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Research Paper

Asymptotic behavior of first order delay difference equation with a forcing term

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Abstract

In this paper, we study the asymptotic behavior of solutions of the following first order forced delay difference equation

$$\Delta x(n) + p(n)f(x(n-\tau)) + r(n) = 0, \quad n \geq 0. \qquad (*)$$

Some sufficient conditions for every solution of (*) to tend to zero are established.

Keywords: *Asymptotic behavior, delay difference equation, oscillatory solution.*

1. Introduction

Recently there has been many investigations into the study of delay difference equations. In particular, an extensive literature now exists on the global stability for delay difference equations and various applications have been found, as we refer to [1, 3, 4, 5] and the references cited therein. However, concerning the asymptotic behavior of solutions for nonlinear delay difference equations are very few.

In this paper, we study the asymptotic behavior of solutions of the forced delay difference equation

$$\Delta x(n) + p(n)f(x(n-\tau)) + r(n) = 0, \quad n \geq 0, \qquad (1)$$

where Δ is the forward difference operator defined by $\Delta x(n) = x(n+1) - x(n)$, $\{p(n)\}$ is a sequence of positive real numbers, $\{r(n)\}$ is a sequence of real numbers, τ is a positive integer and $f : R \to R$ is an increasing function such that

(H_1) $uf(u) > 0 \quad \text{for} \quad u \neq 0$;

(H_2) $\lim_{u\to 0} \frac{f(u)}{u} = b \in (0, \infty)$; and

(H_3) $|f(u)| \leq |u|$, $u \in R$.

In [6], it was proved that if $p(n) > 0$ for all n and

$$\sum_{n=1}^{\infty} p(n) = +\infty, \quad \sum_{s=n-\tau}^{n} p(s) < 1$$

for sufficiently large n, then every solution of the equation

$$\Delta x(n) + p(n)x(n-\tau) = 0 \tag{2}$$

tends to zero as n tends to infinity.

In [1], it was shown that if $|f(u)| \leq f(|u|)$ for all $u \in R$ and

$$\sum_{n=1}^{\infty} |p(n)| < \infty, \quad \sum_{n=1}^{\infty} |r(n)| < \infty,$$

then every solutions of the equation.

$$\Delta x(n) + p(n)f(x(n-\tau)) = r(n) \tag{3}$$

tends to zero as n tends to infinity. Parhi [7] studied the equation

$$\Delta y(n) + q(n)G(y(n-k)) = b(n), \quad n \geq 0, \tag{4}$$

where $G \in C(R,R)$ is nondecreasing and $uG(u) > 0$ for all $u \neq 0$, $\{q(n)\}$ and $\{b(n)\}$ are sequences of real numbers. It was proved that if $q(n) \geq 0$, $b(n) \geq 0$ with $\sum_{n=0}^{\infty} b(n) < \infty$ or $q(n) \geq 0$, $b(n) \leq 0$ with $\sum_{n=0}^{\infty} b(n) > -\infty$, then every solution of equation (4) oscillates or tends to zero as n tends to infinity if and only if $\sum_{n=0}^{\infty} q(n) = +\infty$.

In [2], Graef and Qian studied the following difference equation

$$\Delta x(n) + px(n-k) = r(n), \quad n \geq 0, \tag{5}$$

where p is a real number, k is a positive integer, $\{r(n)\}$ is a real sequence. They proved that if

$$0 < p < \frac{k^k}{(k+1)^{k+1}},$$

then every solution of (5) tends to zero as n tends to infinity if and only if

$$\lim_{n\to\infty} r(n) = 0.$$

Although the equation (1) was studied by Yuji Lie et al. in [8], the results and their proofs are different.

By a solution of (1), we mean a nontrivial real sequence $\{x(n)\}$ which is defined for $n \geq -\tau$ and satisfies (1) for $n \geq 0$. The initial condition of (1) is $x(i) = a_i$; $i = -k, -k+1, ..., 0$ with $a_i \in (-\infty, +\infty)$ for $i = -k, -k+1, ..., 0$. A solution $\{x(n)\}$ of (1) is said to be oscillatory if for every positive integer $N_0 > 0$ there exits $n > N_0$ such that $x(n)x(n+1) \leq 0$, otherwise $\{x(n)\}$ is said to be nonoscillatory.

