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Piecewise analytic method (solving any nonlinear ordinary differential equation of 1st order with any initial condition)

Tamer A. Abassy*

Department of Scientific Computing, Faculty of Computers and Informatics,
Benha University, Benha 13518, Egypt.
Department of mathematics,Faculty of Arts and Science in Wadi Addwasir,
Salman Bin Abdulaziz University, KSA.
**Corresponding author E-mail: tamerabassy@yahoo.com*

Abstract

The paper shows a new method which has the ability to solve any nonlinear 1st order differential equation with any initial conditions. The method is called Piecewise Analytic Method (PAM). The accuracy of the method can be controlled according to our needs. A comparison between PAM and Runge-Kutta method is introduced which enhances the use of PAM. For non-mathematician, they can now test their systems with any initial condition.

Keywords: *Differential equation; Padé Approximants; Power series; Runge-Kutta method; Piecewise analytic method.*

1 Introduction

Mathematical modeling of many engineering and physical systems leads to nonlinear ordinary and partial differential equations. In general, it is very difficult to solve nonlinear problems analytically. An effective method that provides solutions conforming to physical reality is required to analyze the mathematical model. Therefore, we must be able to solve nonlinear ordinary and partial differential equations, in space and time. Some analytic procedures linearize the system or assume that nonlinearities are relatively insignificant. Assumptions have to be made artificially or unnecessarily to make the practical problems solvable, leading to loss of most important information. Such procedures change the actual problem to make it tractable by the conventional methods. These approaches sometimes change the solution seriously [1, 2].

Generally, The ability to solve nonlinear equations by analytical methods is important because linearization changes the problem being analyzed to a different problem, perturbation methods are only reasonable when nonlinear effects are very small, and the numerical methods need a substantial amount of computations but only lead to limited information [3].

In this paper, the piecewise analytic method (PAM) is introduced for solving any initial value ordinary nonlinear differential equation. The PAM is based on dividing the solution interval into subintervals and obtaining an approximate analytic solution which is very accurate and can be applied to each subinterval successively. The approximate analytic solution is based on truncated Taylor series [4] or Padé approximants [5, 6]. In PAM, the solution accuracy can be controlled according to needs. The PAM gives the exact solution in some special cases [7]. A comparison between PAM and Runge-Kutta method is introduced. The Runge-Kutta method is one of the most famous and popular method, which is used for solving ordinary differential equations. The Runge-Kutta method is named for its' creators Carl Runge(1856-1927) and Wilhelm Kutta (1867-1944). The Runge-Kutta formulas are available from order 2 up to order 10. It should be noted that no Runge-Kutta formula of order 11 is available at present [8, 9, 10, 11, 12, 13]. The comparison between PAM and Runge-Kutta method enhances the use of PAM, especially, when high order of accuracy and analytic form are needed.

2 Piecewise Analytic Method

Consider the general 1^{st} order differential equation:

$$u' = \phi(t,u), \qquad u(t_0) = f_0, \qquad t_0 \le t \le b. \tag{1}$$

For solving (1) using piecewise analytic method (PAM), The interval $t_0 \le t \le b$ is divided into n equal parts, each of length h, by the points $t_m = mh$, $m = 0, 1, 2, \ldots, n$. The value $h = \frac{b - t_0}{n}$ is called the subinterval length. The points t_m are called interval points see Fig. **1**.

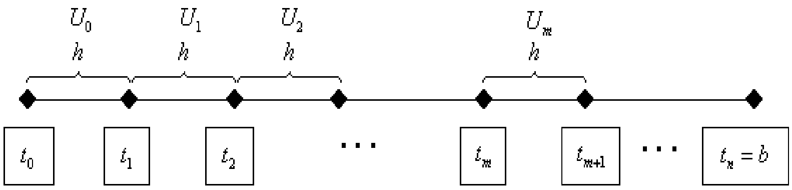


Fig. 1

U_m denotes to the approximate analytic solution in the m^{th} subinterval $[t_m, t_{m+1}]$. U_m can be applied to any subinterval m ($t \in [t_m, t_{m+1}]$, $\quad m = 0,1,2,\ldots,(n-1)$).

