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Research Paper

On the number of paths of length 5 in a graph

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Abstract

In this paper, we obtain an explicit formula for the total number of paths of length 5 in a simple graph G. We also determine some formulae for the number of paths of length 5 each of which starts from a specific vertex v_i and for the number of $v_i - v_j$ paths of length 5 in a simple graph G, in terms of the adjacency matrix and with the help of combinatorics.

Keywords: *Adjacency Matrix; Cycle; Graph Theory; Path; Subgraph, Walk.*

1. Introduction

In a simple graph G, a walk is a sequence of vertices and edges of the form $v_0, e_1, v_1, ..., e_k, v_k$ such that the edge e_i has ends v_{i-1} and v_i. A walk is called closed if $v_0 = v_k$. If the vertices of a walk are distinct then the walk is called a path. A cycle is a non-trivial closed walk in which all the vertices are distinct except the end vertices.

It is known that if a graph G has adjacency matrix A=$[a_{ij}]$, then for k = 0, 1, ... , the ij-entry of A^k is the number of $v_i - v_j$ walks of length k in G. It is also known that tr(A^n) is the sum of the diagonal entries of A^n and d_i is the degree of the vertex v_i.

In 1971, Frank Harary and Bennet Manvel [2], gave formulae for the number of cycles of lengths 3 and 4 in simple graphs as given by the following theorems:

Theorem 1.1 [2] *If G is a simple graph with adjacency matrix A, then the number of 3-cycles in G is* $\frac{1}{6}$ *tr(A^3).*
(It is known that $tr(A^3) = \sum_{i=1}^{n} a_{ii}^{(3)} = \sum_{j \neq i} a_{ij}^{(2)} a_{ij}$*).*

Theorem 1.2 [2] *If G is a simple graph with adjacency matrix A, then the number of* $4-$*cycles in G is* $\frac{1}{8}[\ tr(A^4) - 2q - 2 \sum_{j \neq i} a_{ij}^{(2)}]$*, where q is the number of edges in G.*

(It is obvious that the above formula is also equal to $\frac{1}{8}\ [trA^4 - trA^2 - 2 \sum_{j \neq i} a_{ij}^{(2)}]$ *)*

They also gave a formula for the number of 5-cycles in a simple graph. Their proofs are based on the following fact: The number of n-cycles (n= $3, 4, 5$) in a graph G is equal to $\frac{1}{2n}(\mathrm{tr}(A^n) - x)$ where x is the number of closed walks of length n, which are not n-cycles.

In 1986, Tomescu [4], gave some formulae for the number of paths of length s, having k edges in common with a fixed s-path of a complete graph. In 1994, Bax [5], gave an algorithm to count number of all paths and $v_i - v_j$ paths in a graph. His algorithm cannot count the number of paths of a specific size.
In 1996, Eric Bax and Joel Franklin [7], gave an algorithm to count paths and cycles of a given length in a directed graph. In [6, 8, 9, 10, 12, 13, 15], we have also some bounds to estimate the total time complexity for finding or counting paths and cycles in a graph.
In the previous works there is no formula to count the exact number of paths of a specific size in a graph.
In our recent work [1], we obtained some formulae and propositions to find the exact number of paths of lengths 3 and 4, in a simple graph G, given below:

Proposition 1.3 [1] *In a simple graph G with n vertices and the adjacency matrix A= $[a_{ij}]$, the number of paths of length n is* $\sum_{j\neq i} a_{ij}^{(n)} - x$, *where x is the number of non-closed walks of length n in G, which are not paths.*

Proposition 1.4 [1] *In a simple graph G with n vertices and the adjacency matrix A= $[a_{ij}]$, the number of paths of length n, each of which begins with a specific vertex v_i is* $\sum_{j=1, j\neq i}^{n} a_{ij}^{(n)} - x$, *where x is the number of non-closed walks of length n in G, starting from the vertex v_i, which are not paths.*

Proposition 1.5 [1] *In a simple graph G with n vertices and the adjacency matrix A= $[a_{ij}]$, the number of $v_i - v_j$ $(j \neq i)$ paths of length n is $a_{ij}^{(n)} - x$, where x is the number of non-closed $v_i - v_j$ walks of length n in G, which are not paths.*

Theorem 1.6 [1] *Let G be a simple graph with n vertices and the adjacency matrix A= $[a_{ij}]$. The number of paths of length 3 in G is* $\sum_{j\neq i} a_{ij}^{(2)}(d_j - a_{ij} - 1)$.

Theorem 1.7 [1] *Let G be a simple graph with n vertices and the adjacency matrix A= $[a_{ij}]$. The number of paths of length 4 in G is* $\sum_{j\neq i}[a_{ij}^{(4)} - 2a_{ij}^{(2)}(d_j - a_{ij})] - \sum_{i=1}^{n}[(2d_i - 1)a_{ii}^{(3)} + 6\binom{d_i}{3}]$.

Theorem 1.8 [1] *Let G be a simple graph with n vertices and the adjacency matrix A= $[a_{ij}]$. The number of paths of length 3 in G, each of which starts from a specific vertex v_i is* $\sum_{j=1, j\neq i}^{n} a_{ij}^{(2)}(d_j - a_{ij} - 1)$.

Theorem 1.9 [1] *Let G be a simple graph with n vertices and the adjacency matrix A= $[a_{ij}]$. The number of paths of length 4 in G, each of which starts from a specific vertex v_i is* $\sum_{j=1, j\neq i}^{n}[a_{ij}^{(4)} - (d_i + d_j - 3a_{ij})a_{ij}^{(2)} - (a_{ii}^{(3)} + a_{jj}^{(3)} + 2\binom{d_j - 1}{2})a_{ij}]$.

Theorem 1.10 [1] *Let G be a simple graph with n vertices and the adjacency matrix A= $[a_{ij}]$. The number of $v_i - v_j$ $(j \neq i)$ paths of length 3 in G is* $\sum_{k=1, k\neq i,j}^{n}(a_{ik}^{(2)} - a_{ij})a_{jk}$.

In this paper we give some formulae to count the exact number of paths of length 5 in a simple graph G, in terms of the adjacency matrix of G and with the help of combinatorics.

2. Number of paths of length 5

In this section, we give formulae to count the number of paths of length 5 in a simple graph G. We first give a result below which is useful to prove our other theorems.

Theorem 2.1 *If G is a simple graph with n vertices and the adjacency matrix A= $[a_{ij}]$, then the number of 4-cycles each of which contains a specific vertex v_i of G is* $\frac{1}{2}\left[a_{ii}^{(4)} - a_{ii}^{(2)} - 2\binom{d_i}{2} - \sum_{j=1, j\neq i}^{n} a_{ij}^{(2)}\right]$.

Proof: The number of 4−cycles each of which contains a specific vertex v_i of the graph G is equal to $\frac{1}{2}(a_{ii}^{(4)} - x)$, where x is the number of closed walks of length 4 from the vertex v_i to v_i that are not 4−cycles. To find x, we have 3 cases as considered below; the cases are based on the configurations-(subgraphs) that generate $v_i - v_i$ walks of length 4 that are not cycles. In each case, N denote the number of walks of length 4 from v_i to v_i that are not cycles in the corresponding subgraph, M denote the number of subgraphs of G of the same configuration and F denote the total number of $v_i - v_i$ walks of length 4 that are not cycles in all possible subgraphs of G of the same configuration. It is clear that F is equal to N× M. To find N in each case, we have to include in any walk, all the edges and the vertices of the corresponding subgraphs at least once.

Case 1: For the configuration of Fig 1, N= 1, M= $a_{ii}^{(2)}$, F= $a_{ii}^{(2)}$.

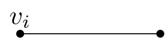


Fig 1

Case 2: For the configuration of Fig 2, N= 2, M= $\binom{d_i}{2}$, F= $2\binom{d_i}{2}$.

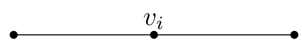


Fig 2

Case 3: For the configuration of Fig 3, N= 1, M= $\sum_{j=1, j\neq i}^{n} a_{ij}^{(2)}$, F= $\sum_{j=1, j\neq i}^{n} a_{ij}^{(2)}$.

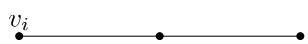


Fig 3

Consequently, $x = a_{ii}^{(2)} + 2\binom{d_i}{2} + \sum_{j=1, j\neq i}^{n} a_{ij}^{(2)}$ and we get the required result. □

Example 2.2 *In the graph of Fig 4, we have* $a_{11}^{(4)} = 21$, $a_{11}^{(2)} = 3$, $2\binom{d_1}{2} = 6$, $\sum_{j=2}^{4} a_{1j}^{(2)} = 6$. *So, by Theorem 2.1, the number of 4-cycles each of which contains the vertex v_1 in the graph of Fig 4 is 3.*

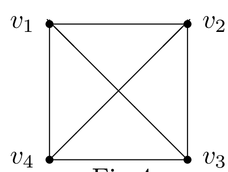


Fig 4

Theorem 2.3 *Let G be a simple graph with n vertices and the adjacency matrix A= $[a_{ij}]$. The number of paths of length 5 in G is* $\sum_{j\neq i} a_{ij}^{(5)} - 2\sum_{j\neq i} a_{ij}^{(4)} + 2\sum_{i=1}^{n} a_{ii}^{(3)}(d_i - 2) + 4\sum_{j\neq i} a_{ij}^{(2)} - 2\sum_{j\neq i} a_{ij}^{(2)}(d_j - a_{ij} - 1) - 4\sum_{j\neq i} a_{ij}^{(2)}\binom{d_i - a_{ij} - 1}{2}$ $+ 6\sum_{j\neq i} a_{ij}\binom{a_{ij}^{(2)}}{2} - 2\sum_{j\neq i} a_{ii}^{(3)} a_{ij}^{(2)} - 2\sum_{i=1}^{n} a_{ii}^{(3)}\binom{d_i - 2}{2} - 2\sum_{i=1}^{n}\left(a_{ii}^{(4)} - a_{ii}^{(2)} - 2\binom{d_i}{2} - \sum_{j=1, j\neq i}^{n} a_{ij}^{(2)}\right)(d_i - 2) - \sum_{j\neq i} a_{ij} -$ *3 tr* A^4 *+ 6 tr* A^3 *+ 3 tr* A^2.

