



Inverse of invertible standard multi-companion matrices with applications

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Abstract

The inverse of invertible standard multi-companion matrices will be derived and introduced as a new technique for generation of periodic autoregression models to get the desired spectrum and extract the parameters of the model from it when the information of the standard multi-companion matrices is not enough for the extracting of the parameters of the model.

We will find explicit expressions for the generalized eigenvectors of the inverse of invertible standard multi-companion matrices such that each generalized eigenvector depends on the corresponding eigenvalue therefore we obtain a parameterization of the inverse of invertible standard multi-companion matrix through the eigenvalues and these additional quantities. The results can be applied to statistical estimation, simulation and theoretical studies of periodically correlated and multivariate time series in both discrete- and continuous-time series.

Keywords: Standard multi-companion matrix, Inverse of invertible standard multi-companion matrix, Factorization, Jordan decomposition, Generalized eigenvectors.

1. Introduction

Time series arise as recordings of processes which vary over time. A recording can either be a continuous trace or a set of discrete observations.

Boshnakov (2001) generate the matrix F from the spectral parameters and then reconstruct the parameters for the required parameterization of the models. The main idea of the multi-companion method for generation of periodic autoregression models is to generate a multi-companion matrix with the desired spectrum and extract the parameters of the model from it.

$$F = \begin{bmatrix} f_{1,1} & f_{1,2} & \dots & f_{1,m-d} & f_{1,m-d+1} & \dots & f_{1,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ f_{d,1} & f_{d,2} & \dots & f_{d,m-d} & f_{d,m-d+1} & \dots & f_{d,m} \\ 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \end{bmatrix}_{m \times m} \quad (1)$$

The backward leading minors of the upper right block of a d -companion matrix F are denoted by δ_j , $j = 1, 2, \dots, d$ and the corresponding determinants by $\Delta_j(F)$. Now we can compute the determinant of the d -companion matrix F by $\det(F) = (-1)^{(m+1)d} \Delta_d(F)$.

If we use all of the above notations and symbols we can rewrite F in the new form as blocks to be

$$F = \begin{bmatrix} F_{1:d,1:m-d} & \delta_d \\ I_{m-d} & 0_{m-d,d} \end{bmatrix}. \tag{2}$$

In this paper, we introduce to the class of multi-companion matrices which is the inverse of invertible standard multi-companion matrices when the information of the standard multi-companion matrices is not enough for the extracting of the parameters of the model. The results can be applied to statistical estimation, simulation and theoretical studies of periodically correlated and multivariate time series in both discrete- and continuous-time series.

Theorem 1.1 The standard multi-companion matrix F is invertible iff F is non singular.

Corollary 1.2 The standard multi-companion matrix F is invertible iff δ_d is non singular.

Direct calculation shows that the inverse of an invertible standard multi-companion matrix F is

$$F^{-1} = \begin{bmatrix} 0_{m-d,d} & I_{m-d} \\ \delta_d^{-1} & -\delta_d^{-1} F_{1:d,1:m-d} \end{bmatrix}, \tag{3}$$

and in the next section we need to study the matrix $G = F^{-1}$.

2. Inverse of invertible standard multi-companion matrices

Definition 2.1 The $m \times m$ invertible matrix G is said to be inverse of invertible standard multi-companion of order d (or d -companion) if

1. the first $m - d$ rows of G consists of $\{e_i, i = m - d + 1, \dots, m\}$ where $e_m = [0, \dots, 0, 1, 0, \dots, 0]$ is the standard basis vector with 1 in the i -th position and 0 elsewhere;
2. the last d rows of G are arbitrary;
3. $1 \leq d < m$.

The new form of the inverse of invertible standard multi-companion can be written as

$$G = \begin{bmatrix} 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 \\ g_{m-d+1,1} & \dots & g_{m-d+1,d} & g_{m-d+1,d+1} & \dots & g_{m-d+1,m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ g_{m,1} & \dots & g_{m,d} & g_{m,d+1} & \dots & g_{m,m} \end{bmatrix}_{m \times m}. \tag{4}$$

