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Research Paper

Asymptotic behavior of oscillatory solutions of first order functional delay difference equations

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Abstract

In this paper, we study the asymptotic behavior of oscillatory solutions of the first order functional delay difference equation

$$\Delta x(n) = f(n, x(n-\tau)), \quad n \geq n_0. \tag{$*$}$$

A new sufficient condition is established under which every oscillatory solution of (*) tends to zero asymptotically.

Keywords: *Asymptotic behavior, delay difference equation, oscillatory solution.*

1. Introduction

In this paper, we consider the following first order functional delay difference equation of the form

$$\Delta x(n) = f(n, x(n-\tau)), \quad n \in N(n_0) \tag{1}$$

where Δ is the forward difference operator given by $\Delta x(n) = x(n+1) - x(n)$, τ is a positive integer, n_0 is a fixed integer, $N(n_0) = \{n_0, n_0+1, n_0+2, ...\}$, $f : N(n_0) \times R \to R$ is a real valued function and for any $n \in N(n_0)$, $f(n, .)$ is a continuous function with the following properties:

(H_1) $f(n, 0) = 0$;

(H_2) $uf(n, u) > 0$ for $u \neq 0$; and

(H_3) there exists a sequence $\{q(n)\}$ of positive real numbers defined on $N(n_0)$ such that

$$|f(n, u)| \leq q(n)\,|u|\,.$$

Qualitative theory of discrete processes has drawn considerable attention in recent years. In particular, oscillation properties of discrete analogs of delay differential equations have been studied recently by a number of authors (see e.g., [6,7,10,11]). On the other hand, relatively little is known about the asymptotic behavior of all solutions of these discrete equations, see for example [3,8,12], and the references cited therein. For the general background of difference equations, one can refer to [1,2,5,9].

In [3], Chen et al. obtained sufficient conditions which ensure that all solutions of the first order nonlinear delay difference equation

$$\Delta x(n) + F(n, x(n-k)) = 0, \quad n \geq n_0 \tag{2}$$

tend to zero as $n \to \infty$.

In [8], Liu et al. established sufficient conditions under which every solutions of the equation

$$\Delta x(n) = p(n)f(x(n-k)) + r(n), \quad n = 0, 1, 2, ... \tag{3}$$

converges to zero. The asymptotic behavior of the solutions of the equation

$$\Delta x(n) + p(n)x(n-\tau) = 0, \quad n = 0, 1, 2, ... \tag{4}$$

has been extensively investigated in the literature, see for example, [4,12,13].

The purpose of this paper is to give a new sufficient condition under which every oscillatory solution of (1) tends to zero as $n \to \infty$. By a solution of (1), we mean a nontrivial real sequence $\{x(n)\}$ which is defined on $N(n_0 - \tau) = \{n_0 - \tau, n_0 - \tau + 1, ...\}$ and which satisfies (1) for $n \in N(n_0)$. A solution $\{x(n)\}$ of (1) on $N(n_0)$ is said to be oscillatory if for every positive integer $N_0 > n_0$, there exists $n \geq N_0$ such that $x(n)x(n+1) \leq 0$, otherwise $\{x(n)\}$ is said to be nonoscillatory.

Throughout this paper we use the following notations:

For any $a, b \in N$, define

$$N(a) = \{a, a+1, a+2, ...\},$$

$$N(a, b) = \{a, a+1, a+2, ..., b\},$$

$$Q(N) := \sup_{n \geq N} \sum_{s=n-\tau}^{n} q(s), \quad for \quad n \geq n_0 + \tau$$

and

$$Q_\infty := \lim_{N\to\infty} Q(N) = \limsup_{n\to\infty} \sum_{n-\tau}^{n} q(s).$$

2. Main Results

Lemma 2.1 *Let $\{x(n)\}$ be a solution of (1) and $n_0 + \tau < n_1 < n_2 - 1$. If $x(n) > 0$ for all $n \in N(n_1 + 1, n_2 - 1)$ or $x(n) < 0$ for all $n \in N(n_1 + 1, n_2 - 1)$ and $x(n_2)x(n) \leq 0$ for all $n \in N(n_1 + 1, n_2 - 1)$, then $n_1 \geq n_2 - \tau$.*

Proof. Assume the contrary, that is, $n_1 < n_2 - \tau$. Without loss of generality, we may suppose that $x(n) > 0$ for $n \in N(n_1 + 1, n_2 - 1)$. Then $x(n_2) \leq 0$ and there exists an integer n^* satisfying $n_1 \leq n^* - \tau < n^* < n_2$. Then

$$\Delta x(n) = f(n, x(n-\tau)) > 0$$

for $n \in N(n^*, n_2)$, which implies $x(n_2) > x(n^*) > 0$. This is a contradiction.

The proof is complete.