Proof: By Proposition 1.3, the number of paths of length 5 in a graph G is equal to $\sum_{j\neq i} a_{ij}^{(5)} - x$, where x is the number of non-closed walks of length 5, that are not paths. To find x, we have 13 cases as considered below;

the cases are based on the configurations-(subgraphs) that generate all non-closed walks of length 5, that are not paths. In each case, N denote the number of non-closed walks of length 5, that are not paths in the corresponding subgraph, M denote the number of subgraphs of G of the same configuration and F denote the total number of non-closed walks of length 5, that are not paths in all possible subgraphs of G of the same configuration. However, in the cases with more than one figure (cases 7, 12), N, M and F are based on the first graph of the respective figures and P_1, P_2,... denote the number of subgraphs of G which do not have the same configuration as the first graph but are counted in M. It is clear that F is equal to N $\times$ (M$-$ $P_1$$-$$P_2$$-$...). To find N in each case, we have to include in any walk, all the edges and the vertices of the corresponding subgraphs at least once.

Case 1: For the configuration of Fig 5, N= 2, M= $\frac{1}{2}\sum_{j\neq i} a_{ij}$ and F= $\sum_{j\neq i} a_{ij}$.

Fig 5

Case 2: For the configuration of Fig 6, N= 12, M= $\frac{1}{2}\sum_{j\neq i} a_{ij}^{(2)}$ and F= $6\sum_{j\neq i} a_{ij}^{(2)}$.

Fig 6

Case 3: For the configuration of Fig 7, N= 12, M= $\frac{1}{2}\sum_{j\neq i} a_{ij}^{(2)}(d_j - a_{ij} - 1)$ and F= $6\sum_{j\neq i} a_{ij}^{(2)}(d_j - a_{ij} - 1)$. (See Theorem 1.6)

Fig 7

Case 4: For the configuration of Fig 8, N= 4, M= $\frac{1}{2}\,[\sum_{j\neq i}[a_{ij}^{(4)} - 2a_{ij}^{(2)}(d_j - a_{ij})] - \sum_{i=1}^{n}[(2d_i - 1)a_{ii}^{(3)} + 6\binom{d_i}{3}]]$ and F= $2\sum_{j\neq i}[a_{ij}^{(4)} - 2a_{ij}^{(2)}(d_j - a_{ij})] - 2\sum_{i=1}^{n}[(2d_i - 1)a_{ii}^{(3)} + 6\binom{d_i}{3}]$. (See Theorem 1.7)

Fig 8

Case 5: For the configuration of Fig 9, N= 24, M= $\frac{1}{6}$ trA^3 and F= 4 trA^3. (See Theorem 1.1)

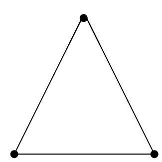


Fig 9

Case 6: For the configuration of Fig 10, N= 12, M= $\frac{1}{2}\sum_{i=1}^{n} a_{ii}^{(3)}(d_i - 2)$ and F= $6\sum_{i=1}^{n} a_{ii}^{(3)}(d_i - 2)$.

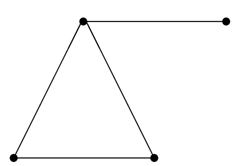


Fig 10

Case 7: For the configuration of Fig 11(a), N= 4, M= $\frac{1}{2}\sum_{j\neq i} a_{ii}^{(3)} a_{ij}^{(2)}$. Let P_1 denote the number of all subgraphs of G that have the same configuration as the graph of Fig 11(b) and are counted in M. Thus P_1= $6\times\frac{1}{6}\times$trA^3, where

$\frac{1}{6}\times \mathrm{tr}A^3$ is the number of subgraphs of G that have the same configuration as the graph of Fig 11(b) (See Theorem 1.1) and 6 is the number of times that this subgraph is counted in M. Let P_2 denote the number of all subgraphs of G that have the same configuration as the graph of Fig 11(c) and are counted in M. Thus $P_2 = 2\times\frac{1}{2}\times\sum_{i=1}^{n} a_{ii}^{(3)}(d_i-2)$, where $\frac{1}{2}\times\sum_{i=1}^{n} a_{ii}^{(3)}(d_i-2)$ is the number of subgraphs of G that have the same configuration as the graph of Fig 11(c) and 2 is the number of times that this subgraph is counted in M. Let P_3 denote the number of all subgraphs of G that have the same configuration as the graph of Fig 11(d) and are counted in M. Thus $P_3 = 4\times\frac{1}{2}\times\sum_{j\neq i}\binom{a_{ij}^{(2)}}{2}a_{ij}$, where $\frac{1}{2}\times\sum_{j\neq i}\binom{a_{ij}^{(2)}}{2}a_{ij}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 11(d) and 4 is the number of times that this subgraph is counted in M.

Consequently, $F = 2\sum_{j\neq i} a_{ii}^{(3)}a_{ij}^{(2)} - 4\,\mathrm{tr}A^3 - 4\sum_{i=1}^{n} a_{ii}^{(3)}(d_i-2) - 8\sum_{j\neq i}\binom{a_{ij}^{(2)}}{2}a_{ij}$.

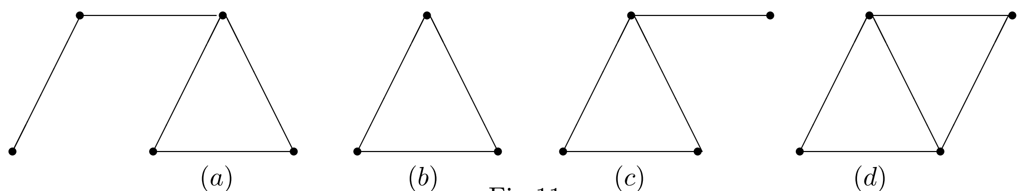


Fig 11

Case 8: For the configuration of Fig 12, N= 4, $M = \frac{1}{2}\sum_{i=1}^{n} a_{ii}^{(3)}\binom{d_i-2}{2}$ and $F = 2\sum_{i=1}^{n} a_{ii}^{(3)}\binom{d_i-2}{2}$.

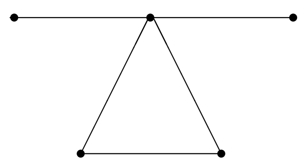


Fig 12

Case 9: For the configuration of Fig 13, N= 12, $M = \sum_{i=1}^{n}\binom{d_i}{3}$ and $F = 12\sum_{i=1}^{n}\binom{d_i}{3}$.

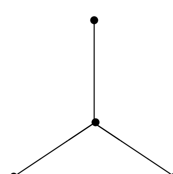


Fig 13

Case 10: For the configuration of Fig 14, N= 12, $M = \frac{1}{2}\sum_{j\neq i}\binom{a_{ij}^{(2)}}{2}a_{ij}$ and $F = 6\sum_{j\neq i}\binom{a_{ij}^{(2)}}{2}a_{ij}$.

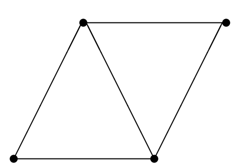


Fig 14

Case 11: For the configuration of Fig 15, N= 24, M= $\frac{1}{8}$ (trA^4− tr$A^2 - 2\sum_{j\neq i} a_{ij}^{(2)}$) and F= 3 (tr$A^4$− tr$A^2 - 2 \sum_{j\neq i} a_{ij}^{(2)}$). (See Theorem 1.2)

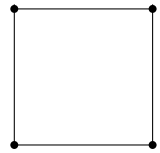
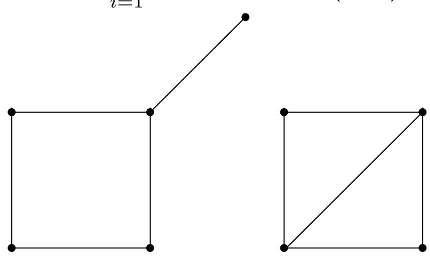


Fig 15

Case 12: For the configuration of Fig 16(a), N= 4, M= $\frac{1}{2}\sum_{i=1}^{n}(a_{ii}^{(4)} - a_{ii}^{(2)} - 2\begin{pmatrix} d_i \\ 2 \end{pmatrix} - \sum_{j=1,j\neq i}^{n} a_{ij}^{(2)})(d_i - 2)$ (See Theorem 2.1). Let P_1 denote the number of all subgraphs of G that have the same configuration as the graph of Fig 16(b) and are counted in M. Thus $P_1 = 2 \times \frac{1}{2}\sum_{j\neq i}\begin{pmatrix} a_{ij}^{(2)} \\ 2 \end{pmatrix} a_{ij}$, where $\frac{1}{2}\sum_{j\neq i}\begin{pmatrix} a_{ij}^{(2)} \\ 2 \end{pmatrix} a_{ij}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 16(b) and 2 is the number of times that this subgraph is counted in M. Consequently, F= $2\sum_{i=1}^{n}(a_{ii}^{(4)} - a_{ii}^{(2)} - 2\begin{pmatrix} d_i \\ 2 \end{pmatrix} - \sum_{j=1,j\neq i}^{n} a_{ij}^{(2)})(d_i - 2) - 4\sum_{j\neq i}\begin{pmatrix} a_{ij}^{(2)} \\ 2 \end{pmatrix} a_{ij}$.



(a) Fig 16 (b)

Case 13: For the configuration of Fig 17, N= 4, M= $\sum_{j\neq i} a_{ij}^{(2)}\begin{pmatrix} d_i - a_{ij} - 1 \\ 2 \end{pmatrix}$ and F= $4\sum_{j\neq i} a_{ij}^{(2)}\begin{pmatrix} d_i - a_{ij} - 1 \\ 2 \end{pmatrix}$.