Now, for calculating U_m, equation (1) is written in the form

$$\frac{dU_m}{dt} = \phi(t, U_m),$$
$$U_m(t_m) = f_m, \qquad t \in [t_m, t_{m+1}], \qquad m = 0,1,2,\ldots,(n-1). \tag{2}$$

Then using any symbolic mathematical program like Mathematica for obtaining the approximate solution U_m. I have two forms of approximate solutions, one is the truncated Taylor series solution and the other is the Padé approximants solution.

In the case of truncated Taylor solution [4], U_m is defined according to the needed accuracy. If we need the accuracy to be of $O(h^s)$, U_m will take the form

$$U_m(t) = \sum_{n=0}^{s-1} c_n \left(t - t_m\right)^n = \sum_{n=0}^{s-1} \frac{\left(t - t_m\right)^n}{n!} \left(\left. \frac{d^n U_m}{dt^n} \right|_{t=t_m} \right) \qquad t \in \left[t_m, t_{m+1}\right] \tag{3}$$

In the case of Padé approximants solution[5, 14, 6], U_m will take the form

$$U_m(t) = \frac{P_l}{Q_k} = \frac{\sum_{n=0}^{l} p_n \left(t - t_m\right)^n}{\sum_{n=0}^{k} q_n \left(t - t_m\right)^n} \quad where \quad l + k = s - 1, \ t \in \left[t_m, t_{m+1}\right] \tag{4}$$

if we need the accuracy to be of $O(h^s)$.

Another approximate solution is under study.

The final step is applying the approximate analytic solution formula U_m to each subinterval successively with the initial value $f_m = U_{m-1}(t_m)$, $U_{-1}(t_0) = f_0$.

Notes:

- We have many methods for calculating(3). The first method is the substitution by $U_m(t)=\sum_{n=0}^{s-1} c_n\left(t-t_m\right)^n$ and its derivatives into the ODE, then equating the coefficients of each power of $\left(t-t_m\right)^n$ to zero to get a recurrence relation. The recurrence relation expresses a coefficient c_n in terms of the coefficients c_m where $m<n$. The second method is calculating $U_m(t)=\sum_{n=0}^{s-1}\frac{\left(t-t_m\right)^n}{n!}\left(\left.\frac{dU_m}{dt}\right|_{t=t_m}\right)$ which is based on the manipulation of the function formula by classical differential calculus techniques. The results are constants that represent the value of derivatives at the point of evaluation. The other methods are based on the new methods which give the truncated series solution like Adomian decomposition method [15], Improved Adomian decomposition method[16, 17], modified variational iteration method [16, 18], homotopy perturbation method[19], homotopy analysis method[20] and others.
  I prefer the first method because it is the origin and all the others are synthesis from it.
- The PAM gives the exact solution in two cases:
  1. If the exact solution is a polynomial with order w and the truncated series approximation (3) is used with $s-1\geq w$.
  2. If the exact solution is a rational function $\frac{\sum_{n=0}^{z} p_n\left(t-t_m\right)^n}{\sum_{n=0}^{w} q_n\left(t-t_m\right)^n}$ and the Padé approximants (4) is used with $l\geq z$ and $k\geq w$.
- The truncated series (3) is suitable if the solution has no poles and bounded, if not, the Padé approximants (4) is more suitable than truncated series (3).
- I don't know the best form for Padé approximants (4) but by experience I prefer $l=k=even$.
- The Padé approximants coefficients $p_n(n=0,1,2,...,l)$ and $q_n(0,1,2,...,k)$ are determined by [5, 14, 6]
-

$$\sum_{n=0}^{S\geq l+k} c_n\left(t-t_m\right)^n-\frac{\sum_{n=0}^{l} p_n\left(t-t_m\right)^n}{\sum_{n=0}^{k} q_n\left(t-t_m\right)^n}=O\left((t-t_m)^{l+k+1}\right), \tag{5}$$

setting $q_0=1$ and multiply (5) by $\sum_{n=0}^{k} q_n\left(t-t_m\right)^n$, which linearizes the equations coefficient. It can be written out in more detail as

$$\left.\begin{array}{ll} c_{l+1}+c_l q_1+\cdots+c_{l-k+1}q_k & =0\\ c_{l+2}+c_{l+1}q_1+\cdots+c_{l-k+2}q_k & =0\\ \vdots & \\ c_{l+k}+c_{l+k-1}q_1+\cdots+c_l q_k & =0 \end{array}\right\} \tag{6}$$

$$\left.\begin{array}{ll} c_0 & =p_0\\ c_1+c_0q_1 & =p_1\\ c_2+c_1q_1+c_0q_2 & =p_2\\ \vdots & \\ c_l+c_{l-1}q_1+\cdots+c_0q_l & =p_l \end{array}\right\} \tag{7}$$