Throughout this paper, we define

$$N(a) = \{a, a+1, a+2, ...\}$$

and

$$N(a,b) = \{a, a+1, a+2, ...b\}$$

where a and b are integers with $a \leq b$.

2. Main result

In this section, we give the sufficient conditions so that every solution of equation (1) tends to zero as n tends to infinity.

Theorem 2.1 *Suppose that*

$$\sum_{s=0}^{\infty} p(s) = +\infty, \tag{6}$$

$$\mu = \limsup_{n\to\infty} \sum_{s=n-\tau}^{n} p(s) < \frac{\sqrt{7}}{2}, \tag{7}$$

and

$$\lim_{n\to\infty} \frac{r(n)}{p(n)} = 0. \tag{8}$$

Then every solution of Eq. (1) tends to zero as $n \to +\infty$.

3. Some lemmas

Clearly, conditions (H_2) and (8) imply that there exists $\alpha > 0$ such that $\frac{f(u)}{u} > \frac{b}{2}$ for $|u| < \alpha$, and for any $\epsilon \in (0, \alpha)$, there is $N_0 > 0$ such that

$$\left|\frac{r(n)}{p(n)}\right| < \frac{b\epsilon}{2}, \quad n > N_0. \tag{9}$$

In order to prove Theorem 2.1, we need the following lemmas.

Lemma 3.1 *Suppose that (8) hold. If $\{x(n)\}$ is an oscillatory solution of Eq. (1) and $A > 0$, $\delta > 1$ such that $\{x(n)\}$ satisfies that*

$$\Delta x(n) \le Ap(n) + r(n), \quad n \ge N_0, \tag{10}$$

$$\Delta x(n) \le -p(n)x(n-\tau) + r(n) \quad if \quad x(n-\tau) \le 0, \quad and \quad n \ge N_0 + \tau, \tag{11}$$

$$\sum_{s=n-\tau}^{n} p(s) \le \delta \quad for \quad all \quad n \ge N_0 + \tau. \tag{12}$$

If $n^* > N_0 + 2\tau$ such that $x(n^*) > 0$ and $\Delta x(n^*) \ge 0$, then we have

$$x(n^*) \le \left(\left(\delta + \frac{1}{2}\right)^2 - \frac{3}{4}\right) A + \epsilon \left(b\delta + \frac{b\delta^2}{2} + 1\right). \tag{13}$$

Proof. By (H_2), (8), we know (9) holds. Since $\Delta x(n^*) \ge 0$, we claim that $x(n^* - \tau) \le \epsilon$. In fact if $x(n^* - \tau) > \epsilon$, then by (1), noting that f is increasing, we get

$$0 \le \Delta x(n^*) = p(n^*)\left(-f(x(n^* - \tau)) + \frac{r(n^*)}{p(n^*)}\right)$$

$$< \epsilon p(n^*)\left(-\frac{f(\epsilon)}{\epsilon} + \frac{b}{2}\right)$$

$$< \epsilon p(n^*)\left(-\frac{b}{2}+\frac{b}{2}\right)=0.$$

This is impossible. Now we consider two cases.

Case 1. Let $0 \le x(n^*-\tau) \le \epsilon$. For $n \in N(n^*-\tau-1, n^*-1)$, we have $n-\tau \le n^*-\tau-1$. Summing (10) from $n-\tau$ to $n^*-\tau-1$, we get

$$-x(n-\tau) \le -x(n^*-\tau) + A\sum_{s=n-\tau}^{n^*-\tau-1} p(s) + \sum_{s=n-\tau}^{n^*-\tau-1} r(s)$$

$$\le A\sum_{s=n-\tau}^{n^*-\tau-1} p(s) + \frac{b\epsilon}{2}\sum_{s=n-\tau}^{n^*-\tau-1} p(s)$$

$$\le A\sum_{s=n-\tau}^{n^*-\tau-1} p(s) + \frac{b\epsilon\delta}{2}.$$

If $x(n-\tau) \le 0$, then by (11) we get

$$\Delta x(n) \le Ap(n)\sum_{s=n-\tau}^{n^*-\tau-1} p(s) + p(n)\frac{b\epsilon\delta}{2} + r(n), \quad n \in N(n^*-\tau-1, n^*-1) \tag{14}$$

If $x(n-\tau) > 0$, then (1) implies $\Delta x(n) \le r(n)$, and hence (14) is also valid.