Let M and N in the standard multi-companion matrix F be equal to the left and right upper corner blocks respectively(i.e. $M = F_{1:d,1:m-d}$ and $N = \delta_d$). We will rewrite new form of the inverse of invertible standard multi-companion in Equation (4), to get

$$G = \begin{bmatrix} 0 & I \\ B & A \end{bmatrix}_{m \times m} = \begin{bmatrix} 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 \\ b_{1,1} & \dots & b_{1,d} & a_{1,1} & \dots & a_{1,m-d} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b_{d,1} & \dots & b_{d,d} & a_{d,1} & \dots & a_{d,m-d} \end{bmatrix}_{m \times m} = \begin{bmatrix} e_{m-d+1} \\ \vdots \\ e_m \\ G_{bot} \end{bmatrix}_{m \times m}$$

where $0, I, B = N^{-1}$ and $A = -N^{-1}M$ are matrices of size $(m-d) \times d$, $(m-d) \times (m-d)$, $d \times d$, $d \times (m-d)$ respectively, for some integer d , $1 \leq d < m$, N is invertible, G_{bot} is the block matrix $[B, A]$, and $e_i = [0, \dots, 0, 1, 0, \dots, 0]$ is the standard basis vector with 1 in the i -th position and 0 elsewhere.

The companion matrix C from the matrix G can be written as

$$C_G = [\phi_1, \phi_2, \dots, \phi_m]_{C_G} = \begin{bmatrix} 0 & I \\ B & A \end{bmatrix} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ \phi_1 & \phi_2 & \dots & \phi_m \end{bmatrix}$$

which has the following properties:

Determinant	$(-1)^{m+1}\phi_{m1}$
Characteristic polynomial	$\lambda^m + \phi_m\lambda^{m-1} + \dots + \phi_2\lambda + \phi_1$
Number of its linearly independent eigenvectors	The vector $\{1, \lambda, \dots, \lambda^{m-1}\}$ is an eigenvector corresponding to the eigenvalue λ

3. Properties of inverse of invertible standard multi-companion matrices

The standard multi-companion matrices have properties which can be as a generalization of the corresponding properties of companion matrices, multi-companion matrices.

Now we will see in the next subsections some important properties of the inverse of invertible standard multi-companion matrices that we need for our works, for more see [3], [7] and [8].

3.1. Multiplication by multi-companion matrices

For instance, a d -companion matrix G is non-singular if and only if its lower left block $B_{d \times d}$ has a non-zero determinant. Now if $d = 1$, then G is companion, and the corresponding determinants of $B_{d \times d}$ is equal to $g_{m,d}$, which is a scalar.

Theorem 3.1 Let G be a d -companion invertible $m \times m$ matrix, and A an arbitrary matrix.

1. The left multiplication of any matrix A by a d -companion matrix G (i.e. GA) moves the last $m - d$ rows of A , d rows upwards without change.
2. The right multiplication of any matrix A by the transposed of a d -companion matrix G (i.e. AG') moves the last $m - d$ columns of A , d columns leftwards without change.
3. The symmetric product of any matrix A with G and its transpose (i.e. GAG') moves each element of the upper left $(m - d) \times (m - d)$ block of A , d rows upwards and d columns leftwards (i.e., d positions in north-west direction).

Corollary 3.2 Let $G = AG_{d-1}$ with size $m \times m$, where A is 1-companion, G_{d-1} is $(d - 1)$ -companion, and $1 + (d - 1) < m$. Then G is d -companion.

Corollary 3.3 The product $A_d \dots A_1$ of companion matrices $A_i = [\phi_{i1}, \dots, \phi_{im}]_{C_G}, i = 1, \dots, d, d < m$ is multi-companion of order d . This product is non-singular if and only if $\phi_{dm} \dots \phi_{1m} \neq 0$.

Note that for any d -companion matrix it is not always possible of writing it as a product of companion matrices, so the converse is not true even in the invertible case. Actually, we can find a permutation of the non-trivial rows which allows for fully factorization. The next theorem shows what we say.

Theorem 3.4 Let G be an invertible multi-companion matrix of order d , then it can be factored as products of d companion matrices to be as

$$G = PA_d A_{d-1} \dots A_1$$

where P is a (row) permutation matrix and $A_i, i = 1, \dots, d$, are companion matrices.

We will use these useful properties of multiplication by multi-companion matrices and certain facts about the factorization of multi-companion matrix for a generation matrix, for more details see [1] and [8].