Lemma 2.2 *Given $\delta > 0$, there exists an increasing sequence $\{h(n)\}$ of nonnegative real numbers such that*

$$h(n) - h(n-\tau) = \frac{\delta}{2}, \quad n \geq n_0 + \tau. \tag{5}$$

Proof. Choose a sequence $\{N_k\}$ of integers such that $N_0 = n_0$ and for $k = 0, 1, 2, ..., N_{k+1} = N_k + \tau$. Then $\lim_{k\to\infty} N_k = \infty$. Let us define

$$h(n) = \frac{\delta}{2}\left(\frac{n - N_k}{\tau} + k\right), \quad for \quad n \in N\left(N_k, N_{k+1} - 1\right)$$

for $k = 0, 1, 2, ...$ We see that $h(N_k) = \frac{k\delta}{2}$ for all k and $\{h(n)\}$ is an increasing sequence on $N(n_0)$. For any $n \in N\left(N_k, N_{k+1} - 1\right)$, $k = 1, 2, 3, ...,$

$$h(n) < h(N_{k+1}) \quad and \quad h(n - \tau) \geq h(N_k - \tau),$$

which implies

$$h(n) - h(n - \tau) = \frac{\delta}{2}\left(\frac{n - N_k}{\tau} + k\right) - \frac{\delta}{2}\left(\frac{n - \tau - N_{k-1}}{\tau} + k - 1\right) = \frac{\delta}{2}.$$

Therefore (5) holds for $n \geq N_1$.

Theorem 2.3 *Let* $\{h(n)\}$ *be an increasing sequence of positive real numbers satisfying (5) for some* $\delta > 0$. *If* $Q(N) \leq \frac{\sqrt{11}-1}{2}$ *for some* $N \geq n_0+\tau$, *then for any oscillatory solution* $\{x(n)\}$ *of (1), there exists a* $K = K(\beta, h, x) > 0$ *such that*

$$|x(n)| < K\bar{e}^{\beta h(n)}, \quad n \in N(n_0), \tag{6}$$

where

$$\left.\begin{array}{ll} \beta = \dfrac{2}{3\delta} log \dfrac{4}{(Q(N) + 1)^2 - 1}, & Q(N) < 1 \\ \beta = \dfrac{2}{3\delta} log \dfrac{2}{(Q(N) + \frac{1}{2})^2 - \frac{3}{4}}, & 1 \leq Q(N) \leq \dfrac{\sqrt{11} - 1}{2}. \end{array}\right\} \tag{7}$$

Proof. Since $\{x(n)\}$ is an oscillatory solution of (1), there exists a sufficiently large $n^* > N_0+\tau$ such that $x(n^*) \leq 0$. We will show that (6) holds for a positive constant K such that

$$K > \max_{n_0 \leq n \leq n^*} e^{\beta h(n)} |x(n)|.$$

Assume that (6) does not hold. Then there exists an integer $\xi > n^*$ such that

$$|x(n)| < K\bar{e}^{\beta h(n)} \quad for \quad n \in N(n_0, \xi - 1) \quad and \quad |x(\xi)| \geq K\bar{e}^{\beta h(\xi)}. \tag{8}$$

Then $x(\xi) \neq 0$. Since $\{x(n)\}$ is oscillatory and $\xi > n^*$, we can define two integers n_1, $n_2 \in N(n_0)$ by

$$n_1 = \sup\{n : n < \xi, \quad x(n)x(\xi) \leq 0\}$$

and

$$n_2 = \inf\{n : n > \xi, \quad x(n)x(\xi) \leq 0\}.$$

We see that $n^* \leq n_1 < \xi < n_2$ and $x(n) > 0$ for all $n \in N(n_1 + 1, n_2 - 1)$ or $x(n) < 0$ for all $n \in N(n_1 + 1, n_2 - 1)$. Also, we have $x(n_1)x(n) \leq 0$ for all $n \in N(n_1 + 1, n_2 - 1)$ and $x(n_2)x(n) \leq 0$ for all $n \in N(n_1 + 1, n_2 - 1)$. Lemma 2.1 leads to $n_1 \geq n_2 - \tau$ and hence $n_1 \geq u - \tau$ for $u \in N(n_1, n_2)$. Then

$$\begin{aligned} |x(u - \tau)| &= |x(u - \tau) - x(n_1) + x(n_1)| \\ &\leq |x(n_1) - x(u - \tau)| + |x(n_1)| \\ &= \left|\sum_{s=u-\tau}^{n_1-1} \Delta x(s)\right| + |x(n_1)| \\ &= \left|\sum_{s=u-\tau}^{n_1-1} f(s, x(s - \tau))\right| + |x(n_1)| \end{aligned}$$

which implies that for $u \in N(n_1, n_2)$,

$$|x(u-\tau)| \leq \sum_{s=u-\tau}^{n_1-1} q(s)\,|x(s-\tau)| + |x(n_1)|\,. \tag{9}$$