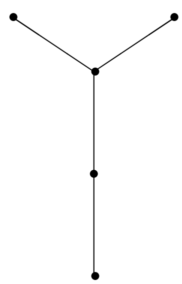


Fig 17

Now we add the values of F arising from the above cases and determine x. Substituting the value of x in $\sum_{j\neq i} a_{ij}^{(5)} - x$ and simplifying, we get the desired result. □

Example 2.4 *In the graph of Fig 39,* $\sum_{j\neq i} a_{ij}^{(5)} = 15630$, $\sum_{j\neq i} a_{ij}^{(4)} = 3120$, $\sum_{i=1}^{6} a_{ii}^{(3)}(d_i - 2) = 360$, $\sum_{j\neq i} a_{ij}^{(2)} = 120$, $\sum_{j\neq i} a_{ij}^{(2)}(d_j - a_{ij} - 1) = 360$, $\sum_{j\neq i} a_{ij}^{(2)}\begin{pmatrix} d_i - a_{ij} - 1 \\ 2 \end{pmatrix} = 360$, $\sum_{j\neq i} a_{ij}\begin{pmatrix} a_{ij}^{(2)} \\ 2 \end{pmatrix} = 180$, $\sum_{j\neq i} a_{ii}^{(3)} a_{ij}^{(2)} = 2400$, $\sum_{i=1}^{6} a_{ii}^{(3)}\begin{pmatrix} d_i - 2 \\ 2 \end{pmatrix} = 360$, $\sum_{i=1}^{6}(a_{ii}^{(4)} - a_{ii}^{(2)} - 2\begin{pmatrix} d_i \\ 2 \end{pmatrix} - \sum_{j=1,j\neq i}^{6} a_{ij}^{(2)})(d_i - 2) = 1080$, $\sum_{j\neq i} a_{ij} = 30$, *tr* $A^4 = 630$, *tr* $A^3 = 120$, *tr* $A^2 = 30$.

So by Theorem 2.3, the number of paths of length 5 in K_6 is 720.

Theorem 2.5 *Let G be a simple graph with n vertices and the adjacency matrix A= $[a_{ij}]$. The number of paths of length 5 in G, each of which starts from a specific vertex v_i is* $\sum_{j=1,j\neq i}^{n} a_{ij}^{(5)} - \sum_{j=1,j\neq i}^{n} a_{ij}^{(4)} - \sum_{j=1,j\neq i}^{n} a_{ij}^{(2)} - \sum_{j=1,j\neq i}^{n} a_{ij} - \sum_{j=1,j\neq i}^{n} a_{ij}^{(2)} a_{ij} - 2\sum_{j=1,j\neq i}^{n} \binom{d_j-1}{2} a_{ij}^{(2)} + 6\sum_{j=1,j\neq i}^{n} \binom{a_{ij}^{(2)}}{2} a_{ij} + \sum_{j=1,j\neq i}^{n} a_{jj}^{(3)} a_{ij} + \sum_{j=1,j\neq i}^{n} a_{ii}^{(3)} a_{ij} - 3\sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}(d_j - a_{ij} - 1) - \sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}(d_i - a_{ij} - 1) + 2\sum_{j=1,j\neq i}^{n} a_{ij}^{(2)} a_{ij}(d_j-2) - \sum_{j=1,j\neq i}^{n} a_{ii}^{(3)} a_{ij}^{(2)} - \sum_{j=1,j\neq i}^{n} a_{jj}^{(3)} a_{ij}^{(2)} - \sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}(d_j - a_{ij} - 1)(d_i-1) - 2\sum_{j=1,j\neq i}^{n} (\frac{1}{2} a_{jj}^{(3)} a_{ij} - a_{ij}^{(2)} a_{ij})(d_j-3) - (a_{ii}^{(4)} - a_{ii}^{(2)} - 2\binom{d_i}{2}) - \sum_{j=1,j\neq i}^{n} a_{ij}^{(2)})(d_i-2) - \sum_{j=1,j\neq i}^{n} (a_{jj}^{(4)} - a_{jj}^{(2)} - 2\binom{d_j}{2} - \sum_{k=1,k\neq j}^{n} a_{jk}^{(2)}) a_{ij} - \sum_{j\neq k, j,k\neq i} (a_{ij} a_{jk}^{(2)} - a_{ij} a_{ik})(d_j-2) - 6\binom{d_i}{2} - 6\binom{d_i}{3}$.

Proof : By Proposition 1.4, the number of paths of length 5 in a graph G, each of which starts from a specific vertex v_i is $\sum_{j=1,j\neq i}^{n} a_{ij}^{(5)} - x$, where x is the number of non-closed walks of length 5 , that begin from v_i and are not paths. To find x, we have 21 cases as considered below; the cases are based on the configurations-(subgraphs) that generate all non-closed walks of length 5, each of which starts from the specific vertex v_i, that are not paths. In each case, N denote the number of non-closed walks of length 5, which start from the vertex v_i and are not paths in the corresponding subgraph, M denote the number of subgraphs of G of the same configuration, F denote the total number of non-closed walks of length 5, which start from the vertex v_i and are not paths in all possible subgraphs of G of the same configuration. However, in the cases with more than one figure (cases 7, 9,12, 13, 18,19, 20, 21), N, M and F are based on the first graph of the respective figures and P_1, P_2,... denote the number of subgraphs of G which do not have the same configuration as the first graph but are counted in M. It is clear that F is equal to N $\times$ (M− P_1−P_2−...). To find N in each case, we have to include in any walk, all the edges and the vertices of the corresponding subgraphs at least once.

Case 1: For the configuration of Fig 18, N= 1, M= $\sum_{j=1,j\neq i}^{n} a_{ij}$ and F= $\sum_{j=1,j\neq i}^{n} a_{ij}$.

Fig 18

Case 2: For the configuration of Fig 19, N= 3, M= $\sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}$ and F= $3\sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}$.

Fig 19

Case 3: For the configuration of Fig 20, N= 6, M= $\binom{d_i}{2}$ and F= $6\binom{d_i}{2}$.

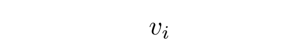


Fig 20

Case 4: For the configuration of Fig 21, N= 4, M= $\sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}(d_j - a_{ij} - 1)$ and F= $4\sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}(d_j - a_{ij} - 1)$.

(See Theorem 1.8)

Fig 21

Case 5: For the configuration of Fig 22, N= 2, M= $\sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}(d_i - a_{ij} - 1)$ and F= $2\sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}(d_i - a_{ij} - 1)$.

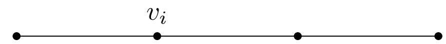


Fig 22

Case 6: For the configuration of Fig 23, N= 1, M= $\sum_{j=1,j\neq i}^{n} [a_{ij}^{(4)} - (d_i + d_j - 3a_{ij})a_{ij}^{(2)} - (a_{ii}^{(3)} + a_{jj}^{(3)} + 2\binom{d_j - 1}{2})a_{ij}]$ and F= $\sum_{j=1,j\neq i}^{n} [a_{ij}^{(4)} - (d_i + d_j - 3a_{ij})a_{ij}^{(2)} - (a_{ii}^{(3)} + a_{jj}^{(3)} + 2\binom{d_j - 1}{2})a_{ij}]$. (See Theorem 1.9)

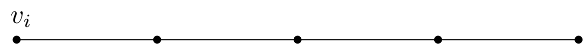
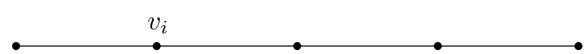
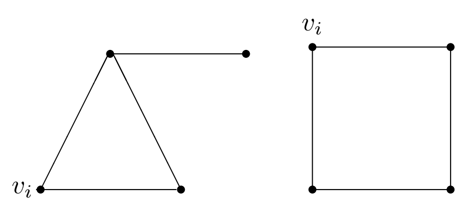


Fig 23

Case 7: For the configuration of Fig 24(a), N= 1, M= $\sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}(d_j - a_{ij} - 1)(d_i - 1)$ (See Theorem 1.8). Let P_1 denote the number of all subgraphs of G that have the same configuration as the graph of Fig 24(b) and are counted in M. Thus $P_1 = 1 \times \sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}a_{ij}(d_j - 2)$, where $\sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}a_{ij}(d_j - 2)$ is the number of subgraphs of G that have the same configuration as the graph of Fig 24(b) and this subgraph is counted only once in M. Let P_2 denote the number of all subgraphs of G that have the same configuration as the graph of Fig 24(c) and are counted in M. Thus $P_2 = 2 \times \frac{1}{2}\,[a_{ii}^{(4)} - a_{ii}^{(2)} - 2\binom{d_i}{2} - \sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}]$, where $\frac{1}{2}[a_{ii}^{(4)} - a_{ii}^{(2)} - 2\binom{d_i}{2} - \sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}]$ is the number of subgraphs of G that have the same configuration as the graph of Fig 24(c) (See Theorem 2.1) and 2 is the number of times that this subgraph is counted in M. Consequently, F= $\sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}(d_j - a_{ij} - 1)(d_i - 1) - \sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}a_{ij}(d_j - 2) - [a_{ii}^{(4)} - a_{ii}^{(2)} - 2\binom{d_i}{2} - \sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}]$.



(a)



(b) (c)

Fig 24

Case 8: For the configuration of Fig 25, N= 8, M= $\frac{1}{2}\sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}a_{ij}$ and F= $4\sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}a_{ij}$.

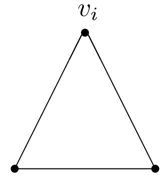
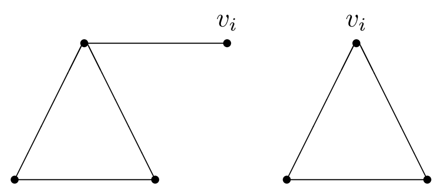


Fig 25

Case 9: For the configuration of Fig 26, N= 6, M= $\frac{1}{2}\sum\limits_{j=1,j\neq i}^{n} a_{jj}^{(3)}a_{ij}$. Let P_1 denote the number of all subgraphs of G that have the same configuration as the graph of Fig 26(b) and are counted in M. Thus $P_1 = 2\times\frac{1}{2}\sum\limits_{j=1,j\neq i}^{n} a_{ij}^{(2)}a_{ij}$, where $\frac{1}{2}\sum\limits_{j=1,j\neq i}^{n} a_{ij}^{(2)}a_{ij}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 26(b) and 2 is the number of times that this subgraph is counted in M. Consequently, F= $3\sum\limits_{j=1,j\neq i}^{n} a_{jj}^{(3)}a_{ij} - 6\sum\limits_{j=1,j\neq i}^{n} a_{ij}^{(2)}a_{ij}$.



(a) (b)

Fig 26

Case 10: For the configuration of Fig 27, N= 3, M= $\sum\limits_{j=1,j\neq i}^{n} a_{ij}^{(2)}a_{ij}(d_j-2)$ and F= $3\sum\limits_{j=1,j\neq i}^{n} a_{ij}^{(2)}a_{ij}(d_j-2)$.