Once, the q's are known from equations (6), equations (7) can be solved easily. If equations (6) and (7) are nonsingular, then they can be solved directly as follows;

$$U_m(t) = \frac{\det \begin{vmatrix} c_{l-k+1} & c_{l-k+2} & \cdots & c_{l+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_l & c_{l+1} & \cdots & c_{l+k} \\ \sum_{j=k}^{l} c_{j-k} t^j & \sum_{j=k-1}^{l} c_{j-k+1} t^j & \cdots & \sum_{j=0}^{l} c_j t^j \end{vmatrix}}{\det \begin{vmatrix} c_{l-k+1} & c_{l-k+2} & \cdots & c_{l+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_l & c_{l+1} & \cdots & c_{l+k} \\ t^k & t^{k-1} & \cdots & 1 \end{vmatrix}}, \tag{8}$$

For $l = 1$ and $k = 1$

$$p_0 = c_0, \qquad q_0 = 1,$$
$$p_1 = \frac{c_1^2 - c_0 c_2}{c_1}, \qquad q_1 = -\frac{c_2}{c_1}. \tag{9}$$

For $l = 1$ and $k = 2$

$$p_0 = c_0, \qquad q_0 = 1,$$
$$p_1 = \frac{c_1^3 - 2c_0 c_1 c_2 + c_0^2 c_3}{c_1^2 - c_1 c_2}, \qquad q_1 = \frac{-c_1 c_2 + c_0 c_3}{c_1^2 - c_0 c_2}, \tag{10}$$
$$q_2 = \frac{c_2^2 - c_1 c_3}{c_1^2 - c_0 c_2}.$$

For $l = 2$ and $k = 1$

$$p_0 = c_0, \qquad q_0 = 1,$$
$$p_1 = \frac{c_1 c_2 - c_0 c_3}{c_2}, \qquad q_1 = \frac{-c_3}{c_2}. \tag{11}$$
$$P_2 = \frac{c_2^2 - c_1 c_3}{c_2},$$

For $l = 2$ and $k = 2$

$$p_0 = c_0, \qquad q_0 = 1,$$
$$p_1 = \frac{c_1 c_2^2 - c_1^2 c_3 - c_0 c_2 c_3 + c_0 c_1 c_4}{c_2^2 - c_1 c_3}, \qquad q_1 = \frac{-c_2 c_3 + c_1 c_4}{c_2^2 - c_1 c_3}, \tag{12}$$
$$P_2 = \frac{c_2^3 - 2c_1 c_2 c_3 + c_0 c_3^2 + c_1^2 c_4 - c_0 c_2 c_4}{c_2^2 - c_1 c_3}, \qquad q_2 = \frac{c_3^2 - c_2 c_4}{c_2^2 - c_1 c_3}.$$