3.2. Factorization of multi-companion matrices

Factoring into companion times multi-companion

We are looking for a companion matrix A for which the following equation holds:

$$G_d = AG_{d-1}. \tag{5}$$

Hence, the expanded form of (5) is

$$\begin{bmatrix} 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 \\ g_{m-d+1,1}^{(d)} & \dots & g_{m-d+1,d}^{(d)} & g_{m-d+1,d+1}^{(d)} & \dots & g_{m-d+1,m}^{(d)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ g_{m,1}^{(d)} & \dots & g_{m,d}^{(d)} & g_{m,d+1}^{(d)} & \dots & g_{m,m}^{(d)} \end{bmatrix}_{m \times m} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ a_1 & a_2 & \dots & a_m \end{bmatrix}_{m \times m} \times \begin{bmatrix} 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 \\ g_{m-d+2,1}^{(d-1)} & \dots & g_{m-d+2,d}^{(d-1)} & g_{m-d+2,d+1}^{(d-1)} & \dots & g_{m-d+2,m}^{(d-1)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ g_{m,1}^{(d-1)} & \dots & g_{m,d}^{(d-1)} & g_{m,d+1}^{(d-1)} & \dots & g_{m,m}^{(d-1)} \end{bmatrix}_{m \times m}$$

And so, we can write the elements of these matrices in one of the following equations

$$g_{m,j}^{(d)} = \sum_{i=m-d+2}^m a_i g_{i,j}^{(d-1)}, \quad j = 1, \dots, d-1, \tag{6}$$

$$g_{m,j}^{(d)} = \sum_{i=m-d+2}^m a_i g_{i,j}^{(d-1)} + a_{j-d+1}, \quad j = d, \dots, m, \tag{7}$$

$$g_{i-1,j}^{(d)} = g_{i,j}^{(d-1)}, \quad i = m-d+2, \dots, m; j = 1, \dots, m. \tag{8}$$

We can solved (7) explicitly for a_{j-d+1} , $j = d, \dots, m$ (i.e., a_d, \dots, a_m),

$$a_{j-d+1} = g_{m,j}^{(d)} - \sum_{i=m-d+2}^m a_i g_{i,j}^{(d-1)}, \quad j = d, \dots, m.$$

The remaining equations involve operations on parts of the rows of G_{d-1} . Say, $g_{i\bullet}^{(d)}$ be the i -th ($1 : d$)-row of G_{d-1} , and so $g_{i\bullet}^{(d)} = (g_{i1}^{(d)}, \dots, g_{id}^{(d)})$. Now from (6) and (8), we have

$$g_{m\bullet}^{(d)} = \sum_{i=m-d+2}^m a_i g_{i\bullet}^{(d-1)}; \quad g_{i-1\bullet}^{(d)} = g_{i\bullet}^{(d-1)}, \quad i = m-d+2, \dots, m.$$

Factoring into multi-companion times companion

Now the expanded form of $G_d = G_{d-1}A$ where A a companion matrix is

$$\begin{bmatrix} 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 \\ g_{m-d+1,1}^{(d)} & \dots & g_{m-d+1,d}^{(d)} & g_{m-d+1,d+1}^{(d)} & \dots & g_{m-d+1,m}^{(d)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ g_{m,1}^{(d)} & \dots & g_{m,d}^{(d)} & g_{m,d+1}^{(d)} & \dots & g_{m,m}^{(d)} \end{bmatrix}_{m \times m} =$$

$$\begin{bmatrix} 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 \\ g_{m-d+2,1}^{(d-1)} & \dots & g_{m-d+2,d}^{(d)} & g_{m-d+2,d+1}^{(d-1)} & \dots & g_{m-d+2,m}^{(d-1)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ g_{m,1}^{(d-1)} & \dots & g_{m,d}^{(d-1)} & g_{m,d+1}^{(d-1)} & \dots & g_{m,m}^{(d-1)} \end{bmatrix}_{m \times m} \times \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ a_1 & a_2 & \dots & a_m \end{bmatrix}_{m \times m}$$

So, we can write the elements of these matrices in one of the following equations

$$\begin{aligned} g_{m-d+1,j}^{(d)} &= a_j, & j &= 1, \dots, m. \\ g_{i,1}^{(d)} &= g_{i,m}^{(d-1)} a_1, & i &= m-d+2, \dots, m. \\ g_{i,j}^{(d)} &= g_{i,m}^{(d-1)} a_j + g_{i,j-1}^{(d-1)}, & j &= d-1, \dots, m, \quad i = m-d+2, \dots, m, \end{aligned}$$

3.3. Eigenvalues and eigenvectors of inverse of invertible standard multi-companion matrices

The Jordan canonical forms of inverse of invertible standard multi-companion matrices provide a way to generate eigenvalues and eigenvectors to construct the matrix G and then extract the parameters of the corresponding PAR model from it.