Subcase 1.1 Let $\sum_{s=n^*-\tau}^{n^*-2} p(s) \le 1$. Summing (14) from $n^*-\tau$ to n^*-1, and applying (9), (12) we get

$$x(n^*) \le x(n^*-\tau) + A\sum_{n=n^*-\tau}^{n^*-1} p(n)\sum_{s=n-\tau}^{n^*-\tau-1} p(s) + \frac{b\epsilon\delta}{2}\sum_{s=n^*-\tau}^{n^*-1} p(s) + \sum_{n=n^*-\tau}^{n^*-1} r(n)$$

$$\le \quad \epsilon + A\sum_{n=n^*-\tau}^{n^*-1} p(n)\left(\sum_{s=n-\tau}^{n} p(s) - \sum_{s=n^*-\tau}^{n} p(s)\right) + \frac{b\epsilon\delta^2}{2} + \frac{b\epsilon\delta}{2}$$

$$\le \quad \epsilon\left(1+\frac{b\delta}{2}+\frac{b\delta^2}{2}\right) + A\sum_{n=n^*-\tau}^{n^*-1} p(n)\left(\delta - \sum_{s=n^*-\tau}^{n} p(s)\right)$$

$$= \quad \epsilon\left(1+\frac{b\delta}{2}+\frac{b\delta^2}{2}\right) + A\delta\sum_{n=n^*-\tau}^{n^*-1} p(n) - A\sum_{n=n^*-\tau}^{n^*-1} p(n)\sum_{s=n^*-\tau}^{n} p(s)$$

$$\le \quad \epsilon\left(1+\frac{b\delta}{2}+\frac{b\delta^2}{2}\right) + A\delta\sum_{n=n^*-\tau}^{n^*-1} p(n) - \frac{A}{2}\left(\sum_{s=n^*-\tau}^{n^*-1} p(s)\right)^2$$

$$\le \quad \epsilon\left(1+\frac{b\delta}{2}+\frac{b\delta^2}{2}\right) + A\delta^2 + A\delta\sum_{n=n^*-\tau}^{n-2} p(n) - \frac{A}{2}\left(\sum_{s=n^*-\tau}^{n^*-2} p(s)\right)^2.$$

Since $\delta x - \frac{1}{2}x^2$ is increasing for $0 \le x \le 1 < \delta$, then

$$x(n^*) \le \epsilon\left(\frac{b\delta^2}{2} + \frac{b\delta}{2} + 1\right) + A\left(\delta^2 + \delta - \frac{1}{2}\right)$$

$$\le \quad \epsilon\left(1 + b\delta + \frac{b\delta^2}{2}\right) + A\left(\left(\delta + \frac{1}{2}\right)^2 - \frac{3}{4}\right).$$

Subcase 1.2. Let $\sum_{s=n^*-\tau}^{n^*-2} p(s) > 1$. Choosing $\eta \in N(n^* - \tau, n^* - 2)$ such that $\sum_{s=\eta}^{n^*-2} p(s) \ge 1$, we get in applying (10), (14), (9), (12), that

$$x(n^*) = x(n^* - \tau) + \sum_{s=n^*-\tau}^{\eta} \Delta x(s) + \sum_{s=\eta+1}^{n^*-1} \Delta x(s)$$

$$\le x(n^* - \tau) + \sum_{s=n^*-\tau}^{\eta} (Ap(s) + r(s)) + \sum_{n=\eta+1}^{n^*-1} \left(Ap(n) \sum_{s=n-\tau}^{n^*-\tau-1} p(s) + p(n)\frac{b\epsilon\delta}{2} + r(n)\right)$$

$$= x(n^* - \tau) + A \sum_{n=n^*-\tau}^{\eta} p(n) + \sum_{n=n^*-\tau}^{\eta} r(n) + A \sum_{n=\eta+1}^{n^*-1} p(n) \sum_{s=n-\tau}^{n^*-\tau-1} p(s)$$

$$+\frac{b\epsilon\delta}{2} \sum_{n=\eta+1}^{n^*-1} p(n) + \sum_{n=\eta+1}^{n^*-1} r(n)$$

$$\le \epsilon\left(1 + b\delta + \frac{b\delta^2}{2}\right) + A \sum_{n=n^*-\tau}^{\eta} p(n) + A \sum_{n=\eta}^{n^*-1} p(n) \sum_{s=n-\tau}^{n^*-\tau-1} p(s)$$