For $l = 3$ and $k = 3$

$$\begin{aligned} p_0 &= c_0, \\ p_1 &= (-c_1 c_3^3 + 2c_1 c_2 c_3 c_4 + c_0 c_3^2 c_4 - c_1^2 c_4^2 - c_0 c_2 c_4^2 - c_1 c_2^2 c_5 + c_1^2 c_3 c_5 - \\ &\quad c_0 c_2 c_3 c_5 + c_0 c_1 c_4 c_5 + c_0 c_2^2 c_6 - c_0 c_1 c_3 c_6) / (-c_3^3 + 2c_2 c_3 c_4 - c_1 c_4^2 - \\ &\quad c_2^2 c_5 + c_1 c_3 c_5), \\ p_2 &= (c_2 c_3^3 - 2c_2^2 c_3 c_4 - c_1 c_3^2 c_4 + 2c_1 c_2 c_4^2 + c_0 c_3 c_4^2 + c_2^3 c_5 - c_0 c_3^2 c_5 - \\ &\quad c_1^2 c_4 c_5 - c_0 c_2 c_4 c_5 - c_0 c_1 c_5^2 - c_1 c_2^2 c_6 + c_1^2 c_3 c_6 + c_0 c_2 c_3 c_6 - c_0 c_1 c_4 c_6) / \\ &\quad (c_3^3 - 2c_2 c_3 c_4 + c_1 c_4^2 + c_2^2 c_5 - c_1 c_3 c_5), \\ p_3 &= (c_3^4 - 3c_2 c_3^2 c_4 + c_2^2 c_4^2 + 2c_1 c_3 c_4^2 - c_0 c_4^3 + 2c_2^2 c_3 c_5 - 2c_1 c_3^2 c_5 - 2c_1 c_2 c_4 c_5 + \\ &\quad 2c_0 c_3 c_4 c_5 + c_1^2 c_5^2 - c_0 c_2 c_5^2 - c_2^3 c_6 + 2c_1 c_2 c_3 c_6 - c_0 c_3^2 c_6 - c_1^2 c_4 c_6 + c_0 c_2 c_4 c_6) / \\ &\quad (c_3^3 - 2c_2 c_3 c_4 + c_1 c_4^2 + c_2^2 c_5 - c_1 c_3 c_5), \end{aligned} \tag{13}$$

$$q_0 = 1,$$
$$q_1 = \frac{-c_3^{\,2}c_4 + c_2c_4^{\,2} + c_2c_3c_5 - c_1c_4c_5 - c_2^{\,2}c_6 + c_1c_3c_6}{c_3^{\,3} - 2c_2c_3c_4 + c_1c_4^{\,2} + c_2^{\,2}c_5 - c_1c_3c_5},$$
$$q_2 = \frac{c_3c_4^{\,2} - c_3^{\,2}c_5 - c_2c_4c_5 + c_1c_5^{\,2} + c_2c_3c_6 - c_1c_4c_6}{c_3^{\,3} - 2c_2c_3c_4 + c_1c_4^{\,2} + c_2^{\,2}c_5 - c_1c_3c_5},$$
$$q_3 = \frac{-c_4^{\,3} + 2c_3c_4c_5 - c_2c_5^{\,2} - c_3^{\,2}c_6 + c_2c_4c_6}{c_3^{\,3} - 2c_2c_3c_4 + c_1c_4^{\,2} + c_2^{\,2}c_5 - c_1c_3c_5}.$$

And so on, where we can calculate any desired Padé approximants by using any symbolic mathematical program for any series $U_m(t) = \sum_{n=0}^{s} c_n \left(t - t_m\right)^n$ and then all what we will do is only substituting by c_n's in the suitable p's and q's for obtaining the desired Padé approximants .

3 Case-Studies

3.1 Case-study 1:

Consider the differential equation

$$u'(t) = -u^3, \qquad u(0) = 1. \tag{14}$$

This is a nonlinear problem which has the exact solution

$$u(t) = \frac{1}{\sqrt{1+2t}}. \tag{15}$$

Defining a differential equation for each subinterval m from (14)

$$\frac{dU_m}{dt} = -U_m^{\,3}, \qquad U_m(t_m) = f_m, \qquad t \in \left[t_m, t_{m+1}\right]. \tag{16}$$

where $f_m = U_{m-1}(t_m)$, $U_{-1}(t_0) = u(0) = 0$, $m = 0,1,2,\ldots,n-1$

Substituting $U_m(t) = \sum_{n=0}^{s} c_n \left(t - t_m\right)^n$ and its derivatives into (16) leads to

$$\sum_{n=1}^{s} n\, c_n \left(t - t_m\right)^{n-1} = -\left(\sum_{n=0}^{s} c_n \left(t - t_m\right)^n\right)^3, \qquad c_0 = f_m,\ t \in \left[t_m, t_{m+1}\right]. \tag{17}$$

