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A New Difference Scheme for Fractional Cable Equation and Stability Analysis

Ibrahim Karatay and Nurdane Kale*

Fatih University

**nurdaneguduk@fatih.edu.tr*

Abstract

We consider the fractional cable equation. For solution of fractional Cable equation involving Caputo fractional derivative, a new difference scheme is constructed based on Crank Nicholson difference scheme. We prove that the proposed method is unconditionally stable by using spectral stability technique.

Keywords: *Caputo fractional derivative, Difference scheme, Stability.*

1. Introduction

In this study, we consider the following time fractional cable equation;

$$\begin{cases} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} - \mu^2 u(x,t) + f(x,t), (0 < x < 1, 0 < t < 1), \\ u(x,0) = r(x),\ 0 < x < 1, \\ u(0,t) = 0,\ \ u(1,t) = 0,\ \ 0 \le t \le 1. \end{cases} \tag{1}$$

Here, the term $\frac{\partial^\alpha u(t,x)}{\partial t^\alpha}$ denotes α-order Caputo derivative with the formula:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u_t(x,\tau)}{(t-\tau)^\alpha} d\tau,\ \text{where } 0 < \alpha < 1, \tag{2}$$

where $\Gamma(.)$ is the Gamma function.

2. Discretization of Problem

We introduce the basic ideas for the numerical solution of the Time Fractional Cable equation by Crank-Nicholson difference scheme.

For some positive integers M and N, the grid sizes in space and time for the finite difference algorithm are defined by $h = 1/M$ and $\tau = 1/N$, respectively. The grid points in the space interval $[0,1]$ are the numbers

$x_j = jh$, $j = 0, 1, 2, ..., M$, and the grid points in the time interval $[0, 1]$ are labeled $t_k = k\tau, k = 0, 1, 2, ..., N$. The values of the functions u and f at the grid points are denoted $u_j^k = u(x_j, t_k)$ and $f_j^k = f(x_j, t_k)$, respectively. Let $u(x, t), u_t(x, t)$ and $u_{tt}(x, t)$ are continuous on $[0, 1]$.

As in the classical Crank-Nicholson difference scheme, a discrete approximation to the fractional derivative $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ at $(x_j, t_{k+\frac{1}{2}})$ can be obtained by the following approximation[12]:

$$\frac{\partial^\alpha u(x_j, t_{k+\frac{1}{2}})}{\partial t^\alpha} = \left[w_1 u^k + \sum_{m=1}^{k-1} (w_{k-m+1} - w_{k-m}) u^m - w_k u^0 + \sigma \frac{(u_j^{k+1} - u_j^k)}{2^{1-\alpha}} \right] + O(\tau^{2-\alpha}). \tag{3}$$

Where $\sigma = \frac{1}{\Gamma(2-\alpha)} \frac{1}{\tau^\alpha}$ and $w_j = \sigma \left((j + 1/2)^{1-\alpha} - (j - 1/2)^{1-\alpha} \right)$ In addition for k = 0 there is no these terms $w_1 u_k$ and $w_k u_0$. On the other hand, we have

$$\frac{\partial^2 u(x_j, t_{k+\frac{1}{2}})}{\partial x^2} = \frac{1}{2} \left[\frac{u_{j+1}^{k+1} - 2u_j^{k+1} + u_{j-1}^{k+1}}{h^2} + \frac{u_{j+1}^k - 2u_j^k + u_{j-1}^k}{h^2} \right] + O(h^2). \tag{4}$$

3. The Proposed Difference Scheme

Using these approximations (3) and (4) into (1), we obtain the following difference scheme for (1) which is accurate of order $O(\tau^{2-\alpha} + h^2)$;

$$w_1 u^k + \sum_{m=1}^{k-1} (w_{k-m+1} - w_{k-m}) u^m - w_k u^0 + \sigma \frac{(u_j^{k+1} - u_j^k)}{2^{1-\alpha}} = \frac{1}{2} \left[\frac{u_{j+1}^{k+1} - 2u_j^{k+1} + u_{j-1}^{k+1}}{h^2} + \frac{u_{j+1}^k - 2u_j^k + u_{j-1}^k}{h^2} \right] -\mu^2 \left(\frac{u_j^k + u_j^{k+1}}{2} \right) + f(x_j, t_k + \tfrac{\tau}{2})$$