Moreover, because of $x(n_1)x(\xi) \leq 0$, we have

$$\begin{aligned} |x(\xi)| &\leq |x(\xi) - x(n_1)| \\ &= \left|\sum_{u=n_1}^{\xi-1} \Delta x(u)\right| \\ &= \left|\sum_{u=n_1}^{\xi-1} f(u, x(u-\tau))\right| \\ &\leq \sum_{u=n_1}^{\xi-1} q(u)\,|x(u-\tau)|\,, \end{aligned}$$

and

$$\begin{aligned} |x(\xi)| &\leq |x(n_2) - x(\xi)| \\ &= \left|\sum_{u=\xi}^{n_2-1} \Delta x(u)\right| \\ &\leq \left|\sum_{u=\xi}^{n_2-1} f(u, x(u-\tau))\right| \\ &\leq \sum_{u=\xi}^{n_2-1} q(u)\,|x(u-\tau)|\,. \end{aligned}$$

Thus we obtain

$$|x(\xi)| \leq \frac{1}{2}\sum_{u=n_1}^{n_2-1} q(u)\,|x(u-\tau)|\,. \tag{10}$$

Here we consider two cases.

Case 1: $Q(N) < 1$. We note that $\beta > 0$. By (9) and (10),

$$|x(\xi)| \leq \frac{1}{2}\sum_{u=n_1}^{n_2-1} q(u)\left(\sum_{s=u-\tau}^{n_1-1} q(s)\,|x(s-\tau)| + |x(n_1)|\right)$$

or

$$|x(\xi)| \leq \frac{1}{2}\sum_{u=n_1}^{n_2-1} q(u)\sum_{s=u-\tau}^{n_1-1} q(s)\,|x(s-\tau)| + \frac{1}{2}\,|x(n_1)|\sum_{u=n_1}^{n_2-1} q(u) \tag{11}$$

By (8), we see that $|x(s-\tau)| < K\bar{e}^{\beta h(s-\tau)}$ for $s \in N(n_0+\tau, n_1)$. Then by (11),

$$|x(\xi)| < \frac{1}{2}\sum_{u=n_1}^{n_2-1} q(u)\sum_{s=u-\tau}^{n_1-1} q(s)K\bar{e}^{\beta h(s-\tau)} + \frac{1}{2}\,|x(n_1)|\sum_{u=n_1}^{n_2-1} q(u).$$

Since $\{h(n)\}$ is increasing and $\beta > 0$,

$$|x(\xi)| < \frac{K}{2}\sum_{u=n_1}^{n_2-1} q(u)\bar{e}^{\beta h(u-2\tau)}\sum_{s=u-\tau}^{n_1-1} q(s) + \frac{1}{2}\,|x(n_1)|\sum_{u=n_1}^{n_2-1} q(u)$$

$$\leq \frac{K}{2}\bar{e}^{\beta h(n_1-2\tau)}\sum_{u=n_1}^{n_2-1}q(u)\sum_{s=u-\tau}^{n_1-1}q(s)+\frac{1}{2}|x(n_1)|\sum_{u=n_1}^{n_2-1}q(u).$$

Then

$$|x(\xi)|<\frac{K}{2}\bar{e}^{\beta h(n_1-2\tau)}\left\{\sum_{u=n_1}^{n_2-1}q(u)\sum_{s=u-\tau}^{u}q(s)-\sum_{u=n_1}^{n_2-1}q(u)\sum_{s=n_1}^{u}q(s)\right\}+|x(n_1)|\sum_{u=n_1}^{n_2-1}q(u).$$

Since

$$\sum_{u=n_1}^{n_2-1}q(u)\sum_{s=n_1}^{u}q(s)\geq\frac{1}{2}\sum_{u=n_1}^{n_2-1}q(u)\sum_{s=n_1}^{n_2-1}q(s)=\frac{1}{2}\left(\sum_{u=n_1}^{n_2-1}q(u)\right)^2, \tag{12}$$

$$|x(\xi)|<\frac{K}{2}\bar{e}^{\beta h(n_1-2\tau)}\left\{Q(N)\sum_{u=n_1}^{n_2-1}q(u)-\frac{1}{2}\left(\sum_{u=n_1}^{n_2-1}q(u)\right)^2\right\}+\frac{1}{2}|x(n_1)|\,Q(N). \tag{13}$$

The right side of (13) is a quadratic function of

$$\sum_{u=n_1}^{n_2-1}q(u)\quad and\quad 0<\sum_{u=n_1}^{n_2-1}q(u)\leq\sum_{u=n_2-\tau}^{n_2-1}q(u)\leq Q(N).$$

Then

$$|x(\xi)|<\frac{K}{2}\bar{e}^{\beta h(n_1-2\tau)}\left\{Q^2(N)-\frac{Q^2(N)}{2}\right\}+\frac{K}{2}\bar{e}^{\beta h(n_1-2\tau)}Q(N)$$

$$=\frac{K}{2}\bar{e}^{\beta h(n_1-2\tau)}\left\{\frac{Q^2(N)}{2}+Q(N)\right\}$$

$$=\frac{K}{4}\bar{e}^{\beta h(n_1-2\tau)}\left\{Q^2(N)+2Q(N)\right\}$$

$$\leq\frac{K}{4}\bar{e}^{\beta h(\xi-3\tau)}\left\{(Q(N)+1)^2-1\right\}$$

$$\leq\frac{K}{4}\left\{(Q(N)+1)^2-1\right\}\bar{e}^{\beta h(\xi-3\tau)}$$

$$=\frac{K}{4}\left\{(Q(N)+1)^2-1\right\}e^{\beta(h(\xi)-h(\xi-3\tau))}e^{-\beta h(\xi)}.$$