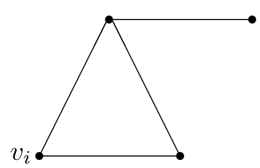


Fig 27

Case 11: For the configuration of Fig 28, N= 2, M= $\sum\limits_{j=1,j\neq i}^{n} (\frac{1}{2}a_{jj}^{(3)}a_{ij} - a_{ij}^{(2)}a_{ij})(d_j-3)$ and F= $2\sum\limits_{j=1,j\neq i}^{n} (\frac{1}{2}a_{jj}^{(3)}a_{ij} - a_{ij}^{(2)}a_{ij})(d_j-3)$. (See Case 9)

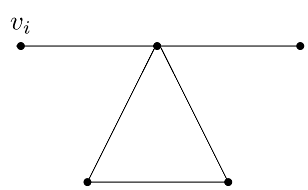


Fig 28

Case 12: For the configuration of Fig 29(a), N= 2, M= $\frac{1}{2}\sum\limits_{j=1,j\neq i}^{n} a_{ii}^{(3)}a_{ij}^{(2)}$ (See Theorem 1.1). Let P_1 denote the number of all subgraphs of G that have the same configuration as the graph of Fig 29(b) and are counted in M. Thus $P_1 = 2\times\frac{1}{2}\sum\limits_{j=1,j\neq i}^{n} a_{ij}^{(2)}a_{ij}$, where $\frac{1}{2}\sum\limits_{j=1,j\neq i}^{n} a_{ij}^{(2)}a_{ij}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 29(b) and 2 is the number of times that this subgraph is counted in M. Let P_2 denote the number of all subgraphs of G that have the same configuration as the graph of Fig 29(c) and are counted in M. Thus $P_2 = 1\times\sum\limits_{j=1,j\neq i}^{n} a_{ij}^{(2)}a_{ij}(d_j-2)$, where $\sum\limits_{j=1,j\neq i}^{n} a_{ij}^{(2)}a_{ij}(d_j-2)$ is the number of subgraphs of G that have the same configuration as the graph of Fig 29(c) and this subgraph is counted only once in M. Let P_3 denote the number of all subgraphs of G that have the same configuration as the graph of Fig 29(d) and are counted in M. Thus $P_3 = 2\sum\limits_{j=1,j\neq i}^{n} \binom{a_{ij}^{(2)}}{2} a_{ij}$, where $\sum\limits_{j=1,j\neq i}^{n} \binom{a_{ij}^{(2)}}{2} a_{ij}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 29(d) and 2 is the number of times that this subgraph is counted in M.

Consequently, F= $\sum_{j=1,j\neq i}^{n} a_{ii}^{(3)}a_{ij}^{(2)} - 2\sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}a_{ij} - 2\sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}a_{ij}(d_j-2) - 4\sum_{j=1,j\neq i}^{n}\binom{a_{ij}^{(2)}}{2}a_{ij}$.

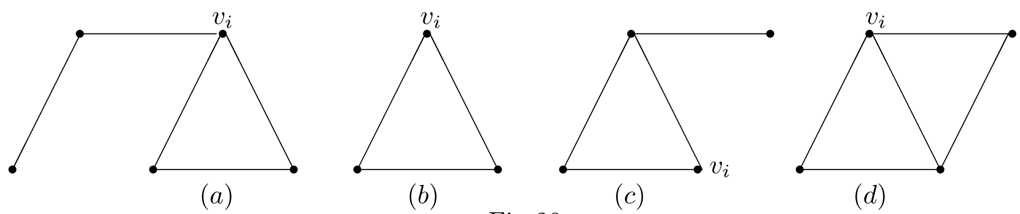


Fig 29

Case 13: For the configuration of Fig 30(a), N= 2, M= $\frac{1}{2}\sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}a_{jj}^{(3)}$. Let P_1 denote the number of all subgraphs of G that have the same configuration as the graph of Fig 30(b) and are counted in M. Thus $P_1 = 2\times\frac{1}{2}\sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}a_{ij}$, where $\frac{1}{2}\sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}a_{ij}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 30(b) and 2 is the number of times that this subgraph is counted in M. Let P_2 denote the number of all subgraphs of G that have the same configuration as the graph of Fig 30(c) and are counted in M. Thus $P_2 = 2\times(\frac{1}{2}\sum_{j=1,j\neq i}^{n} a_{jj}^{(3)}a_{ij} - \sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}a_{ij})$, where $\frac{1}{2}\sum_{j=1,j\neq i}^{n} a_{jj}^{(3)}a_{ij} - \sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}a_{ij}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 30(c) (See Case 9) and 2 is the number of times that this subgraph is counted in M. Let P_3 denote the number of all subgraphs of G that have the same configuration as the graph of Fig 30(d) and are counted in M. Thus $P_3 = 2\sum_{j=1,j\neq i}^{n}\binom{a_{ij}^{(2)}}{2}a_{ij}$, where $\sum_{j=1,j\neq i}^{n}\binom{a_{ij}^{(2)}}{2}a_{ij}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 30(d) and 2 is the number of times that this subgraph is counted in M. Consequently, F= $\sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}a_{jj}^{(3)} + 2\sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}a_{ij} - 2\sum_{j=1,j\neq i}^{n} a_{jj}^{(3)}a_{ij} - 4\sum_{j=1,j\neq i}^{n}\binom{a_{ij}^{(2)}}{2}a_{ij}$.

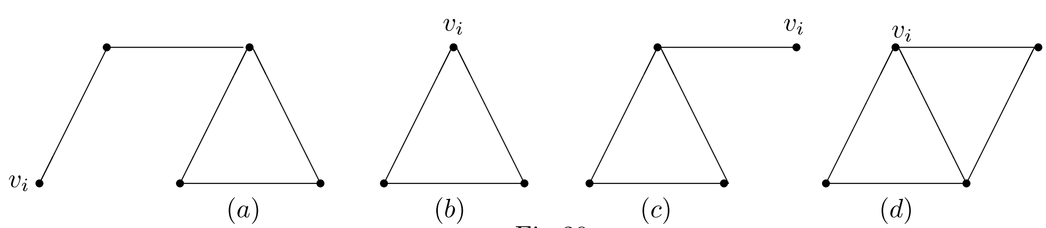


Fig 30

Case 14: For the configuration of Fig 31, N= 2, M= $\sum_{j=1,j\neq i}^{n} a_{ij}\binom{d_j-1}{2}$ and F= $2\sum_{j=1,j\neq i}^{n} a_{ij}\binom{d_j-1}{2}$.

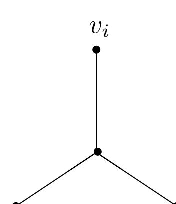


Fig 31

Case 15: For the configuration of Fig 32, N= 6, M= $\binom{d_i}{3}$ and F= $6\binom{d_i}{3}$.

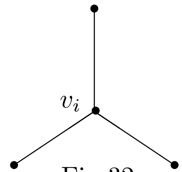


Fig 32

Case 16: For the configuration of Fig 33, N= 6, M= $\sum_{j=1,j\neq i}^{n} \binom{a_{ij}^{(2)}}{2} a_{ij}$ and F= 6 $\sum_{j=1,j\neq i}^{n} \binom{a_{ij}^{(2)}}{2} a_{ij}$.

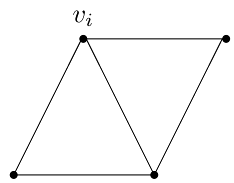


Fig 33

Case 17: For the configuration of Fig 34, N= 6, M= $\frac{1}{2}[a_{ii}^{(4)} - a_{ii}^{(2)} - 2\binom{d_i}{2} - \sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}]$ and F= 3 $[a_{ii}^{(4)} - a_{ii}^{(2)} - 2\binom{d_i}{2} - \sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}]$. (See Theorem 2.1)

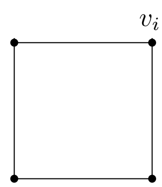
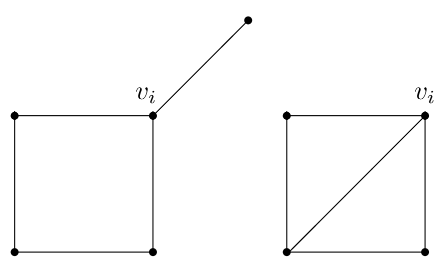


Fig 34

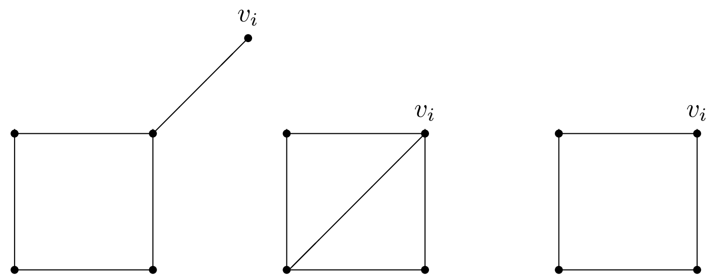
Case 18: For the configuration of Fig 35(a), N= 2, M= $\frac{1}{2}(a_{ii}^{(4)} - a_{ii}^{(2)} - 2\binom{d_i}{2} - \sum_{j=1,j\neq i}^{n} a_{ij}^{(2)})(d_i - 2)$ (See Theorem 2.1). Let P_1 denote the number of all subgraphs of G that have the same configuration as the graph of Fig 35(b) and are counted in M. Thus $P_1 = 1 \times \sum_{j=1,j\neq i}^{n} \binom{a_{ij}^2}{2} a_{ij}$, where $\sum_{j=1,j\neq i}^{n} \binom{a_{ij}^2}{2} a_{ij}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 35(b) and this subgraph is counted only once in M. Consequently, F= $(a_{ii}^{(4)} - a_{ii}^{(2)} - 2\binom{d_i}{2} - \sum_{j=1,j\neq i}^{n} a_{ij}^{(2)})(d_i - 2) - 2 \sum_{j=1,j\neq i}^{n} \binom{a_{ij}^{(2)}}{2} a_{ij}$.