$$= \epsilon\left(1 + b\delta + \frac{b\delta^2}{2}\right) + A \sum_{n=n^*-\tau}^{\eta} p(n) + A \sum_{n=\eta}^{n^*-1} p(n) \left[\sum_{s=n-\tau}^{n} p(s) - \sum_{s=n^*-\tau}^{n} p(s)\right]$$

$$\le \epsilon\left(1 + b\delta + \frac{b\delta^2}{2}\right) + A \sum_{n=n^*-\tau}^{\eta} p(n) + A \sum_{n=\eta}^{n^*-1} p(n) \left[\delta - \sum_{s=n^*-\tau}^{n} p(s)\right]$$

$$= \epsilon\left(1 + b\delta + \frac{b\delta^2}{2}\right) + A \sum_{n=n^*-\tau}^{\eta} p(n) + A\delta \sum_{n=\eta}^{n^*-1} p(n) - A \sum_{n=\eta}^{n^*-1} p(n) \sum_{s=n^*-\tau}^{n} p(s)$$

$$\le \epsilon\left(1 + b\delta + \frac{b\delta^2}{2}\right) + A \sum_{n=n^*-\tau}^{\eta} p(n) + A\delta \sum_{n=\eta}^{n^*-1} p(n) - \frac{A}{2} \left(\sum_{n=n^*-\tau}^{n^*-1} p(n)\right)^2$$

$$\le \epsilon\left(1 + b\delta + \frac{b\delta^2}{2}\right) + A\left(\delta^2 + \delta - \frac{1}{2}\right)$$

$$= \epsilon\left(1 + b\delta + \frac{b\delta^2}{2}\right) + A\left(\left(\delta + \frac{1}{2}\right)^2 - \frac{3}{4}\right).$$

Case 2. Let $x(n^*-\tau)<0$. There exists $\xi\in N(n^*-\tau+1,n^*)$ such that $x(\xi)\geq 0$. Then for $n\in N(\xi,n^*)$, we have $n-\tau\leq\xi$. By use of (9), we get from (10)

$$-x(n-\tau)\leq A\sum_{s=n-\tau}^{\xi-1}p(s)+\frac{b\xi\delta}{2}. \tag{15}$$

If $x(n-\tau)\leq 0$, we get based on (11),

$$\Delta x(n)\leq Ap(n)\sum_{s=n-\tau}^{\xi-1}p(s)+\frac{b\delta}{2}\epsilon p(n)+r(n),\quad n\in N(\xi,n^*). \tag{16}$$

If $x(n-\tau)>0$, then by (1), we have $\Delta x(n)\leq r(n)$, and (16) is also valid. By the method of that in the proof of Subcase 1.1 and 1.2, we get

$$x(n^*)\leq\epsilon\left(1+b\delta+\frac{b\delta^2}{2}\right)+\left(\left(\delta+\frac{1}{2}\right)^2-\frac{3}{4}\right)A. \tag{17}$$

This completes the proof.

Lemma 3.2 *Suppose that (8) hold. If $\{x(n)\}$ is an oscillatory solution of Eq. (1), $B<0$, such that*

$$\Delta x(n)\geq Bp(n)+r(n),\quad n\geq N_1,$$

$$\Delta x(n)\geq -p(n)x(n-\tau)+r(n),\quad if\quad x(n-\tau)\geq 0,\quad and\quad n\geq N_1+\tau,$$

(12) holds, $x(n^*)<0$ and $\Delta x(n^*)\leq 0$, then we have that

$$x(n^*)\geq\left(\left(\delta+\frac{1}{2}\right)^2-\frac{3}{4}\right)B-\epsilon\left(1+b\delta+\frac{b\delta^2}{2}\right).$$

Proof. We omit the proof since it is similar to that of Lemma 3.1.

Lemma 3.3 *Suppose that $\{x(n)\}$ is an eventually nonnegative solution of Eq. (1), and (6), (8) hold. Then $x(n)\to 0$ as $n\to\infty$.*

Proof. Let $\limsup_{n\to+\infty}x(n)=L$. If $L=0$, then the proof is complete. If $L>0$, we have two cases to consider.