By Lemma 2.2, $h(\xi)-h(\xi-3\tau)=\frac{3\delta}{2}$.
Then, we have

$$|x(\xi)|<\frac{1}{4}\left\{(Q(N)+1)^2-1\right\}e^{\frac{3\delta\beta}{2}}\left(Ke^{-\beta h(\xi)}\right).$$

Thus, (7) implies $|x(\xi)|<K\bar{e}^{\beta h(\xi)}$.
Then we have a contradiction to the assumption that $|x(\xi)|\geq K\bar{e}^{\beta h(\xi)}$.

Case 2: $1\leq Q(N)\leq\frac{\sqrt{11}-1}{2}$. We note that $\beta\geq 0$. There are two possibilities.

Case 2.1: $1\leq\sum_{n=n_1}^{n_2-1}q(n)\leq\frac{\sqrt{11}-1}{2}$. There exists an integer η such that $n_1\leq\eta\leq n_2-1$ and $\sum_{n=\eta}^{n_2-1}q(n)\geq 1$. By (9) and (10), we have

$$|x(\xi)|\leq\frac{1}{2}\sum_{u=n_1}^{n_2-1}q(u)\,|x(u-\tau)|$$

$$\le \frac{1}{2}\sum_{u=n_1}^{\eta-1} q(u)\,|x(u-\tau)| + \frac{1}{2}\sum_{u=\eta}^{n_2-1} q(u)\,|x(u-\tau)|$$

$$\le \frac{1}{2}\sum_{u=n_1}^{\eta-1} q(u)\,|x(u-\tau)| + \frac{1}{2}\sum_{u=\eta}^{n_2-1} q(u)\left\{\sum_{s=u-\tau}^{n_1-1} q(s)\,|x(s-\tau)| + |x(n_1)|\right\}$$

$$= \frac{1}{2}\sum_{u=n_1}^{\eta-1} q(u)\,|x(u-\tau)| + \frac{1}{2}\sum_{u=\eta}^{n_2-1} q(u)\sum_{s=u-\tau}^{n_1-1} q(s)\,|x(s-\tau)| + \frac{1}{2}\sum_{u=\eta}^{n_2-1} q(u)\,|x(n_1)|$$

$$= \frac{1}{2}\sum_{u=n_1}^{\eta-1} q(u)\,|x(u-\tau)| + \frac{1}{2}\sum_{u=\eta}^{n_2-1} q(u)\left[\sum_{s=u-\tau}^{\eta-1} q(s)\,|x(s-\tau)| - \sum_{s=n_1}^{\eta-1} q(s)\,|x(s-\tau)|\right] + \frac{1}{2}\sum_{u=\eta}^{n_2-1} q(u)\,|x(n_1)|$$

$$= \frac{1}{2}\sum_{u=n_1}^{\eta-1} q(u)\,|x(u-\tau)| + \frac{1}{2}\sum_{u=\eta}^{n_2-1} q(u)\sum_{s=u-\tau}^{\eta-1} q(s)\,|x(s-\tau)| - \frac{1}{2}\sum_{u=\eta}^{n_2-1} q(u)\sum_{s=n_1}^{\eta-1} q(s)\,|x(s-\tau)| + \frac{1}{2}\sum_{u=\eta}^{n_2-1} q(u)\,|x(n_1)|.$$

Since $\sum_{n=\eta}^{n_2-1} q(n) \ge 1$,

$$\sum_{u=\eta}^{n_2-1} q(u)\sum_{s=n_1}^{\eta-1} q(s)\,|x(s-\tau)| = \left(\sum_{u=\eta}^{n_2-1} q(u)\right)\left(\sum_{s=n_1}^{\eta-1} q(s)\,|x(s-\tau)|\right) \ge \sum_{s=n_1}^{\eta-1} q(s)\,|x(s-\tau)|$$

which implies

$$|x(\xi)| \le \frac{1}{2}\sum_{u=\eta}^{n_2-1} q(u)\sum_{s=u-\tau}^{\eta-1} q(s)\,|x(s-\tau)| + \frac{1}{2}\sum_{u=\eta}^{n_2-1} q(u)\,|x(n_1)|. \tag{14}$$