Solving (17) leads to:

$$\begin{aligned}
&c_0 = f_m, && c_1 = -f_m^{\,3},\\
&c_2 = \frac{3}{2} f_m^{\,5}, && c_3 = -\frac{5}{2} f_m^{\,7},\\
&c_4 = \frac{35}{8} f_m^{\,9}, && c_5 = -\frac{63}{8} f_m^{\,11},\\
&c_6 = \frac{231}{16} f_m^{\,13}, && c_6 = \frac{231}{16} f_m^{\,13},\\
&c_7 = -\frac{429}{16} f_m^{\,15}, && {}_8 = \frac{6435}{128} f_m^{\,17},\\
&\vdots
\end{aligned} \tag{18}$$

Substituting by (18) into(3), if we need accuracy $O(h^5)$, we obtain the approximate analytic Taylor series solution.

$$U_m(t) \simeq f_m - f_m{}^3(t-t_m) + \frac{3}{2} f_m{}^5(t-t_m)^2 - \frac{5}{2} f_m{}^7(t-t_m)^3 + \frac{35}{8} f_m{}^9(t-t_m)^4, \qquad t \in [t_m, t_{m+1}] \tag{19}$$

Substituting by (18) into (12) for obtaining p's and q's then substituting by them into(4), we obtain the Padé approximants

$$U_m(t) \simeq \frac{4f_m + 6f_m{}^3(t-t_m) + f_m{}^5(t-t_m)^2}{4 + 10f_m{}^2(t-t_m) + 5f_m{}^4(t-t_m)^2}, \quad t \in [t_m, t_{m+1}] \tag{20}$$

if we need accuracy $O(h^9)$, we can obtain the approximate analytic Taylor series solution

$$U_m(t) \simeq f_m - f_m{}^3(t-t_m) + \frac{3}{2} f_m{}^5(t-t_m)^2 - \frac{5}{2} f_m{}^7(t-t_m)^3 + \frac{35}{8} f_m{}^9(t-t_m)^4 - \frac{63}{8} f_m{}^{11}(t-t_m)^5 + \frac{231}{16} f_m{}^{13}(t-t_m)^6 - \frac{429}{16} f_m{}^{15}(t-t_m)^7 + \frac{6435}{128} f_m{}^{17}(t-t_m)^8, \qquad t \in [t_m, t_{m+1}] \tag{21}$$

and Padé approximants

$$U_m(t) \simeq \frac{16f_m + 56f_m{}^3(t-t_m) + 60f_m{}^5(t-t_m)^2 + 20f_m{}^7(t-t_m)^3 + f_m{}^9(t-t_m)^4}{16 + 72f_m{}^2(t-t_m) + 108f_m{}^4(t-t_m)^2 + 60f_m{}^6(t-t_m)^3 + 9f_m{}^8(t-t_m)^4}, \qquad t \in [t_m, t_{m+1}] \tag{22}$$

Fig 2 shows the exact solution of (14).

Table 1 shows the absolute error between (14) exact solution and Padé approximants and Taylor series for different values of h .

Table 2 shows the absolute error between (14) exact solution and different forms of Padé approximants and Taylor series for $h = 0.1$.