(a) Fig 35 (b)

Case 19: For the configuration of Fig 36(a), N= 2, M= $\frac{1}{2} \sum_{j=1,j\neq i}^{n} (a_{jj}^{(4)} - a_{jj}^{(2)} - 2\binom{d_j}{2} - \sum_{k=1,k\neq j}^{n} a_{jk}^{(2)}) a_{ij}$ (See Theorem 2.1). Let P_1 denote the number of all subgraphs of G that have the same configuration as the graph of Fig 36(b) and are counted in M. Thus $P_1 = 1 \times \sum_{j=1,j\neq i}^{n} \binom{a_{ij}^{(2)}}{2} a_{ij}$, where $\sum_{j=1,j\neq i}^{n} \binom{a_{ij}^{(2)}}{2} a_{ij}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 36(b) and this subgraph is counted only once in M. Let P_2 denote the number of all subgraphs of G that have the same configuration as the graph of Fig 36(c) and are counted in M. Thus $P_2 = 2 \times \frac{1}{2}[a_{ii}^{(4)} - a_{ii}^{(2)} - 2\binom{d_i}{2} - \sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}]$, where $\frac{1}{2}[a_{ii}^{(4)} - a_{ii}^{(2)} - 2\binom{d_i}{2} - \sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}]$

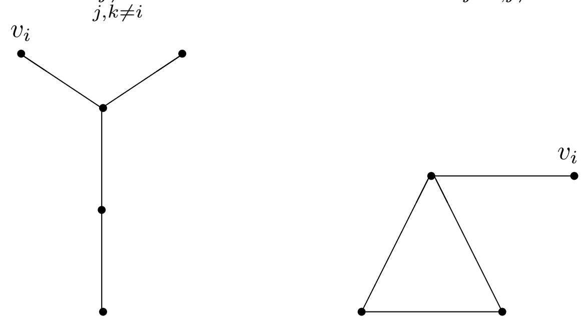
is the number of subgraphs of G that have the same configuration as the graph of Fig 36(c) (See Theorem 2.1) and 2 is the number of times that this subgraph is counted in M. Consequently, F= $\sum_{j=1,j\neq i}^{n} (a_{jj}^{(4)} - a_{jj}^{(2)} - 2\binom{d_j}{2} - \sum_{k=1,k\neq j}^{n} a_{jk}^{(2)})a_{ij} - 2\sum_{j=1,j\neq i}^{n} \binom{a_{ij}^{(2)}}{2} a_{ij} - 2\,[a_{ii}^{(4)} - a_{ii}^{(2)} - 2\binom{d_i}{2} - \sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}]$.



(a) (b) (c)

Fig 36

Case 20: For the configuration of Fig 37(a), N= 1, M= $\sum_{\substack{j\neq k\\ j,k\neq i}} (a_{ij}a_{jk}^{(2)} - a_{ij}a_{ik})(d_j - 2)$. Let P_1 denote the number of all subgraphs of G that have the same configuration as the graph of Fig 37(b) and are counted in M. Thus $P_1 = 2 \times [\frac{1}{2}\sum_{j=1,j\neq i}^{n} a_{jj}^{(3)}a_{ij} - \sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}a_{ij}]$, where $\frac{1}{2}\sum_{j=1,j\neq i}^{n} a_{jj}^{(3)}a_{ij} - \sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}a_{ij}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 37(b) (See Case 9) and 2 is the number of times that this subgraph is counted in M. Consequently, F= $\sum_{\substack{j\neq k\\ j,k\neq i}} (a_{ij}a_{jk}^{(2)} - a_{ij}a_{ik})(d_j - 2) - \sum_{j=1,j\neq i}^{n} a_{jj}^{(3)}a_{ij} + 2\sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}a_{ij}$.



(a) Fig 37 (b)

Case 21: For the configuration of Fig 38(a), N= 2, M= $\sum_{j=1,j\neq i}^{n} a_{ij}^{(2)} \binom{d_j - 1}{2}$. Let P_1 denote the number of all subgraphs of G that have the same configuration as the graph of Fig 38(b) and are counted in M. Thus $P_1 = 1 \times \sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}a_{ij}(d_j - 2)$, where $\sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}a_{ij}(d_j - 2)$ is the number of subgraphs of G that have the same configuration as the graph of Fig 38(b) and this subgraph is counted only once in M. Consequently, F= $2\sum_{j=1,j\neq i}^{n} a_{ij}^{(2)} \binom{d_j - 1}{2} - 2\sum_{j=1,j\neq i}^{n} a_{ij}^{(2)}a_{ij}(d_j - 2)$.

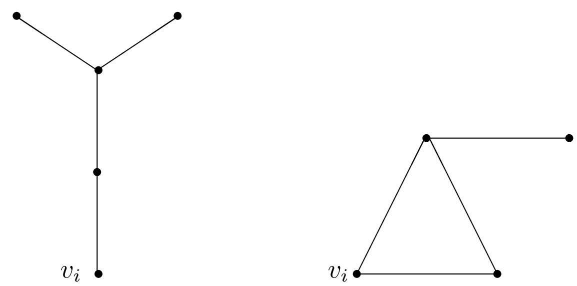


Fig 38

Now we add the values of F arising from the above cases and determine x. Substituting the value of x in $\sum_{j=1,j\neq i} a_{ij}^{(5)} - x$ and simplifying, we get the desired result. $\square$

Example 2.6 *In the graph of Fig 39,* $\sum_{j=2}^{6} a_{1j}^{(5)} = 2605$, $\sum_{j=2}^{6} a_{1j}^{(4)} = 520$, $\sum_{j=2}^{6} a_{1j}^{(2)} = 20$, $\sum_{j=2}^{6} a_{1j} = 5$, $\sum_{j=2}^{6} a_{1j}^{(2)} a_{1j} = 20$, $\sum_{j=2}^{6} \binom{d_j - 1}{2} a_{1j}^{(2)} = 120$, $\sum_{j=2}^{6} \binom{a_{1j}^{(2)}}{2} a_{1j} = 30$, $\sum_{j=2}^{6} a_{jj}^{(3)} a_{1j} = 100$, $\sum_{j=2}^{6} a_{11}^{(3)} a_{1j} = 100$, $\sum_{j=2}^{6} a_{1j}^{(2)} (d_j - a_{1j} - 1) = 60$, $\sum_{j=2}^{6} a_{1j}^{(2)} (d_1 - a_{1j} - 1) = 60$, $\sum_{j=2}^{6} a_{1j}^{(2)} a_{1j} (d_j - 2) = 60$, $\sum_{j=2}^{6} a_{11}^{(3)} a_{1j}^{(2)} = 400$, $\sum_{j=2}^{6} a_{jj}^{(3)} a_{1j}^{(2)} = 400$, $\sum_{j=2}^{6} a_{1j}^{(2)} (d_j - a_{1j} - 1)(d_1 - 1) = 240$, $\sum_{j=2}^{6} (\frac{1}{2} a_{jj}^{(3)} a_{1j} - a_{1j}^{(2)} a_{1j})(d_j - 3) = 60$, $(a_{11}^{(4)} - a_{11}^{(2)} - 2\binom{d_1}{2} - \sum_{j=2}^{6} a_{1j}^{(2)})(d_1 - 2) = 180$, $\sum_{j=2}^{6} (a_{jj}^{(4)} - a_{jj}^{(2)} - 2\binom{d_j}{2} - \sum_{k=1,k\neq j}^{6} a_{jk}^{(2)}) a_{1j} = 300$, $\sum_{j\neq k,j,k\neq 1} (a_{1j} a_{jk}^{(2)} - a_{1j} a_{1k})(d_j - 2) = 180$, $\binom{d_1}{2} = 10$, $\binom{d_1}{3} = 10$. *So, by Theorem 2.5, the number of paths of length 5, starting from the vertex* v_1 *in the graph of Fig 39 is 120.*

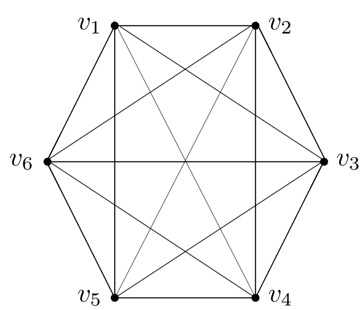


Fig 39

Theorem 2.7 *Let G be a simple graph with n vertices and the adjacency matrix A= $[a_{ij}]$. The number of $v_i - v_j$ paths of length 5 in G is*

$$a_{ij}^{(5)} - \left(2d_i + 2d_j + d_i d_j + a_{ii}^{(4)} + a_{jj}^{(4)} - a_{ii}^{(2)} - a_{jj}^{(2)} - a_{ij}^{(2)} - 2\binom{d_i}{2} - 2\binom{d_j}{2} + 2\binom{d_i - 1}{2} + 2\binom{d_j - 1}{2} - 6\binom{a_{ij}^{(2)}}{2} - 4\right) a_{ij} - \left(a_{ii}^{(3)} + a_{jj}^{(3)}\right) a_{ij}^{(2)} + \sum_{k=1,k\neq i}^{n} a_{ik}^{(2)} a_{ij} + \sum_{k=1,k\neq j}^{n} a_{jk}^{(2)} a_{ij} - \sum_{k=1,k\neq i,j}^{n} a_{kk}^{(3)} a_{ik} a_{jk}$$
$$- \sum_{k=1,k\neq i,j}^{n} \left(a_{ik}^{(2)} + a_{jk}^{(2)} - a_{ik} - a_{jk} - 2a_{ik} a_{jk}\right) a_{ij} + \sum_{k=1,k\neq i,j}^{n} \left(a_{ik}^{(2)} - a_{ij}\right)(3a_{ij} + 3a_{ik} - d_i - d_j - d_k + 1)\, a_{jk}$$
$$+ \sum_{k=1,k\neq i,j}^{n} \left(a_{jk}^{(2)} - a_{ij}\right)(3a_{jk} - d_k + 2)\, a_{ik}.$$

Proof: By Proposition 1.5, the number of $v_i - v_j$ $(j \neq i)$ paths of length 5 in a graph G is $a_{ij}^{(5)} - x$, where x is the number of $v_i - v_j$ $(j \neq i)$ walks of length 5, that are not paths. To find x, we have 23 cases as considered below; the cases are based on the configurations-(subgraphs) that generate all $v_i - v_j$ $(j \neq i)$ walks of length 5, that are not paths. In each case, N denote the number of $v_i - v_j$ $(j \neq i)$ walks of length 5, that are not paths in the corresponding subgraph, M denote the number of subgraphs of G of the same configuration, F denote the total number of $v_i - v_j$ $(j \neq i)$ walks of length 5 that are not paths in all possible subgraphs of G of the same configuration. However, in the cases with more than one figure (Cases 5, 6, 7, 8, 9, 10, 11, 14, 15, 17, 18, 19, 22 and 23), N, M and F are based on the first graph of the respective figures and P_1, P_2,... denote the number of subgraphs of G which do not have the same configuration as the first graph but are counted in M. It is clear that F is equal to N $\times$ (M$-$ $P_1 - P_2 - ...$). To find N in each case, we have to include in any walk, all the edges and the vertices of the corresponding subgraphs at least once.