By (8) and the fact that $s-\tau \le n_1 < \xi$ for $s \in N(n_0+\tau, n_2)$, $|x(s-\tau)| < Ke^{-\beta h(s-\tau)}$ for $s \in N(n_0+\tau, \eta)$. Then by (14),

$$|x(\xi)| < \frac{1}{2}\sum_{u=\eta}^{n_2-1} q(u)\sum_{s=u-\tau}^{\eta-1} q(s)K\bar{e}^{\beta h(s-\tau)} + \frac{K}{2}e^{-\beta h(n_1-\tau)}\sum_{u=\eta}^{n_2-1} q(u)$$

$$\le \frac{K}{2}\sum_{u=\eta}^{n_2-1} q(u)\bar{e}^{\beta(u-2\tau)}\sum_{s=u-\tau}^{\eta-1} q(s) + \frac{K}{2}e^{-\beta h(n_1-\tau)}\sum_{s=u-\tau}^{\eta-1} q(s)$$

$$\le \frac{K}{2}\bar{e}^{\beta h(n_1-2\tau)}\left\{\sum_{u=\eta}^{n_2-1} q(u)\sum_{s=u-\tau}^{\eta-1} q(s) + \sum_{s=u-\tau}^{\eta-1} q(s)\right\}$$

$$\le \frac{K}{2}\bar{e}^{\beta h(n_1-2\tau)}\left\{\sum_{u=\eta}^{n_2-1} q(u)\sum_{s=u-\tau}^{\eta-1} q(s) + Q(N)\right\}$$

$$= \frac{K}{2}\bar{e}^{\beta h(n_1-2\tau)}\left\{\sum_{u=\eta}^{n_2-1} q(u)\sum_{s=u-\tau}^{u} q(s) - \sum_{u=\eta}^{n_2-1} q(u)\sum_{s=\eta}^{u} q(s) + Q(N)\right\}.$$

Since

$$\sum_{u=\eta}^{n_2-1} q(u)\sum_{s=\eta}^{u} q(s) \ge \frac{1}{2}\left(\sum_{u=\eta}^{n_2-1} q(u)\right)^2,$$

$$|x(\xi)| < \frac{K}{2}\bar{e}^{\beta h(n_1-2\tau)}\left\{Q(N)\sum_{u=\eta}^{n_2-1} q(u) - \frac{1}{2}\left(\sum_{u=\eta}^{n_2-1} q(u)\right)^2 + Q(N)\right\}$$

$$\le \frac{K}{2}\bar{e}^{\beta h(n_1-2\tau)}\left\{Q^2(N) - \frac{1}{2} + Q(N)\right\}$$

$$\le \frac{K}{2}\bar{e}^{\beta h(n_1-2\tau)}\left\{\left(Q(N)+\frac{1}{2}\right)^2-\frac{3}{4}\right\}$$

$$\le \frac{K}{2}\bar{e}^{\beta h(\xi-3\tau)}\left\{\left(Q(N)+\frac{1}{2}\right)^2-\frac{3}{4}\right\}$$

$$= \frac{K}{2}e^{\beta(h(\xi)-h(\xi-3\tau))}e^{-\beta h(\xi)}\left\{\left(Q(N)+\frac{1}{2}\right)^2-\frac{3}{4}\right\}$$

$$= \frac{K}{2}e^{\frac{3\delta\beta}{2}}e^{-\beta h(\xi)}\left\{\left(Q(N)+\frac{1}{2}\right)^2-\frac{3}{4}\right\}.$$

Thus, (7) implies $|x(\xi)| < K\bar{e}^{\beta h(\xi)}$. Then we have a contradiction to the assumption that $|x(\xi)| \ge K\bar{e}^{\beta h(\xi)}$.

Case 2.2: $\sum_{n=n_1}^{n_2-1} q(n) < 1$. In the same way as a Case 1, we have

$$|x(\xi)| < \frac{K}{2}\bar{e}^{\beta h(n_1-2\tau)}\left\{Q(N)\sum_{u=n_1}^{n_2-1} q(u) - \frac{1}{2}\left(\sum_{u=n_1}^{n_2-1} q(u)\right)^2 + Q(N)\right\}.$$

Since $Q(N)\sum_{u=n_1}^{n_2-1} q(u) - \frac{1}{2}\left(\sum_{u=n_1}^{n_2-1} q(u)\right)^2$ is a quadratic function of $\sum_{u=n_1}^{n_2-1} q(u)$ and $0 < \sum_{u=n_1}^{n_2-1} q(u) < 1 \le Q(N)$, we have,

$$|x(\xi)| < \frac{K}{2}\bar{e}^{\beta h(n_1-2\tau)}\left\{Q(N).1 - \frac{1}{2}.1^2 + Q(N)\right\}$$

$$\le \frac{K}{2}\bar{e}^{\beta h(n_1-2\tau)}\left\{Q^2(N)+Q(N)-\frac{1}{2}\right\}$$

$$\le \frac{K}{2}\bar{e}^{\beta h(\xi-3\tau)}\left\{\left(Q(N)+\frac{1}{2}\right)^2-\frac{3}{4}\right\}$$

$$= \frac{K}{2}e^{\beta(h(\xi)-h(\xi-3\tau))}e^{-\beta h(\xi)}\left\{\left(Q(N)+\frac{1}{2}\right)^2-\frac{3}{4}\right\}$$

$$= \frac{K}{2}e^{\frac{3\delta\beta}{2}}e^{-\beta h(\xi)}\left\{\left(Q(N)+\frac{1}{2}\right)^2-\frac{3}{4}\right\}.$$

By (7), $|x(\xi)| < K\bar{e}^{\beta h(\xi)}$. Then we have a contradiction to the assumption that $|x(\xi)| \ge K\bar{e}^{\beta h(\xi)}$. Hence, by virtue of the Case 1 and 2, we obtain (6). The proof is complete.