$$\begin{cases} \left[w_1 u_j^k + \sum_{m=1}^{k-1} (w_{k-m+1} - w_{k-m}) u_j^m - w_k u_j^0 + \sigma \frac{(u_j^{k+1} - u_j^k)}{2^{1-\alpha}} \right] \\ - \left[\frac{u_{j+1}^{k+1} - 2u_j^{k+1} + u_{j-1}^{k+1}}{2h^2} + \frac{u_{j+1}^k - 2u_j^k + u_{j-1}^k}{2h^2} \right] + \mu^2 \left(\frac{u_j^k + u_j^{k+1}}{2} \right) = f(x_j, t_k + \frac{\tau}{2}), \\ 0 \le k \le N-1, \ 1 \le j \le M-1, \\ u_j^0 = r(x_j), \ 1 \le j \le M, \\ u_0^k = 0, \quad u_M^k = 0, \ 0 \le k \le N. \end{cases}$$

$$\begin{cases} \left[\left(-\frac{1}{2h^2} \right) u_{j+1}^{k+1} + \left(\frac{\sigma}{2^{1-\alpha}} + \frac{1}{h^2} + \frac{\mu^2}{2} \right) u_j^{k+1} + \left(-\frac{1}{2h^2} \right) u_{j-1}^{k+1} \right] \\ + \left[\left(-\frac{1}{2h^2} \right) u_{j+1}^k + \left(-\frac{\sigma}{2^{1-\alpha}} + \frac{1}{h^2} + \frac{\mu^2}{2} \right) u_j^k + \left(-\frac{1}{2h^2} \right) u_{j-1}^k \right] \\ + \left[w_1 u_j^k + \sum_{m=1}^{k-1} (w_{k-m+1} - w_{k-m}) u_j^m - w_k u_j^0 \right] \\ = f(x_j, t_k + \frac{\tau}{2}), \quad 0 \le k \le N-1, \ 1 \le j \le M-1, \\ u_j^0 = r(x_j), \ 1 \le j \le M, \\ u_0^k = 0, \quad u_M^k = 0, \ 0 \le k \le N. \end{cases}$$

We can arrange the system above to obtain

$$\begin{cases} \left[\left(-\frac{1}{2h^2}\right)u_{j+1}^{k+1}+\left(\frac{\sigma}{2^{1-\alpha}}+\frac{1}{h^2}+\frac{\mu^2}{2}\right)u_j^{k+1}+\left(-\frac{1}{2h^2}\right)u_{j-1}^{k+1}\right] \\ +\left[\left(-\frac{1}{2h^2}\right)u_{j+1}^{k}+\left(-\frac{\sigma}{2^{1-\alpha}}+\frac{1}{h^2}+\frac{\mu^2}{2}\right)u_j^{k}+\left(-\frac{1}{2h^2}\right)u_{j-1}^{k}\right] \\ +\left[\sum_{m=1}^{k} w_m\left(u_j^{k-m+1}-u_j^{k-m}\right)\right] \\ =f(x_j,t_k+\frac{\tau}{2}),\quad 0\le k\le N-1,\ 1\le j\le M-1, \\ u_j^0=r(x_j),\ 1\le j\le M, \\ u_0^k=0,\quad u_M^k=0,\ 0\le k\le N. \end{cases}$$

Then we rewrite the equation following type

$$\begin{cases} \left[\left(-\frac{1}{2h^2}\right)u_{j+1}^{k}+\left(-\frac{1}{2h^2}\right)u_{j+1}^{k+1}\right] \\ +\left[\left(-\frac{\sigma}{2^{1-\alpha}}+\frac{1}{h^2}+\frac{\mu^2}{2}\right)u_j^{k}+\left(\frac{\sigma}{2^{1-\alpha}}+\frac{1}{h^2}+\frac{\mu^2}{2}\right)u_j^{k+1}+\sum_{m=1}^{k} w_m\left(u_j^{k-m+1}-u_j^{k-m}\right)\right] \\ +\left[\left(-\frac{1}{2h^2}\right)u_{j-1}^{k}+\left(-\frac{1}{2h^2}\right)u_{j-1}^{k+1}\right] \\ =f(x_j,t_k+\frac{\tau}{2}),\quad 0\le k\le N-1,\ 1\le j\le M-1, \\ u_j^0=r(x_j),\ 1\le j\le M, \\ u_0^k=0,\quad u_M^k=0,\ 0\le k\le N. \end{cases} \tag{5}$$