Case 1: For the configuration of Fig 40, N= 1, M= a_{ij} and F= a_{ij}.

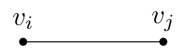


Fig 40

Case 2: For the configuration of Fig 41, N= 3, M= $a_{ij}(d_j - 1)$ and F= $3a_{ij}(d_j - 1)$.

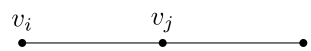


Fig 41

Case 3: For the configuration of Fig 42, N= 3, M= $a_{ij}(d_i - 1)$ and F= $3a_{ij}(d_i - 1)$.

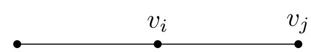


Fig 42

Case 4: For the configuration of Fig 43, N= 3, M= $\sum_{k=1,k\neq i,j}^{n} (a_{ik}^{(2)} - a_{ij})a_{jk}$ and F= $3 \sum_{k=1,k\neq i,j}^{n} (a_{ik}^{(2)} - a_{ij})a_{jk}$.

(See Theorem 1.10)

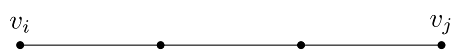


Fig 43

Case 5: For the configuration of Fig 44(a), N= 1, M= $a_{ij}(d_i - 1)(d_j - 1)$. Let P_1 denote the number of subgraphs of G that have the same configuration as the graph of Fig 44(b) and are counted in M. Thus $P_1 = 1 \times a_{ij}^{(2)} a_{ij}$, where $a_{ij}^{(2)} a_{ij}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 44(b) and this subgraph is counted only once in M. Consequently, F= $a_{ij}(d_i - 1)(d_j - 1) - a_{ij}^{(2)} a_{ij}$.

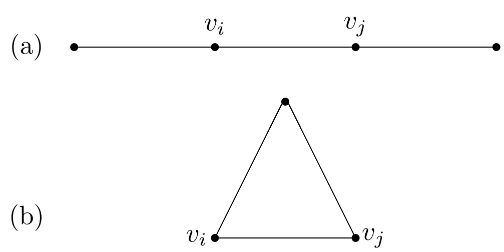


Fig 44

Case 6: For the configuration of Fig 45(a), N= 1, M= $\sum_{k=1,k\neq i,j}^{n} a_{ij}a_{jk}^{(2)}$. Let P_1 denote the number of subgraphs of G that have the same configuration as the graph of Fig 45(b) and are counted in M. Thus $P_1 = 1 \times \sum_{k=1,k\neq i,j}^{n} a_{ij}a_{ik}$, where $\sum_{k=1,k\neq i,j}^{n} a_{ij}a_{ik}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 45(b) and this subgraph is counted only once in M. Consequently, F= $\sum_{k=1,k\neq i,j}^{n} a_{ij}a_{jk}^{(2)} - \sum_{k=1,k\neq i,j}^{n} a_{ij}a_{ik}$.

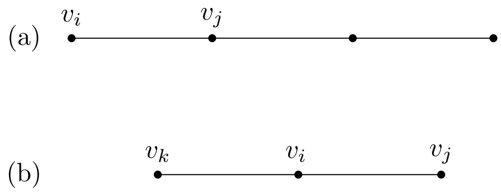


Fig 45

Case 7: For the configuration of Fig 46(a), N= 1, M= $\sum_{k=1,k\neq i,j}^{n} a_{ij}a_{ik}^{(2)}$. Let P_1 denote the number of subgraphs of G that have the same configuration as the graph of Fig 46(b) and are counted in M. Thus $P_1 = 1\times \sum_{k=1,k\neq i,j}^{n} a_{ij}a_{jk}$, where $\sum_{k=1,k\neq i,j}^{n} a_{ij}a_{jk}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 46(b) and this subgraph is counted only once in M. Consequently, F= $\sum_{k=1,k\neq i,j}^{n} a_{ij}a_{ik}^{(2)} - \sum_{k=1,k\neq i,j}^{n} a_{ij}a_{jk}$.

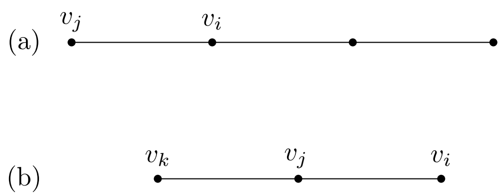
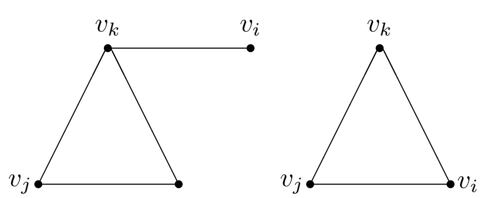


Fig 46

Case 8: For the configuration of Fig 47(a), N= 3, M= $\sum_{k=1,k\neq i,j}^{n} a_{jk}^{(2)}a_{jk}a_{ik}$. Let P_1 denote the number of subgraphs of G that have the same configuration as the graph of Fig 47(b) and are counted in M. Thus $P_1 = 1\times \sum_{k=1,k\neq i,j}^{n} a_{ik}a_{ij}a_{jk}$, where $\sum_{k=1,k\neq i,j}^{n} a_{ik}a_{ij}a_{jk}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 47(b) and this subgraph is counted only once in M. Consequently,

F= $3\sum_{k=1,k\neq i,j}^{n} (a_{jk}^{(2)} - a_{ij})a_{ik}a_{jk}$.



(a) (b)

Fig 47

Case 9: For the configuration of Fig 48(a), N= 3, M= $\sum_{k=1,k\neq i,j}^{n} a_{ik}^{(2)}a_{ik}a_{jk}$. Let P_1 denote the number of subgraphs of G that have the same configuration as the graph of Fig 48(b) and are counted in M. Thus $P_1 = 1\times \sum_{k=1,k\neq i,j}^{n} a_{ik}a_{ij}a_{jk}$, where $\sum_{k=1,k\neq i,j}^{n} a_{ik}a_{ij}a_{jk}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 48(b) and this subgraph is counted only once in M. Consequently,

F= $3\sum_{k=1,k\neq i,j}^{n} (a_{ik}^{(2)} - a_{ij})a_{ik}a_{jk}$.

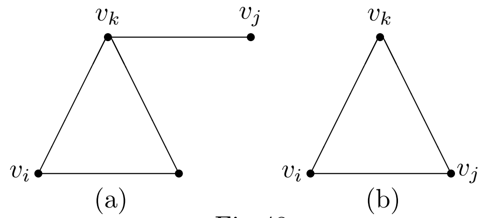


Fig 48

Case 10: For the configuration of Fig 49(a), N= 1, M= $\sum_{k=1,k\neq i,j}^{n}(a_{ik}^{(2)}-a_{ij})(d_j-1)a_{jk}$ (See Theorem 1.10). Let P_1 denote the number of subgraphs of G that have the same configuration as the graph of Fig 49(b) and are counted in M. Thus $P_1 = 1 \times \sum_{k=1,k\neq i,j}^{n}(a_{jk}^{(2)}-a_{ij})a_{ik}a_{jk}$ (See Case 8), where $\sum_{k=1,k\neq i,j}^{n}(a_{jk}^{(2)}-a_{ij})a_{ik}a_{jk}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 49(b) and this subgraph is counted only once in M. Let P_2 denote the number of subgraphs of G that have the same configuration as the graph of Fig 49(c) and are counted in M. Thus $P_2 = 1 \times \sum_{k=1,k\neq i,j}^{n}(a_{ik}^{(2)}-a_{ij})a_{jk}a_{ij}$ (See Theorem 1.10), where $\sum_{k=1,k\neq i,j}^{n}(a_{ik}^{(2)}-a_{ij})a_{jk}a_{ij}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 49(c) and this subgraph is counted only once in M. Consequently, F= $\sum_{k=1,k\neq i,j}^{n}(a_{ik}^{(2)}-a_{ij})(d_j-a_{ij}-1)a_{jk}-\sum_{k=1,k\neq i,j}^{n}(a_{jk}^{(2)}-a_{ij})a_{ik}a_{jk}$.

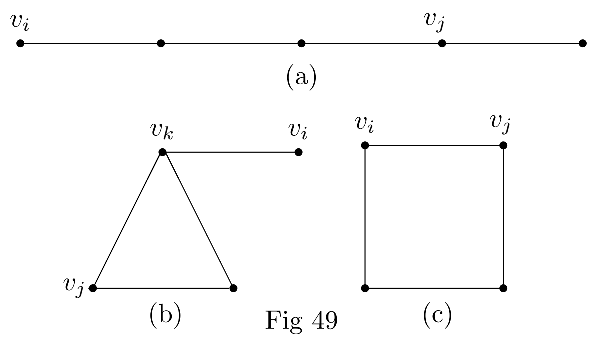


Fig 49

Case 11: For the configuration of Fig 50(a), N= 1, M= $\sum_{k=1,k\neq i,j}^{n}(a_{ik}^{(2)}-a_{ij})(d_i-1)a_{jk}$ (See Theorem 1.10). Let P_1 denote the number of subgraphs of G that have the same configuration as the graph of Fig 50(b) and are counted in M. Thus $P_1 = 1 \times \sum_{k=1,k\neq i,j}^{n}(a_{ik}^{(2)}-a_{ij})a_{ik}a_{jk}$ (See Case 9), where $\sum_{k=1,k\neq i,j}^{n}(a_{ik}^{(2)}-a_{ij})a_{ik}a_{jk}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 50(b) and this subgraph is counted only once in M. Let P_2 denote the number of subgraphs of G that have the same configuration as the graph of Fig 50(c) and are counted in M. Thus $P_2 = 1 \times \sum_{k=1,k\neq i,j}^{n}(a_{ik}^{(2)}-a_{ij})a_{jk}a_{ij}$ (See Theorem 1.10), where $\sum_{k=1,k\neq i,j}^{n}(a_{ik}^{(2)}-a_{ij})a_{jk}a_{ij}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 50(c) and this subgraph is counted only once in M. Consequently, F= $\sum_{k=1,k\neq i,j}^{n}(a_{ik}^{(2)}-a_{ij})(d_i-a_{ij}-1)a_{jk}-\sum_{k=1,k\neq i,j}^{n}(a_{ik}^{(2)}-a_{ij})a_{ik}a_{jk}$.