Using Theorem 2.3, we have two corollaries.

Corollary 2.4 *If* $Q(N) \le \frac{\sqrt{11}-1}{2}$ *for some* $N \ge n_0+\tau$, *then every oscillatory solution of (1) is bounded.*

Corollary 2.5 *If* $Q_\infty < \frac{\sqrt{11}-1}{2}$, *then every oscillatory solution of (1) tends to zero as* $n \to \infty$.

In the case where $\frac{\sqrt{11}-1}{2} < Q(N) < \infty$ for some $N \ge n_0+\tau$ we can prove the following theorem in the same way as case 1 in the proof of Theorem 2.3:

Theorem 2.6 *If* $\{h(n)\}$ *is an increasing sequence of positive real numbers defined on* $N(n_0)$ *and* $Q(N) > \frac{\sqrt{11}-1}{2}$ *for some* $N \ge n_0+\tau$, *then for any oscillatory solution* $\{x(n)\}$ *of (1), there exists a constant* $K = K(h,x) > 0$ *such that*

$$|x(n)| < Ke^{h(n)}, \quad n \ge n_0.$$

Lemma 2.7 *Let $f(n, x(n-\tau) = q(n)x(n-\tau)$, where $\{q(n)\}$ is a sequence of nonnegative and real numbers defined on $N(n_0)$, $\{x(n)\}$ be a solution of (1) and $Q_\infty < \infty$. If $\{x(n)\}$ is not oscillatory, then there exist a $N_0 = N_0(x) > n_0$ and a positive constant $C = C(x)$ such that*

$$|x(n)| \geq C \exp\left\{\frac{1}{Q_\infty + 2}\sum_{s=n_0}^{n-1} q(s)\right\}, \quad n > N_0.$$

Proof. Without loss of generality, we may assume that $\{x(n)\}$ is eventually positive, i.e., there exists $n_1 > n_0$ such that $x(n) > 0$ for any $n > n_1$. Choose $N_0 > n_0$, such that $N_0 - 3\tau > n_1$ and $\sum_{s=n-\tau}^{n} q(s) < Q_\infty + 1, \quad n > N_0$. By (1) and $x(n) \neq 0$ for $n > n_1$, we have

$$x(n) \geq x(N_0) \exp\left\{\sum_{s=N_0}^{n-1} q(s)\frac{x(s-\tau)}{x(s+1)}\right\}, \quad n > N_0.$$

Since $\Delta x(n) \geq 0$ for $n > n_1 + \tau$, we have

$$\begin{aligned} x(n+1) - x(n-\tau) &= \sum_{s=n-\tau}^{n} \Delta x(s) \\ &= \sum_{s=n-\tau}^{n} q(s)x(s-\tau) \\ &\leq x(n-\tau) \sum_{s=n-\tau}^{n} q(s) \\ &\leq x(n-\tau)(Q_\infty + 1), \quad n > N_0. \end{aligned}$$

Then

$$\frac{x(n-\tau)}{x(n+1)} \geq \frac{1}{Q_\infty + 2},$$

which leads to that

$$\begin{aligned} x(n) &\geq x(N_0) \exp\left\{\frac{1}{Q_\infty + 2}\sum_{s=N_0}^{n-1} q(s)\right\} \\ &= x(N_0) \exp\left\{\frac{-1}{Q_\infty + 2}\sum_{s=n_0}^{N_0-1} q(s)\right\} \exp\left\{\frac{1}{Q_\infty + 2}\sum_{s=n_0}^{n-1} q(s)\right\}, \quad n > N_0. \end{aligned}$$

The proof is complete. By this lemma we obtain the following theorem:

Theorem 2.8 *Assume that $Q_\infty < \frac{\sqrt{11}-1}{2}$.*

(*i*) *If a solution $\{x(n)\}$ of (1) satisfies*

$$\lim_{n\to\infty} \frac{x(n)}{\exp\left\{\frac{1}{Q_\infty+2}\sum_{s=n_0}^{n-1} q(s)\right\}} = 0, \tag{15}$$

then $x(n)$ tends to zero as $n \to \infty$.

(*ii*) *If*

$$\sum_{s=n_0}^{\infty} q(s) = \infty, \tag{16}$$

then every bounded solution of (1) tends to zero as $n \to \infty$.

Proof. (i) By Lemma 2.7 and (15), $\{x(n)\}$ is oscillatory. Therefore, by Corollary 2.5, $x(n)$ tends to zero as $n \to \infty$.
(ii) Let $\{x(n)\}$ be a bounded solution of (1). By (16), $\{x(n)\}$ satisfies (15).