3.1. Spectral Stability of the Difference Method

The difference scheme above (5) can be written in matrix form:

$DU_{j+1}+EU_j+DU_{j-1}=\varphi_j$ where $\varphi_j=\left[\varphi_j^0,\varphi_j^1,\varphi_j^2,...,\varphi_j^N\right]^T$, $\varphi_j^0=r(x_j),\varphi_j^k=f(x_j,t_{k+\frac{1}{2}}),1\le j\le M,1\le k\le N$, and $U_j=\left[U_J^0,U_J^1,U_J^2,...,U_J^N\right]^T$.

Here $D_{(N+1)\mathrm{x}(N+1)}$and $E_{(N+1)\mathrm{x}(N+1)}$are the matrices of the form

$$D=\left(-\frac{1}{2h^2}\right)\begin{bmatrix} 0 & & & & & \\ 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & 1 & 1 \end{bmatrix}$$

$$E=\begin{bmatrix} 1 & & & & & \\ b & a & & & & \\ -w_1 & b+w_1 & a & & & \\ -w_2 & w_2-w_1 & b+w_1 & a & & \\ \vdots & & \ddots & \ddots & \ddots & \\ -w_{N-1} & w_{N-1}-w_{N-2} & \cdots & w_2-w_1 & b+w_1 & a \end{bmatrix}$$

where $a=\frac{\sigma}{2^{1-\alpha}}+\frac{1}{h^2}+\frac{\mu^2}{2}$, $b=-\frac{\sigma}{2^{1-\alpha}}+\frac{1}{h^2}+\frac{\mu^2}{2}$

Using the idea on the modified Gauss-Elimination method, we can convert into the following form:

$U_j=\psi_{j+1}U_{j+1}+\mu_{j+1},j=M,...,2,1,0.$

Then, we write

$D+E\psi_{j+1}+D\psi_j\psi_{j+1}=0,$

$E\mu_{j+1}+D\psi_j\mu_{j+1}+D\mu_j=\varphi_j$, where $1\le j\le M.$

So, we obtain the following pair of formulas:

$\psi_{j+1}=-\left(E+D\psi_j\right)^{-1}D$, $\mu_{j+1}=\left(E+D\psi_j\right)^{-1}\left(\varphi_j-D\mu_j\right)$, where $1\le j\le M.$

We will prove that $\rho\ (\psi_j) < 1,\ 1 \leq j \leq M$, by induction. Since ψ_1 is a zero matrix $\rho\ (\psi_1) = 0 < 1$.Moreover, $\psi_2 = -E^{-1}D, \rho\left(\psi_2\right) = \rho\left(-E^{-1}D\right) = \left|\frac{-1}{\frac{\sigma}{2^{1-\alpha}}+\frac{1}{h^2}+\frac{\mu^2}{2}}\right| . \left|\left(-\frac{1}{2h^2}\right)\right| = \frac{1/h^2}{2\left(\frac{\sigma}{2^{1-\alpha}}+\frac{1}{h^2}+\frac{\mu^2}{2}\right)}$,since ψ_2 is of the form

$$\psi_2 = \begin{bmatrix} 0 & & & & \\ * & \frac{1/h^2}{2\left(\frac{\sigma}{2^{1-\alpha}}+\frac{1}{h^2}+\frac{\mu^2}{2}\right)} & & & \\ * & * & \frac{1/h^2}{2\left(\frac{\sigma}{2^{1-\alpha}}+\frac{1}{h^2}+\frac{\mu^2}{2}\right)} & & \\ & & & \ddots & \\ * & * & * & * & \frac{1/h^2}{2\left(\frac{\sigma}{2^{1-\alpha}}+\frac{1}{h^2}+\frac{\mu^2}{2}\right)} \end{bmatrix}_{(M+1)x(M+1)}$$

$\sigma = \frac{1}{\Gamma(2-\alpha)}\frac{1}{\tau^\alpha} > 0$, therefore, $\rho\left(\psi_2\right) < 1$.