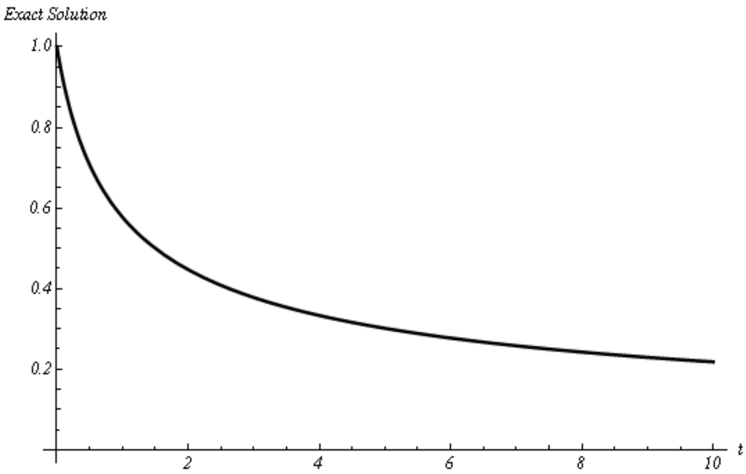
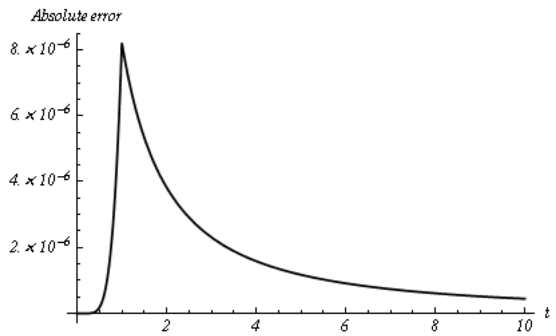
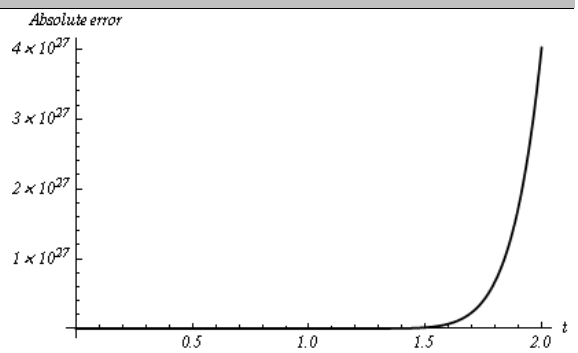
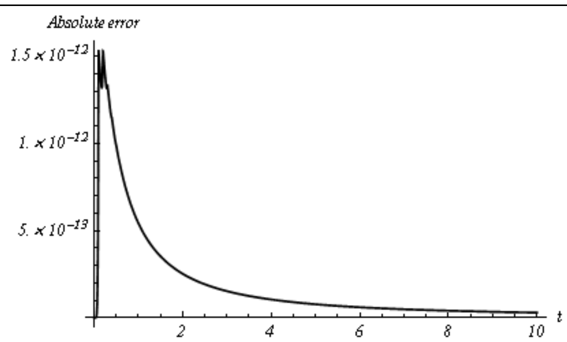
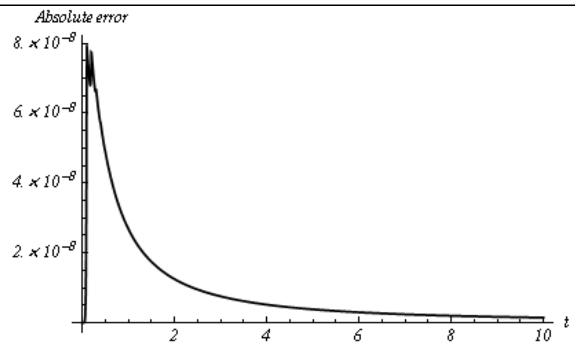


Fig 2: The exact solution of (14)

Table 1: The absolute error between the exact solution (15) and Padé approximants and Taylor series for different values of h .