Consider the equation

$$Gx = \lambda x,$$

that relates G to an eigenvalue λ and a corresponding eigenvector x .

The eigenvalues of the $m \times m$ matrix G are the roots (zeros) of its characteristic polynomial,

$$T(\lambda) = \det(\lambda I - G)$$

where $\lambda I - G$ is the characteristic matrix of G .

If the characteristic polynomial $T(\lambda)$ has distinct roots, then it can be factorized into a product of m linear factors

$$T(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\dots(\lambda - \lambda_m).$$

Also, if $T(\lambda)$ has some s repeated roots, then it can be factorized as follows

$$T(\lambda) = \prod_{i=1}^s (\lambda - \lambda_i)^{q_i}$$

where $\sum_{i=1}^s q_i = m$.

Remark 3.5 If λ_i is an eigenvalue of a matrix A , then the dimension of the linearly independent eigenspace corresponding to λ_i is called the geometric multiplicity of λ_i , and is denoted by $gm(\lambda_i)$.

On the other hand, the number of times $\lambda - \lambda_i$ that appears as a factor in the characteristic polynomial of A is called the algebraic multiplicity of A , and is denoted by $am(\lambda_i)$. Note that from linear algebra $gm(\lambda_i) \leq am(\lambda_i)$, for more details see [1].

Diagonalizable multi-companion matrices

Clearly that, a square matrix G is called diagonalizable if there is an invertible matrix P such that $P^{-1}GP$ is a diagonal matrix; the matrix G is said to diagonalize G . The decomposition of G into the form

$$G = PJP^{-1}$$

is the Jordan matrix decomposition of G where J is a Jordan canonical form of G and P is a non-singular matrix and its columns are the corresponding eigenvectors of G .

Clearly, G is diagonalizable if and only if the geometric multiplicities of all eigenvalues are coincide with the algebraic multiplicities, i.e. $gm(\lambda_i) = am(\lambda_i)$ for every distinct eigenvalue λ_i .

Non-diagonalizable multi-companion matrices

It is important to note that if a matrix has all distinct eigenvalues (whether real or complex), then it is diagonalizable; in other words, only matrices with repeated eigenvalues might be non-diagonalizable.

However, this happens when the Jordan matrix J is blockdiagonal, as the following structure

$$J = \begin{bmatrix} J(\lambda_1) & 0 & \dots & 0 \\ 0 & J(\lambda_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J(\lambda_s) \end{bmatrix}$$

where $J(\lambda_i), i = 1, \dots, s$ is called a Jordan segment associated with the eigenvalue λ which is made up of $g_i = gm(\lambda_i)$ Jordan blocks to get

$$J(\lambda_i) = \begin{bmatrix} J_1(\lambda_i) & 0 & \dots & 0 \\ 0 & J_2(\lambda_i) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_{g_i}(\lambda_i) \end{bmatrix}$$

where

$$J_j(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_i & 1 \\ 0 & \dots & \dots & 0 & \lambda_i \end{bmatrix}, j = 1, \dots, g_i.$$

Each block is associated with a set of columns of P forming a Jordan chain which is called generalized eigenvectors. The sum of dimensions of all Jordan blocks associated with λ_i is equal to $am(\lambda_i)$. The number of Jordan blocks associated with λ_i is equal to $gm(\lambda_i)$, for more details see [5].

Eigenvector and generalized eigenvector of a d -companion matrix

We can add linearly independent vectors to the eigenvectors in order to complete the basis, when G does not have m linearly independent eigenvectors to form the columns of the matrix P .

Suppose that the geometric multiplicity of the eigenvalue is less than its algebraic multiplicity. Choose a single $s \times s$ Jordan block $J_j(\lambda), j = 1, \dots, g$, where $g = gm(\lambda)$. The block $J_j(\lambda)$ is associated with a set of columns of P . Let $P_j = [\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(s)}]$ be the portion of P that correspond to the location of the block $J_j(\lambda)$ in the Jordan matrix J .

