}{10240} + \frac{96}{10240}i = \frac{-1}{320} + \frac{3}{320}i$$

$$\lambda_2^{-1} = \frac{1}{-32-96i} = \frac{-32}{10240} - \frac{96}{10240}i = \frac{-1}{320} - \frac{3}{320}i$$

$$\lambda_3^{-1} = -\frac{1}{80}$$

$$\left\|\boldsymbol{RCIRC_4}^{-\boldsymbol{1}}(\vec{\boldsymbol{g}})\right\|_{\boldsymbol{E}} = \frac{\sqrt{4(3^2+1)}}{|4(3^4-1)|} = \frac{\sqrt{10}}{160}$$

$$\left\|\boldsymbol{RCIRC_4}^{-\boldsymbol{1}}(\vec{\boldsymbol{g}})\right\|_{\boldsymbol{2}} = max\{|\lambda_m^{-1}|\} = |\lambda_m^{-1}| = \frac{1}{80}$$

References

[1] M. Bahsi and S. Solak, “On the Circulant Matrices with Arithmetic Sequence”, International Journal of Contemporary Matematical. Sciences, Vol. 5, 2010, no. 25, 1213 – 1222.

[2] R. M. Gray,” Toeplitz and Circulant Matrices: A Review,’ Technical Report, Stanford University, USA.