Case 1. If $\{\Delta x(n)\}$ is eventually negative, then there is $N_2>N_1+\tau$ such that $\{x(n)\}$ is decreasing for $n\geq N_2$. The assumption $\limsup_{n\to+\infty}x(n)=L$ implies $x(n-\tau)\geq L$ for all $n\geq N_2$. By (1), we have

$$\Delta x(n)\leq -p(n)f(L)+r(n),\quad n\geq N_2. \tag{18}$$

Summing (18) from N_2 to $n-1$, we get

$$x(n)-x(N_2)\leq -f(L)\sum_{s=N_2}^{n-1}p(s)+\sum_{s=N_2}^{n-1}r(s).$$

Since $L>0$, we get $f(L)>0$. Choosing $\epsilon\in(0,f(L))$, (8) implies there is $N_3>N_2$ such that $|r(n)|\leq\epsilon p(n)$ for $n\geq N_3$. Hence

$$x(n)-x(N_2)\leq(-f(L)+\epsilon)\sum_{s=N_3}^{n-1}p(s)-f(L)\sum_{s=N_2}^{N_3-1}p(s)+\sum_{s=N_2}^{N_3-1}r(s). \tag{19}$$

Let $n\to+\infty$, by (19), we get

$L-x(N_2)\leq-\infty$, a contradiction. Therefore $L=0$.

Case 2. Suppose $\{\Delta x(n)\}$ is eventually negative. Choosing $N_2>N_1$ such that $x(n-\tau)\geq 0$ for $n\geq N_2$, we get

$$\Delta x(n)\leq r(n),\quad n\geq N_2 \tag{20}$$

Since $\{\Delta x(n)\}$ is not eventually negative, there exists $m^* > N_2 + \tau$ such that

$\Delta x(m^*) \geq 0.$

From now on, we prove that $x(m^* - \tau) \leq \epsilon$. Otherwise, we have $x(m^* - \tau) > \epsilon$, using $|r(n)| \leq \frac{b\epsilon}{2} p(n)$ and (1), we have

$$\begin{aligned} 0 \leq \Delta x(m^*) &= -p(m^*) f(x(m^* - \tau)) + r(m^*) \\ &< p(m^*) \left(-f(x(m^* - \tau)) + \frac{b\epsilon}{2} \right) \\ &= \epsilon p(m^*) \left(-\frac{f(\epsilon)}{\epsilon} + \frac{b}{2} \right) \\ &< \epsilon p(m^*) \left(-\frac{b}{2} + \frac{b}{2} \right) = 0, \end{aligned}$$

a contradiction. Summing (20) from $m^* - \tau$ to $m^* - 1$, by (9), (12) we get

$$x(m^*) \leq x(m^* - \tau) + \sum_{n=m^*-\tau}^{m^*-1} r(n) \leq \frac{b\tau\epsilon}{2} + \epsilon.$$

This shows that $\{x(n)\}$ is bounded above and then $L < +\infty$. Choosing the sequence $\{n_k\}$ of positive integers such that $N_3 + \tau < n_1 < n_2 < ...$, $\lim_{k\to\infty} n_k = +\infty$, $\Delta x(n_k) \geq 0$, and $\lim_{k\to\infty} x(n_k) = L$, we get $x(n_k - \tau) \leq \epsilon$. By a similar method in Case 2, $f(x(n - \tau)) > 0$ implies $\Delta x(n) \leq r(n)$. Summing this inequality from $n_k - \tau$ to $n_k - 1$, we get

$$x(n_k) \leq x(n_k - \tau) + \sum_{n=n_k-\tau}^{n_k-1} r(n) \leq \epsilon \left(1 + \frac{b\tau}{2} \right).$$

Let $n \to +\infty$, $\epsilon \to 0$, we have $L = 0$. This completes the proof.

Lemma 3.4 *Suppose that $\{x(n)\}$ is an eventually nonpositive solution of Eq. (1) and (6), (8) hold. Then $x(n) \to 0$ as $n \to +\infty$.*

The proof is similar to that of Lemma 3.3 and then omitted.