There exactly one independent eigenvector for each Jordan block which is the first vector in the portion. The following properties are very important and useful for the eigenvectors of any matrix, for more details see [5] and [6].

Proposition 3.6 There can be at most d linearly independent eigenvectors of a d -companion matrix corresponding to a given eigenvalue.

Definition 3.7 The set of vectors $\{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(s)}\}$ is called a chain (or a Jordan chain) of generalized eigenvectors associated with the eigenvalue λ .

Theorem 3.8 A chain of generalized eigenvectors $\mathcal{C} = \{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(s)}\}$ associated with an eigenvalue λ is linearly independent.

Theorem 3.9 The union of chains of generalized eigenvectors of G belonging to distinct eigenvalues is linearly independent.

Theorem 3.10 Let λ be an eigenvalue of G , $\mathbf{x}^{(1)}$ and $\mathbf{y}^{(1)}$ be two independent eigenvectors corresponding to λ . Let $\mathcal{C}_1 = \{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(s_1)}\}$ and $\mathcal{C}_2 = \{\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \dots, \mathbf{y}^{(s_2)}\}$ be the two chains of generalized eigenvectors corresponding to $\mathbf{x}^{(1)}$ and $\mathbf{y}^{(1)}$ respectively. Then the union $\mathcal{C}_1 \cup \mathcal{C}_2$ is linearly independent.

From the above we may be constructed a transition matrix P from the chains of linearly independent generalized eigenvectors of G , and justifies the invertibility of P .

Let $GP = PJ$, we have

$$G[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(s)}] = [\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(s)}] \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda & 1 \\ 0 & \dots & \dots & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{x}^{(1)}\lambda & \mathbf{x}^{(1)}1 & 0 & \dots & 0 \\ 0 & \mathbf{x}^{(2)}\lambda & \mathbf{x}^{(2)} & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \mathbf{x}^{(s-1)}\lambda & \mathbf{x}^{(s-1)}1 \\ 0 & \dots & \dots & 0 & \mathbf{x}^{(s)}\lambda \end{bmatrix}$$

Hence,

$$G\mathbf{x}^{(1)} = \mathbf{x}^{(1)}\lambda$$

$$G\mathbf{x}^{(i)} = \mathbf{x}^{(i)}\lambda + \mathbf{x}^{(i-1)}, i = 2, \dots, s,$$

and the two equations can be squeezed into one if we adopt the convention $\mathbf{x}^{(0)} \equiv 0$.

We called $\mathbf{x}^{(s)}$ a generalized eigenvector of order s associated with the eigenvalue λ if we find a vector $\mathbf{x}^{(s)}$ such that

$$(G - \lambda I)^s \mathbf{x}^{(s)} = 0, \text{ and } (G - \lambda I)^{s-1} \mathbf{x}^{(s)} = \mathbf{x}^{(1)} \neq 0.$$

In particular case, if $s = 1$, then $(G - \lambda I)\mathbf{x}^{(1)} = 0$ and $\mathbf{x}^{(1)} \neq 0$, which is the definition of an eigenvector.

Since $(G - \lambda I)^i \mathbf{x}^{(i)} = (G - \lambda I)^i (G - \lambda I)^{s-i} \mathbf{x}^{(s)} = 0$ and $(G - \lambda I)^{i-1} \mathbf{x}^{(i)} = (G - \lambda I)^{s-1} \mathbf{x}^{(s)} \neq 0$, then one of these can generate the other generalized eigenvectors as follows, (start from $\mathbf{x}^{(s)}$)

$$\mathbf{x}^{(s-1)} = (G - \lambda I)\mathbf{x}^{(s)}$$

$$\mathbf{x}^{(s-2)} = (G - \lambda I)^2 \mathbf{x}^{(s)} = (G - \lambda I)\mathbf{x}^{(s-1)}$$

$$\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots$$

$$\mathbf{x}^{(2)} = (G - \lambda I)^{s-2} \mathbf{x}^{(s)} = (G - \lambda I)\mathbf{x}^{(3)}$$

$$\mathbf{x}^{(1)} = (G - \lambda I)^{s-1} \mathbf{x}^{(s)} = (G - \lambda I)\mathbf{x}^{(2)}$$

This means that $\mathbf{x}^{(i)}, i = 1, \dots, s$, is a generalized eigenvector of order i of G .