Now, assume $\rho\left(\psi_j\right) < 1$. We find that

$\psi_{j+1} = -\left(E + D\psi_j\right)^{-1} D$

$$= \left(\frac{1}{2h^2}\right) \begin{bmatrix} 0 & & & & \\ * & \frac{1}{E_{2,2}-(1/2h^2)\psi_{j_{2,2}}} & & & \\ * & * & \frac{1}{E_{3,3}-(1/2h^2)\psi_{j_{3,3}}} & & \\ & & & \ddots & \\ * & * & * & * & \frac{1}{E_{M+1,M+1}-(1/2h^2)\psi_{j_{M+1,M+1}}} \end{bmatrix}$$

and we already know that $E_{j,j} = \frac{\sigma}{2^{1-\alpha}} + \frac{1}{h^2} + \frac{\mu^2}{2}$ and $\psi_{j_{r,r}} = \rho\left(\psi_j\right)$ for $2 \leq r \leq M+1$:

$$\rho\left(\psi_{j+1}\right) = \left|\frac{1/2h^2}{\frac{\sigma}{2^{1-\alpha}} + \frac{1}{h^2} + \frac{\mu^2}{2} - \frac{1}{2h^2}\rho\left(\psi_j\right)}\right| = \frac{M^2}{2\left[M^2\left(1 - \frac{\rho(\psi_j)}{2}\right) + \frac{\sigma}{2^{1-\alpha}} + \frac{\mu^2}{2}\right]}$$

Since $0 \leq \rho\left(\psi_j\right) < 1$, it follows that $\rho\left(\psi_{j+1}\right) < 1$. So, $\rho\left(\psi_j\right) < 1$ for any j, where $1 \leq j \leq M$.

3.2. Numerical Example

Consider this problem,

$$\begin{cases} \frac{\partial^\alpha u(t,x)}{\partial t^\alpha} = \frac{\partial^2 u(t,x)}{\partial x^2} - u(t,x) + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}(1-x)\sin(x) + 2t^2\left[\cos(x) + (1-x)\sin(x)\right], \\ (0 < x < 1, 0 < t < 1), \\ u(0,x) = 0,\ 0 \leq x \leq 1, \\ u(t,0) = 0,\ \ u(t,1) = 0,\ \ 0 \leq t \leq 1. \end{cases}$$

Exact solution of this problem is $u(t,x) = t^2(1-x)\sin(x)$. The errors for some M and N are given in figure 1. The errors when solving this problem are listed in the table1 for various values of time and space nodes.

Table 1: The errors for some values of M, N and α

		$\alpha = 0.3$		$\alpha = 0.5$		$\alpha = 0.8$	
N	M	$Error(\alpha,\tau)$	Err. rate	$Error(\alpha,\tau)$	Err. rate	$Error(\alpha,\tau)$	Err. rate
8	32	0.001811212	-	0.001688126	-	0.001265365	-
16	32	0.000449950	4.02	0.000409875	4.1	0.000301407	4.1
32	32	0.000111687	4.02	0.000099150	4.1	0.000086960	3.46

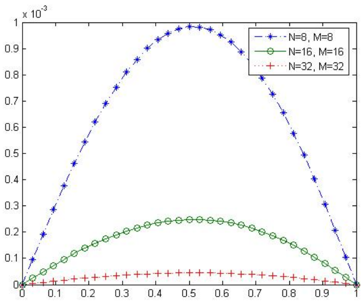

Figure 1: The errors when t=1 for some M and N

4. Conclusion

In this work, $O(\tau^{2-\alpha} + h^2)$ order approximation for the Caputo fractional derivative based on the Crank-Nicholson difference scheme was successfully applied to solve the time-fractional cable equation. It is proven that the time-fractional Crank-Nicholson difference scheme is unconditionally stable by spectral stability analysis.

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