h	Padé $\left[\frac{4}{4}\right] O(h^9)$	Truncated Series $O(h^9)$
1		
0.1		

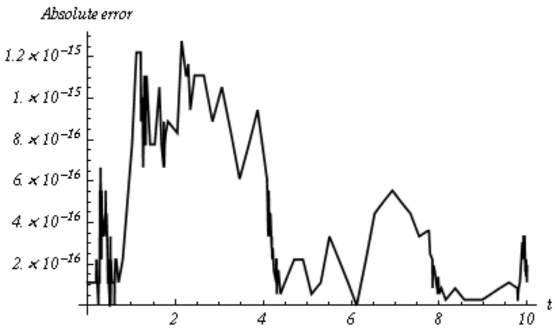
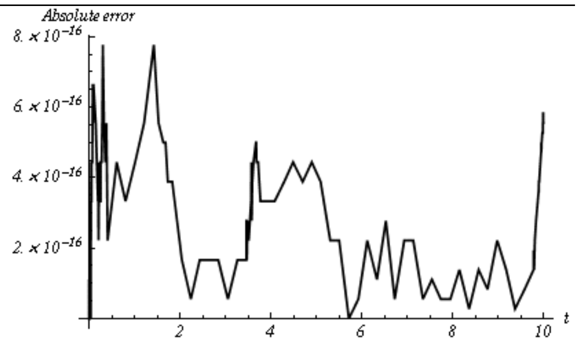
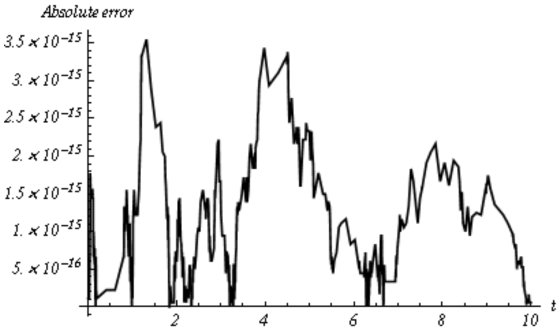
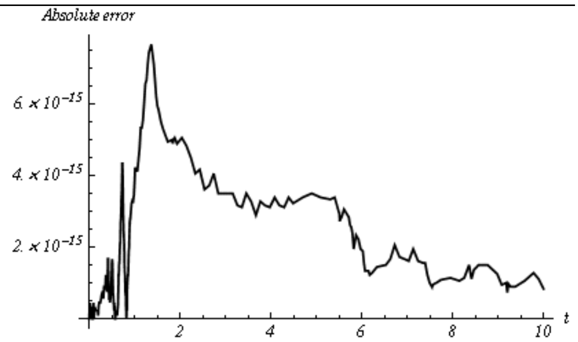
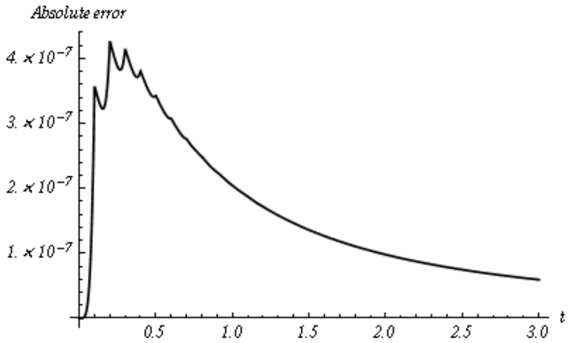
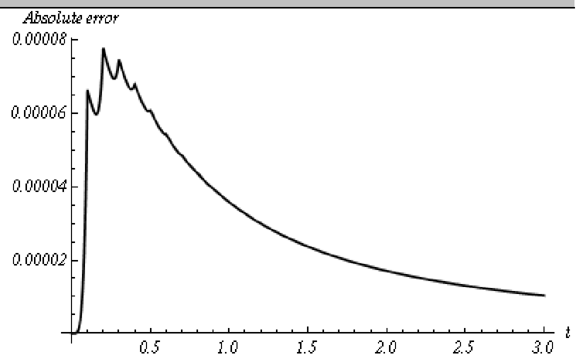
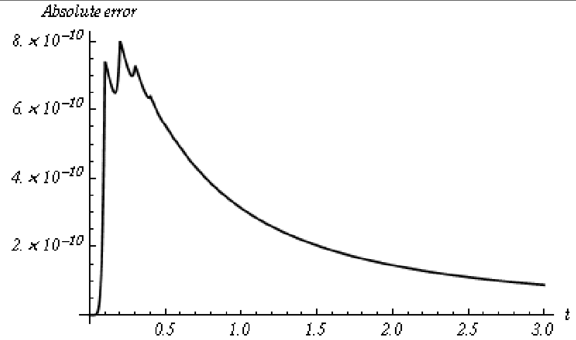
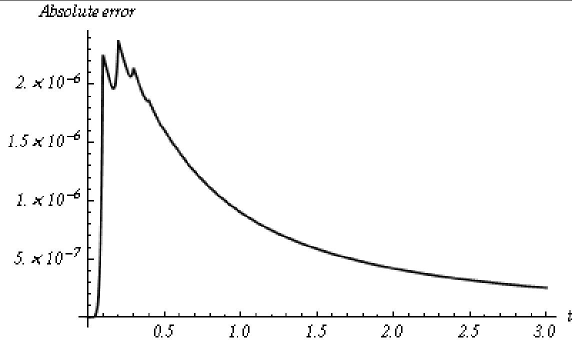
0.01		
.001		

Table 2: The absolute error between the exact solution (15) and different forms of Padé approximants and Taylor series for $h = 0.1$.