4. Applications

Here we outline the periodic autoregressive models where the inverse of invertible standard multi-companion matrices appear and discuss how the results about such matrices may be useful. Exposition of specific results requires a lot of background information from time series analysis and will be published elsewhere.

We say that the process $\{X_t\}$ is a periodically correlated time series, if

$$\exists d \in Z^+; \exists \mu_t = \mu_{t+d}, \gamma_X(s, t) = \gamma_X(s + d, t + d) \quad \forall s, t \in Z.$$

where $\mu_t = EX_t < \infty$ and $\gamma_X(s, t) = E[(X_s - EX_s)(X_t - EX_t)] < \infty$.

We suppose below for simplicity that $\mu_t = 0$. A periodic autoregressive process is a periodically correlated process which satisfies a stochastic difference equation of the form

$$X_t = \sum_{i=1}^{p_t} \phi_{t,i} X_{t-i} + \epsilon_t \tag{9}$$

where $\{\epsilon_t\}$ is an uncorrelated periodic white noise process and normally distributed terms with mean zero and periodic variances $\sigma_\epsilon^2(t)$ and $\phi_{t,i}$ is the autoregression coefficients.

The usual stationary autoregression model can be obtained from equation (9) by putting $d = 1$. In that case the parameters of the model do not depend on t and the polynomial $1 - \sum \phi_i z^i$ or its companion matrix can be used to study the process, e.g. its spectrum. In the general case, $d > 1$, the polynomials $\phi_t(z) = 1 - \sum \phi_{t,i} z^i$ cannot be used with the same success but a natural generalization exists (see [2]). Let $m = \max(d, p_1, \dots, p_d)$, $Z_t = (X_t, X_{t-1}, \dots, X_{t-m+1})$. Define the companion matrices $A_t = C[\phi_{t,1}, \phi_{t,2}, \dots, \phi_{t,m}]$, $t = 1, \dots, d$, and the matrices $G_t = A_t \cdots A_{t-d+1}$. Then $A_t Z_t = Z_{t-1} + E_t$, $G_t Z_t = Z_{t-d} + U_t$, where E_t and U_t are uncorrelated with Z_{t-1} and Z_{t-d} respectively. Without loss of information we can take every d -th element of the sequence Z_t , e.g. $Y_t = Z_{td}$, $t = \dots, -1, 0, 1, \dots$. The process Y_t is multivariate stationary AR(1),

$$G_d Y_t = Y_{t-1} + U_t. \tag{10}$$

Equation (10) can be used to give full description of the properties of the periodic autoregressive process $\{X_t\}$, for more see [2].

The matrix G is an inverse of invertible standard multi-companion F of order d and its decomposition into a product of companion matrices is $A_d \cdots A_1$ and we use it when the information of F is not enough therefore we can get it from (10) by multiply both sides by G^{-1} to get

$$Y_t = F_d Y_{t-1} + U'_t. \tag{11}$$

Some interesting properties of the periodic autoregression model can be derived using the results from Section 3.

Knowledge of the Jordan form of multi-companion matrices provides a way to generate periodic models with specified properties by constructing the matrix G and then deriving the parameters of the model by factorizing G into a product of companion matrices then get the inverse for each one of them or directly take the inverse of G then do the same way. This can be useful for selection of appropriate models in simulation studies and in estimation of restricted models for the purpose of hypothesis testing (for example, to test periodic integration it is necessary to estimate a restricted model with the zero-hypothesis roots on the unit circle). Conditions of this kind are difficult to handle when working with the parameters $\phi_{t,i}$ themselves, for more see [4] and [5].

Here is an example. Suppose that we wish to generate a 6×6 diagonalizable 4-companion matrix G with the following spectral parameters, maybe with the intention of simulating quarterly time series using the generated model.