4. Proof of the theorem

Proof of Theorem 2.1. By (7), (8), we choose $\alpha > 0$, such that $\frac{f(u)}{u} > \frac{b}{2}$ for $|u| < \alpha$. For any $\epsilon \in (0, \alpha)$ with $1 < \sqrt{\mu^2 + \epsilon} - \frac{1}{2} < \mu + 1$, we choose $N_1 > 0$, such that (9) holds and

$$\sum_{s=n-\tau}^{n} p(s) \leq \sqrt{\mu^2 + \epsilon} - \frac{1}{2} = a.$$

By Lemma 3.3 and 3.4, we need to prove that every oscillatory solution $\{x(n)\}$ of Eq. (1) tends to zero. First we prove that $\{x(n)\}$ is bounded, to the contrary, there is $n^* > N_1 + \tau$ such that $|x(n)| < |x(n^*)|$ for $n < n^*$. Without loss of generality, we suppose $x(n^*) > 1$. Then we get

$$\Delta x(n) \leq p(n) x(n^*) + r(n) \quad for \quad n \leq n^*. \tag{21}$$

Then by Lemma 3.1 and (21), we get

$$\begin{aligned} x(n^*) &\leq \left(\left(a + \frac{1}{2} \right)^2 - \frac{3}{4} \right) x(n^*) + \epsilon \left(ab + \frac{b\delta^2}{2} + 1 \right) \\ &\leq \left(\left(a + \frac{1}{2} \right)^2 - \frac{3}{4} \right) x(n^*) + \epsilon M, \end{aligned} \tag{22}$$

where $M = b(1+\mu) + \frac{b(1+\mu)^2}{2} + 1$, (since $a = \delta = \sqrt{\mu^2+\epsilon} - \frac{1}{2} < \mu + 1$). By $\mu < \frac{\sqrt{7}}{2}$, without loss of generality, we suppose that $\epsilon < \frac{\frac{7}{4}-\mu^2}{1+M}$, thus (22) implies that $1 < \mu^2+\epsilon-\frac{3}{4}+\epsilon M = \mu^2-\frac{3}{4}+\epsilon M+\epsilon$. This is impossible. Then $\{x(n)\}$ is bounded. Now we suppose that $\limsup_{n\to+\infty} x(n) = L$, $\liminf_{n\to+\infty} x(n) = l$, then $-\infty < l < 0 \le L < +\infty$. Then there is $N_2 > N_1$ such that, $l_1 = l - \epsilon < x(n-\tau) < L + \epsilon = L_1$ for $n > N_2$. Thus by (1) we get

$$\Delta x(n) \le -p(n)f(l_1) + r(n), \quad n \ge N_2 \tag{23}$$

and

$$\Delta x(n) \ge -p(n)f(L_1) + r(n), \quad n \ge N_2. \tag{24}$$

We choose the sequences $\{n_k\}$ and $\{m_k\}$ of positive integers such that

$$N_2 + \tau < n_1 < n_2 < \ldots, \quad n_k \to +\infty, \quad \Delta x(n_k) \ge 0, \quad x(n_k) \to L \quad as \quad k \to \infty$$

and

$$N_2 + \tau < m_1 < m_2 < \ldots, \quad m_k \to +\infty, \quad \Delta x(m_k) \le 0, \quad x(m_k) \to l \quad as \quad k \to +\infty.$$

If $x(n-\tau) \le 0$, by $|f(u)| \le |u|$ and (1) we get

$$\Delta x(n) \le -p(n)x(n-\tau) + r(n). \tag{25}$$

By Lemma 3.1, we get

$$x(n_k) \le \epsilon \left[b\left(\sqrt{\mu^2+\epsilon} - \frac{1}{2}\right) + \frac{b}{2}\left(\sqrt{\mu^2+\epsilon} - \frac{1}{2}\right)^2 + 1\right] - \left(\mu^2 + \epsilon - \frac{3}{4}\right) f(l_1), \quad n = 1, 2, 3, ...$$

Let $k \to +\infty$ and $\epsilon \to 0$, we get $L \le -\left(\mu^2 - \frac{3}{4}\right) f(l)$. Similarly, we get

$$x(m_k) \ge -\left(\mu^2 + \epsilon - \frac{3}{4}\right) f(L_1) - \epsilon \left[b\left(\sqrt{\mu^2+\epsilon} - \frac{1}{2}\right) + \frac{b}{2}\left(\sqrt{\mu^2+\epsilon} - \frac{1}{2}\right)^2 + 1\right],$$

then $l \ge -\left(\mu^2 - \frac{3}{4}\right) f(L)$. Since $\mu^2 < \frac{7}{4}$, if $L \ne 0$, then $L > 0$. Hence

$$L < -f(l) \le -l \le \left(\mu^2 - \frac{3}{4}\right) f(L) < f(L) \le L,$$

which is impossible. We have $l = L = 0$. The proof is complete.

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