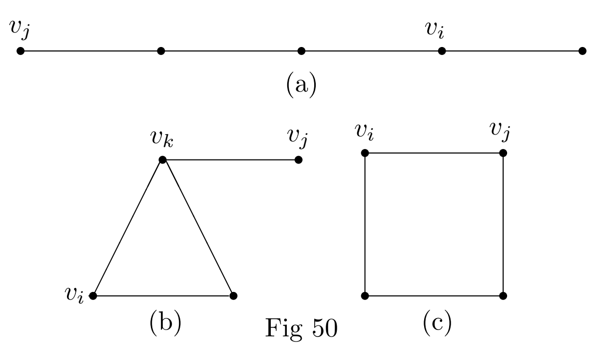


Fig 50

Case 12: For the configuration of Fig 51, N= 2, M= $a_{ij}\begin{pmatrix} d_j-1 \\ 2 \end{pmatrix}$ and F= $2a_{ij}\begin{pmatrix} d_j-1 \\ 2 \end{pmatrix}$.

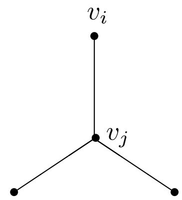


Fig 51

Case 13: For the configuration of Fig 52, N= 2, M= $a_{ij}\begin{pmatrix} d_i-1 \\ 2 \end{pmatrix}$ and F= $2a_{ij}\begin{pmatrix} d_i-1 \\ 2 \end{pmatrix}$.

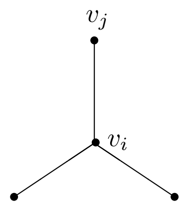
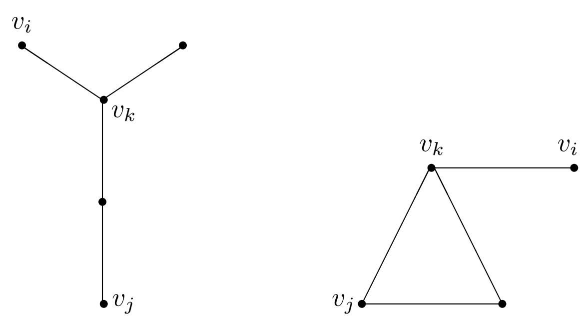


Fig 52

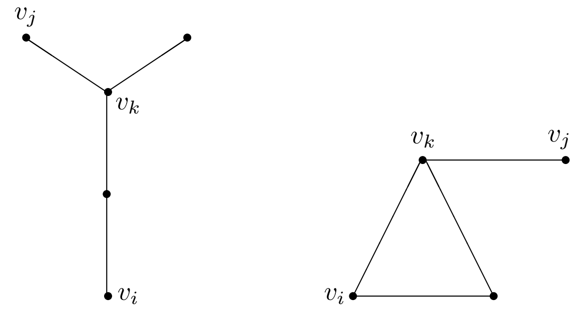
Case 14: For the configuration of Fig 53(a), N= 1, M= $\sum_{k=1,k\neq i,j}^{n}(a_{jk}^{(2)}-a_{ij})(d_k-2)a_{ik}$ (See Theorem 1.10). Let P_1 denote the number of subgraphs of G that have the same configuration as the graph of Fig 53(b) and are counted in M. Thus $P_1 = 1\times\sum_{k=1,k\neq i,j}^{n}(a_{jk}^{(2)}-a_{ij})a_{jk}a_{ik}$, where $\sum_{k=1,k\neq i,j}^{n}(a_{jk}^{(2)}-a_{ij})a_{jk}a_{ik}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 53(b) (See Case 8) and this subgraph is counted only once in M. Consequently, F= $\sum_{k=1,k\neq i,j}^{n}(a_{jk}^{(2)}-a_{ij})(d_k-a_{jk}-2)a_{ik}$.



(a) (b)

Fig 53

Case 15: For the configuration of Fig 54(a), N= 1, M= $\sum_{k=1,k\neq i,j}^{n}(a_{ik}^{(2)}-a_{ij})(d_k-2)a_{jk}$ (See Theorem 1.10). Let P_1 denote the number of subgraphs of G that have the same configuration as the graph of Fig 54(b) and are counted in M. Thus $P_1 = 1\times\sum_{k=1,k\neq i,j}^{n}(a_{ik}^{(2)}-a_{ij})a_{ik}a_{jk}$, where $\sum_{k=1,k\neq i,j}^{n}(a_{ik}^{(2)}-a_{ij})a_{ik}a_{jk}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 54(b) (See Case 9) and this subgraph is counted only once in M. Consequently, F= $\sum_{k=1,k\neq i,j}^{n}(a_{ik}^{(2)}-a_{ij})(d_k-a_{ik}-2)a_{jk}$.



(a) (b)

Fig 54

Case 16: For the configuration of Fig 55, N= 4, M= $a_{ij}^{(2)}a_{ij}$ and F= $4a_{ij}^{(2)}a_{ij}$.

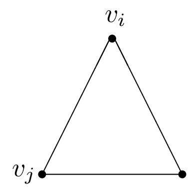
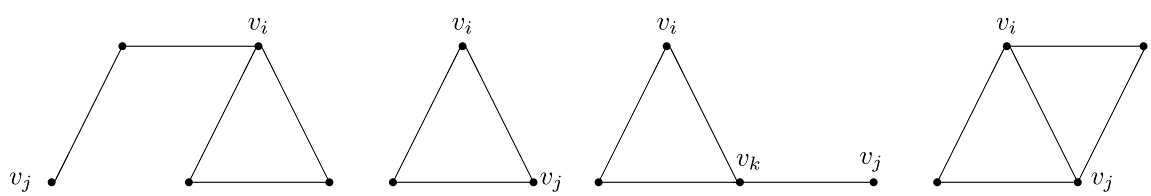


Fig 55

Case 17: For the configuration of Fig 56(a), N= 2, M= $\frac{1}{2}a_{ii}^{(3)}a_{ij}^{(2)}$. Let P_1 denote the number of subgraphs of G that have the same configuration as the graph of Fig 56(b) and are counted in M. Thus $P_1 = 1 \times a_{ij}^{(2)}a_{ij}$, where $a_{ij}^{(2)}a_{ij}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 56(b) and this subgraph is counted only once in M. Let P_2 denote the number of subgraphs of G that have the same configuration as the graph of Fig 56(c) and are counted in M. Thus $P_2 = 1 \times \sum_{k=1,k\neq i,j}^{n} (a_{ik}^{(2)} - a_{ij})a_{ik}a_{jk}$, where $\sum_{k=1,k\neq i,j}^{n} (a_{ik}^{(2)} - a_{ij})a_{ik}a_{jk}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 56(c) (See Case 9) and this subgraph is counted only once in M. Let P_3 denote the number of subgraphs of G that have the same configuration as the graph of Fig 56(d) and are counted in M. Thus $P_3 = 2 \times \left(\begin{array}{c} a_{ij}^{(2)} \\ 2 \end{array}\right) a_{ij}$, where $\left(\begin{array}{c} a_{ij}^{(2)} \\ 2 \end{array}\right) a_{ij}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 56(d) and 2 is the number of times that this subgraph is counted in M. Consequently,

F= $a_{ii}^{(3)}a_{ij}^{(2)} - 2a_{ij}^{(2)}a_{ij} - 4\left(\begin{array}{c} a_{ij}^{(2)} \\ 2 \end{array}\right) a_{ij} - 2\sum_{k=1,k\neq i,j}^{n} (a_{ik}^{(2)} - a_{ij})a_{ik}a_{jk}$.



(a) (b) (c) (d)

Fig 56

Case 18: For the configuration of Fig 57(a), N= 2, M= $\frac{1}{2}a_{jj}^{(3)}a_{ij}^{(2)}$. Let P_1 denote the number of subgraphs of G that have the same configuration as the graph of Fig 57(b) and are counted in M. Thus $P_1 = 1 \times a_{ij}^{(2)}a_{ij}$, where $a_{ij}^{(2)}a_{ij}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 57(b) and this subgraph is counted only once in M. Let P_2 denote the number of subgraphs of G that have the same configuration as the graph of Fig 57(c) and are counted in M. Thus $P_2 = 1 \times \sum_{k=1,k\neq i,j}^{n} (a_{jk}^{(2)} - a_{ij})a_{ik}a_{jk}$, where $\sum_{k=1,k\neq i,j}^{n} (a_{jk}^{(2)} - a_{ij})a_{ik}a_{jk}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 57(c) (See Case 8) and this subgraph is counted only once in M. Let P_3 denote the number of subgraphs of G that have the same configuration as the graph of Fig 57(d) and are counted in M. Thus $P_3 = 2 \times \binom{a_{ij}^{(2)}}{2} a_{ij}$, where $\binom{a_{ij}^{(2)}}{2} a_{ij}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 57(d) and 2 is the number of times that this subgraph is counted in M. Consequently,

F= $a_{jj}^{(3)} a_{ij}^{(2)} - 2a_{ij}^{(2)} a_{ij} - 4 \binom{a_{ij}^{(2)}}{2} a_{ij} - 2 \sum_{k=1,k\neq i,j}^{n} (a_{jk}^{(2)} - a_{ij}) a_{ik} a_{jk}$.