	1	2	3	4	5	6
eigenvalue	0.643	0.542	$0.208 - 1.981i$	$0.208 + 1.981i$	$-0.521 - 0.671i$	$-0.521 + 0.671i$
c_1	-0.881	-0.83	$0.253 - 0.085i$	$0.253 + 0.085i$	$-0.264 + 0.486i$	$-0.264 - 0.486i$
c_2	0.382	0.375	$-0.472 - 0.04i$	$-0.472 + 0.04i$	$0.279 - 0.014i$	$0.279 + 0.014i$
c_3	-0.228	-0.156	-0.608	-0.608	$-0.293 - 0.377i$	$-0.293 + 0.377i$
c_4	-0.074	-0.12	$0.314 - 0.307i$	$0.314 + 0.307i$	$0.069 + 0.16i$	$0.069 - 0.16i$

There are four c -parameters for each eigenvalue since the matrix is 4-companion, i.e. $d = 4$. Here is the generated 4-companion matrix,

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0.5 & 0.7 \\ 0 & 1 & 0 & 0 & -0.749 & -2.561 \\ 0 & 0 & 1 & 0 & -1.541 & -4.199 \\ 0 & 0 & 0 & 1 & 1.2 & 2.1 \end{bmatrix}$$

And here are the periodic autoregression coefficients of the corresponding PAR model,

	1	2	3	4	5	6
ϕ_1	1	0	0	0	-0.5	-0.7
ϕ_2	1	0	0	0.749	2.561	0
ϕ_3	1	0	1.541	4.199	0	0
ϕ_4	1	1.2	2.1	0	0	0

As expected, the companion matrices formed from the parameters for each season provide the companion factorization of our 4-companion matrix,

$$\begin{aligned}
 & \begin{bmatrix} 1 & 1.2 & 2.1 & 0 & 0 & 0 \end{bmatrix}_{C_{A_1}} \begin{bmatrix} 1 & 0 & 1.541 & 4.199 & 0 & 0 \end{bmatrix}_{C_{A_2}} \begin{bmatrix} 1 & 0 & 0 & 0.749 & 2.561 & 0 \end{bmatrix}_{C_{A_3}} \\
 & \times \begin{bmatrix} 1 & 0 & 0 & 0 & -0.5 & -0.7 \end{bmatrix}_{C_{A_4}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0.5 & 0.7 \\ 0 & 1 & 0 & 0 & -0.749 & -2.561 \\ 0 & 0 & 1 & 0 & -1.541 & -4.199 \\ 0 & 0 & 0 & 1 & 1.2 & 2.1 \end{bmatrix}
 \end{aligned}$$

Periodic autoregression moving average models may be generated by applying the above procedure separately to the generation of the autoregression and moving average parts.

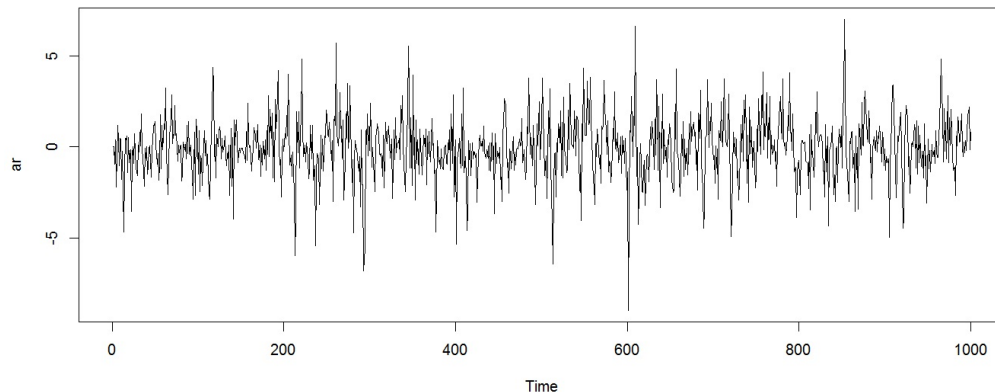


Figure 1: Plot of observed of simulated PAR model for $n = 1000$ with $\phi_{1,i}, \phi_{2,i}, \phi_{3,i}$ and $\phi_{4,i}$ which generation of reversed synthetic river flow data that is important in planning, design and operation of water resources systems.

5. Conclusion

We found explicit expressions for the generalized eigenvectors of the inverse of invertible standard multi-companion matrices such that each generalized eigenvector depends on the corresponding eigenvalue. We will discussed some properties such the other matrices, as the factorization of matrices.

Moreover, we obtained a parametrization of the inverse of invertible standard multi-companion matrix through the eigenvalues and these additional quantities. The number of parameters in this parametrization is equal to the number of non-trivial elements of the inverse of invertible standard multi-companion matrix. The results can be applied to statistical estimation, simulation and theoretical studies of periodically correlated and multivariate time series in both discrete- and continuous-time series.

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