Hence $x(n)$ tends to zero as $n \to \infty$.

The proof is complete.

3. Equations with special forcing term

Let A_0 be the set of all real sequences $\{a(n)\}$ defined on $N(n_0 - \tau)$ such that

$$\lim_{n\to\infty} a(n) = 0.$$

Consider the following equation:

$$\Delta x(n) = q(n)x(n-\tau) + r(n), \quad n \geq n_0, \tag{17}$$

where the sequence $\{r(n)\}$ is given by $r(n) = q(n)a(n-\tau) - \Delta a(n)$ with some $\{a(n)\} \in A_0$. We compare the asymptotic behavior of the oscillatory solution of (17) with that of the equation (1).

Lemma 3.1 *Let $\{y(n)\}$ be a sequence of real numbers defined on $N(n_0)$ and $\{a(n)\} \in A_0$. If $z(n) = y(n) + a(n)$ is a solution of (1) and $\{y(n)\}$ is oscillatory, then $\{z(n)\}$ is also oscillatory.*

Proof. Assume that $\{z(n)\}$ is not oscillatory. Then there exists $n_1 > n_0 + \tau$ such that $|z(n)| > 0$ for $n > n_1$. Without loss of generality, we may assume that $z(n) > 0$. Since $\{z(n)\}$ is a solution of (1) and $q(n) \geq 0$, $\Delta z(n) = q(n)z(n-\tau) \geq 0$ for $n > n_2$ for some $n_2 > n_1 + \tau$. Then $z(n) \geq z(n_2) > 0$ for $n > n_2$. Since $\{a(n)\}$ tends to zero as $n \to \infty$, we have

$$\liminf_{n\to\infty} y(n) = \liminf_{n\to\infty} \{z(n) - a(n)\} \geq z(n_2) > 0.$$

This is a contradiction to the assumption that $\{y(n)\}$ is oscillatory. The proof is complete.

Theorem 3.2 *If every oscillatory solution of (1) tends to zero as $n \to \infty$, then every oscillatory solution of (17) tends to zero as $n \to \infty$.*

Proof. Let $\{x(n)\}$ be an oscillatory solution of (17). Then it follows that

$$\Delta x(n) = q(n)x(n-\tau) + q(n)a(n-\tau) - \Delta a(n),$$

which implies that

$$\Delta(x(n) + a(n)) = q(n)(x(n-\tau) + a(n-\tau)).$$

Set

$$z(n) = x(n) + a(n).$$

Then $\{z(n)\}$ is a solution of (1), we see from Lemma 3.1 that $\{z(n)\}$ is oscillatory. Therefore $z(n)$ tends to zero as $n \to \infty$ by assumption. Hence $x(n) = z(n) - a(n)$ tends to zero as $n \to \infty$.

The proof is complete.

Corollary 3.3 *If $Q_\infty < \frac{\sqrt{11}-1}{2}$, then every oscillatory solution of (17) tends to zero as $n \to \infty$.*

Proof. By Corollary 2.5, every oscillatory solution of (1) tends to zero as $n \to \infty$. By Theorem 3.2, we obtain that every oscillatory solution of (17) tends to zero as $n \to \infty$.

The proof is complete. Now, let us consider the equations:

$$\Delta x(n) = qx(n-\tau) + \mu\lambda^{-n}, \quad n \geq n_0 \tag{18}$$

and

$$\Delta x(n) = qx(n-\tau) \tag{19}$$

where μ is a constant, τ is a positive integer and q, λ are positive real numbers with $\lambda > 1$.

Theorem 3.4 *Every oscillatory solution of (18) tends to zero as $n \to \infty$, if and only if every oscillatory solution of (19) tends to zero as $n \to \infty$.*

Proof. Sufficiency. Suppose that every oscillatory solution of (19) tends to zero as $n \to \infty$. Define a sequence $\{a(n)\}$ on $N(n_0 - \tau)$ by

$$a(n) = \frac{\mu\lambda^{-n}}{1 - \frac{1}{\lambda} + q\lambda^{\tau}}.$$

Then we see that $\{a(n)\} \in A_0$ and $\mu\lambda^{-n} = qa(n-\tau) - \Delta a(n)$. By Theorem 3.2 every oscillatory solution of (18) tends to zero as $n \to \infty$.