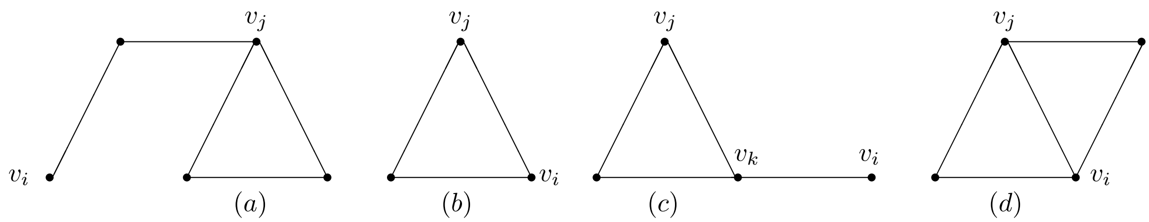


Fig 57

Case 19: For the configuration of Fig 58(a), N= 2, M= $\frac{1}{2} \sum_{k=1,k\neq i,j}^{n} a_{kk}^{(3)} a_{ik} a_{jk}$. Let P_1 denote the number of subgraphs of G that have the same configuration as the graph of Fig 58(b) and are counted in M. Thus $P_1 = 1 \times \sum_{k=1,k\neq i,j}^{n} (a_{ik}^{(2)} - a_{ij}) a_{ik} a_{jk}$, where $\sum_{k=1,k\neq i,j}^{n} (a_{ik}^{(2)} - a_{ij}) a_{ik} a_{jk}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 58(b) (See Case 9) and this subgraph is counted only once in M. Let P_2 denote the number of subgraphs of G that have the same configuration as the graph of Fig 58(c) and are counted in M. Thus $P_2 = 1 \times \sum_{k=1,k\neq i,j}^{n} (a_{jk}^{(2)} - a_{ij}) a_{ik} a_{jk}$, where $\sum_{k=1,k\neq i,j}^{n} (a_{jk}^{(2)} - a_{ij}) a_{ik} a_{jk}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 58(c) (See Case 8) and this subgraph is counted only once in M. Let P_3 denote the number of subgraphs of G that have the same configuration as the graph of Fig 58(d) and are counted in M. Thus $P_3 = 1 \times \sum_{k=1,k\neq i,j}^{n} a_{ij} a_{jk} a_{ik}$, where $\sum_{k=1,k\neq i,j}^{n} a_{ij} a_{jk} a_{ik}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 58(d) and this subgraph is counted only once in M. Consequently, F= $\sum_{k=1,k\neq i,j}^{n} a_{kk}^{(3)} a_{ik} a_{jk} - 2 \sum_{k=1,k\neq i,j}^{n} (a_{ik}^{(2)} - a_{ij}) a_{ik} a_{jk} - 2 \sum_{k=1,k\neq i,j}^{n} (a_{jk}^{(2)} - a_{ij}) a_{ik} a_{jk} - 2 \sum_{k=1,k\neq i,j}^{n} a_{ij} a_{jk} a_{ik}$.

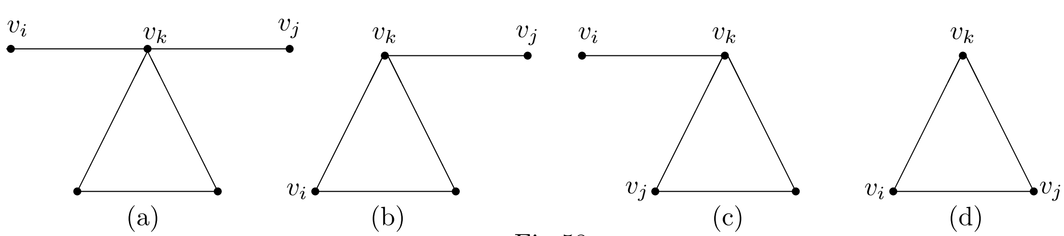


Fig 58

Case 20: For the configuration of Fig 59, N= 6, M= $\binom{a_{ij}^{(2)}}{2} a_{ij}$ and F= $6 \binom{a_{ij}^{(2)}}{2} a_{ij}$.

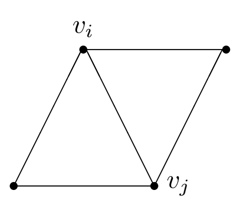


Fig 59

Case 21: For the configuration of Fig 60, N= 3, M= $\sum_{k=1,k\neq i,j}^{n}(a_{ik}^{(2)}-a_{ij})a_{jk}a_{ij}$ and F= $3\sum_{k=1,k\neq i,j}^{n}(a_{ik}^{(2)}-a_{ij})a_{jk}a_{ij}$. (See Theorem 1.10)

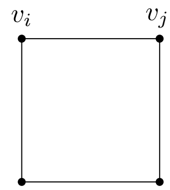
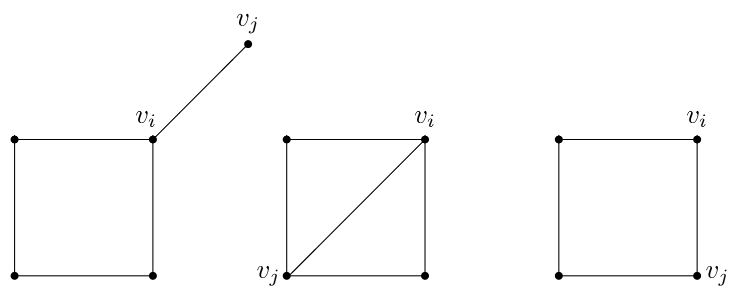


Fig 60

Case 22: For the configuration of Fig 61(a), N= 2, M= $\frac{1}{2}(a_{ii}^{(4)}-a_{ii}^{(2)}-2\binom{d_i}{2}-\sum_{k=1,k\neq i}^{n}a_{ik}^{(2)})a_{ij}$ (See Theorem 2.1). Let P_1 denote the number of subgraphs of G that have the same configuration as the graph of Fig 61(b) and are counted in M. Thus $P_1=1\times\binom{a_{ij}^2}{2}a_{ij}$, where $\binom{a_{ij}^2}{2}a_{ij}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 61(b) and this subgraph is counted only once in M. Let P_2 denote the number of subgraphs of G that have the same configuration as the graph of Fig 61(c) and are counted in M. Thus $P_2=1\times\sum_{k=1,k\neq i,j}^{n}(a_{ik}^{(2)}-a_{ij})a_{jk}a_{ij}$, where $\sum_{k=1,k\neq i,j}^{n}(a_{ik}^{(2)}-a_{ij})a_{jk}a_{ij}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 61(c) (See Theorem 1.10) and this subgraph is counted only once in M. Consequently, F=$(a_{ii}^{(4)}-a_{ii}^{(2)}-2\binom{d_i}{2}-\sum_{k=1,k\neq i}^{n}a_{ik}^{(2)})a_{ij}-2\binom{a_{ij}^2}{2}a_{ij}-2\sum_{k=1,k\neq i,j}^{n}(a_{ik}^{(2)}-a_{ij})a_{jk}a_{ij}$.



(a) (b) (c)

Fig 61

Case 23: For the configuration of Fig 62(a), N= 2, M= $\frac{1}{2}(a_{jj}^{(4)}-a_{jj}^{(2)}-2\binom{d_j}{2}-\sum_{k=1,k\neq j}^{n}a_{jk}^{(2)})a_{ij}$ (See Theorem 2.1). Let P_1 denote the number of subgraphs of G that have the same configuration as the graph of Fig 62(b) and are counted in M. Thus $P_1=1\times\binom{a_{ij}^{(2)}}{2}a_{ij}$, where $\binom{a_{ij}^{(2)}}{2}a_{ij}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 62(b) and this subgraph is counted only once in M. Let P_2 denote the number of subgraphs of G that have the same configuration as the graph of Fig 62(c) and are counted in M. Thus $P_2=1\times\sum_{k=1,k\neq i,j}^{n}(a_{ik}^{(2)}-a_{ij})a_{jk}a_{ij}$, where $\sum_{k=1,k\neq i,j}^{n}(a_{ik}^{(2)}-a_{ij})a_{jk}a_{ij}$ is the number of subgraphs of G that have the same configuration as the graph of Fig 62(c) (See Theorem 1.10) and this subgraph is counted only once in M. Consequently, F=$(a_{jj}^{(4)}-a_{jj}^{(2)}-2\binom{d_j}{2}-\sum_{k=1,k\neq j}^{n}a_{jk}^{(2)})a_{ij}-2\binom{a_{ij}^{(2)}}{2}a_{ij}-2\sum_{k=1,k\neq i,j}^{n}(a_{ik}^{(2)}-a_{ij})a_{jk}a_{ij}$.

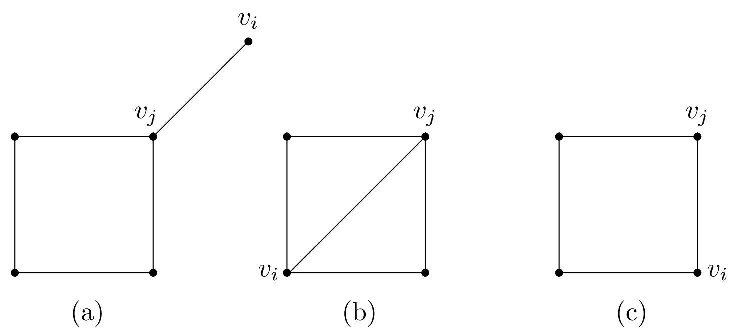


Fig 62

Now we add the values of F arising from the above cases and determine x. Substituting the value of x in $a_{ij}^{(5)} - x$ and simplifying, we get the desired result. $\square$

Example 2.8 *In the graph of Fig 39,* $a_{12}^{(5)} = 521$, $(2d_1 + 2d_2 + d_1 d_2 + a_{11}^{(4)} + a_{22}^{(4)} - a_{11}^{(2)} - a_{22}^{(2)} - a_{12}^{(2)} - 2\binom{d_1}{2} - 2\binom{d_2}{2} + 2\binom{d_1 - 1}{2} + 2\binom{d_2 - 1}{2} - 6\binom{a_{12}^{(2)}}{2} - 4)\ a_{12} = 185$, $(a_{11}^{(3)} + a_{22}^{(3)})\ a_{12}^{(2)} = 160$, $\sum_{k=2}^{6} a_{1k}^{(2)} a_{12} = 20$, $\sum_{k=1, k\neq 2}^{6} a_{2k}^{(2)} a_{12} = 20$, $\sum_{k=3}^{6} a_{kk}^{(3)} a_{1k} a_{2k} = 80$, $\sum_{k=3}^{6} (a_{1k}^{(2)} + a_{2k}^{(2)} - a_{1k} - a_{2k} - 2a_{1k}a_{2k})\ a_{12} = 16$, $\sum_{k=3}^{6} (a_{1k}^{(2)} - a_{12})(3a_{12} + 3a_{1k} - d_1 - d_2 - d_k + 1)\ a_{2k} = -96$, $\sum_{k=3}^{6} (a_{2k}^{(2)} - a_{12})(3a_{2k} - d_k + 2)\ a_{1k} = 0$.
So, by Theorem 2.7, the number of $v_1 - v_2$ *paths of length 5 in the graph of Fig 39 is 24.*

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