Necessity. Suppose that every oscillatory solution of (18) tends to zero as $n \to \infty$ for some $\mu \neq 0$. Let $\{y(n)\}$ be an oscillatory solution of (19) and let $z(n) = y(n) - a(n)$. Then

$$\Delta z(n) = qz(n-\tau) + qa(n-\tau) - \Delta a(n)$$

$$= qz(n-\tau) + \mu\lambda^{-n},$$

which means that $\{z(n)\}$ is a solution of (18), We will prove that $z(n)$ tends to zero as $n \to \infty$. If $\{z(n)\}$ is oscillatory, then $z(n)$ tends to zero as $n \to \infty$ by assumption. Therefore it is enough to consider the case that $\{z(n)\}$ is not oscillatory. We will show that for some $N^* > n_0$,

$$0 < |z(n)| \leq \frac{|\mu|}{1 - \frac{1}{\lambda}}\lambda^{-n}, \quad n > N^*. \tag{20}$$

Let $\mu > 0$. Assume that $z(n) > 0$ for $n > N_1$ for some $N_1 > n_0$, Since $qz(n-\tau) > 0$ for $n > N_1 + \tau$, $\Delta z(n) > 0$ for $n > N_1 + \tau$. Then $\{z(n)\}$ is monotonic increasing as $N(N_1 + \tau)$, which implies that for $N_2 > N_1 + \tau$,

$$\liminf_{n\to\infty} y(n) \geq \liminf_{n\to\infty} z(n) + \liminf_{n\to\infty} a(n)$$

$$\geq z(N_2) > 0.$$

Since $\{y(n)\}$ is oscillatory, we have a contradiction. Hence $z(n) < 0$ for $n > N_3$ for some $N_3 > n_0$. Then

$$\Delta\left(z(n) + \frac{\mu\lambda^{-n}}{1 - \frac{1}{\lambda}}\right) = qz(n-\tau) < 0, \quad n > N_3 + \tau,$$

which implies that $\left\{z(n) + \frac{\mu\lambda^{-n}}{1-\frac{1}{\lambda}}\right\}$ is monotone decreasing on $N(N_3 + \tau)$. Then we have that for any $m > N_3 + \tau$,

$$\limsup_{n\to\infty} y(n) = \limsup_{n\to\infty}(z(n) + a(n))$$

$$\leq \left(\limsup_{n\to\infty} z(n) + \frac{\mu\lambda^{-n}}{1 - \frac{1}{\lambda}}\right) + \limsup_{n\to\infty}\left(a(n) - \frac{\mu\lambda^{-n}}{1 - \frac{1}{\lambda}}\right)$$

$$\leq z(m) + \frac{\mu\lambda^{-m}}{1 - \frac{1}{\lambda}}.$$

Since $\limsup_{n\to\infty} y(n) \geq 0$, $z(m) + \frac{\mu\lambda^{-m}}{1-\frac{1}{\lambda}} \geq 0$ for $m > N_3 + \tau$. Therefore $0 > z(n) \geq \frac{-\mu\lambda^{-m}}{1-\frac{1}{\lambda}}$, $n > N_3 + \tau$. In case $\mu < 0$, we see in the same way that $0 < z(n) \leq \frac{-\mu\lambda^{-m}}{1-\frac{1}{\lambda}}$, $n > N_4$, for some $N_4 > n_0$. Then we have (20), and hence $z(n)$ tends to zero as $n \to \infty$ when $z(n)$ is not oscillatory. Therefore $y(n) = z(n) + a(n)$ tends to zero as $n \to \infty$, which implies that every oscillatory solution of (19) tends to zero as $n \to \infty$.

The proof is complete.

References

[1] R. P. Agarwal, Advanced Topics in Difference Equations, Kluwer Academic Publishers Inc., 1997.

[2] R. P. Agarwal, Difference Equations and Inequalities: Theory, Methods, and Applications, Marcel Dekker, Inc., New York, 1999.

[3] M. P. Chen and B. Liu, Asymptotic behavior of solutions of first order nonlinear delay difference equations, Comput. Math. Appl., 32(1996), 9-13.

[4] L. H. Erbe, H. Xian and J. S. Yu, Global stability of a linear nonautonomous delay difference equation, J. Difference Equ. Appl., 1(1995), 151-161.

[5] I. Györi and G. Ladas, Oscillation Theory of Delay Differential Equations with Applications, Oxford University Press, Oxford, (1991).

[6] G. Ladas, Explicit conditions for the oscillation of difference equations, Math. Anal. Appl., 153(1990), 276-287.

[7] B. S. Lalli, Oscillation theorems for neutral difference equations, Comput. Math. Appl., 28(1994), 191-202.

[8] Y. Liu and W. Ge, Global asymptotic behavior of solutions of a forced delay difference equation, Comput. Math. Appl., 47(2004), 1211-1224.

[9] R. E. Mickens, Difference Equations, Van Nostrand Reinhold Company Inc., New York 1987.

[10] Ch. G. Philos, On oscillations of some difference equations, Funkcialaj Ekvacioj, 34(1991), 157-172.

[11] X. H. Tang and J. S. Yu, A further result on the oscillation of delay difference equations, Comput. Math. Appl., 38(1999), 229-237.

[12] J. S. Yu, Asymptotic stability for a linear difference equation with variable delay, Comput. Math. Appl., 36(1998), 202-210.

[13] Z. Zhou, J. S. Yu and Z. C. Wang, Global attractivity of neutral difference equations, Comput. Math. Appl., 36(6)(1998), 1-10.