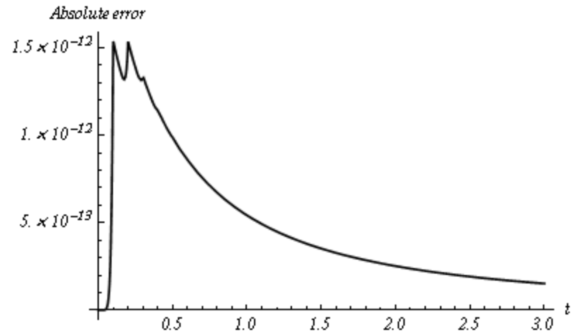
Order of h	Padé [*n/n*]	Series
5		

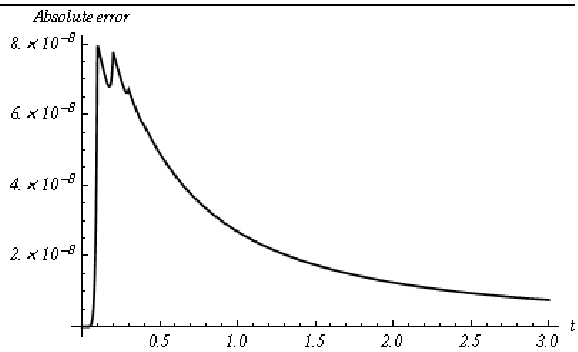
7


Absolute error
8. × 10^−10
6. × 10^−10
4. × 10^−10
2. × 10^−10
0.5 1.0 1.5 2.0 2.5 3.0
t

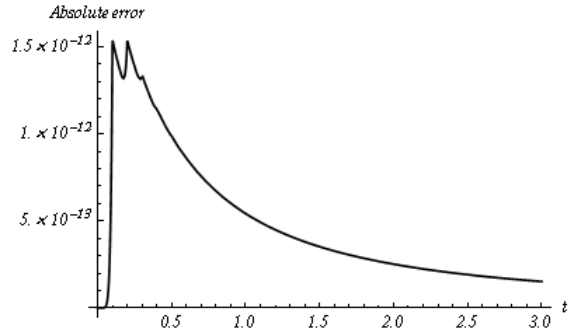

Absolute error
2. × 10^−6
1.5 × 10^−6
1. × 10^−6
5. × 10^−7
0.5 1.0 1.5 2.0 2.5 3.0
t

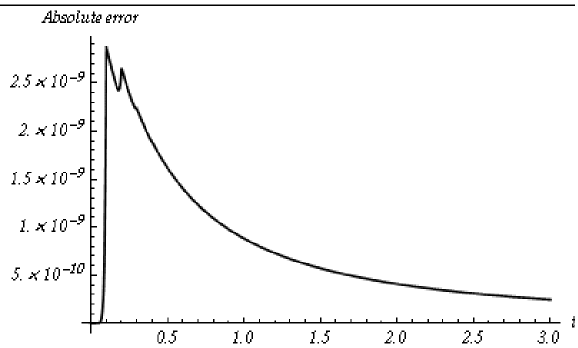
9


Absolute error
1.5 × 10^−12
1. × 10^−12
5. × 10^−13
0.5 1.0 1.5 2.0 2.5 3.0
t

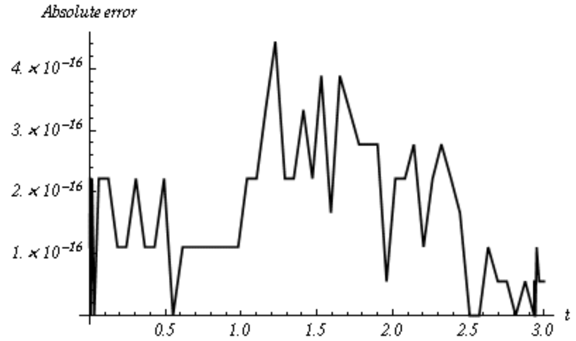

Absolute error
8. × 10^−8
6. × 10^−8
4. × 10^−8
2. × 10^−8
0.5 1.0 1.5 2.0 2.5 3.0
t

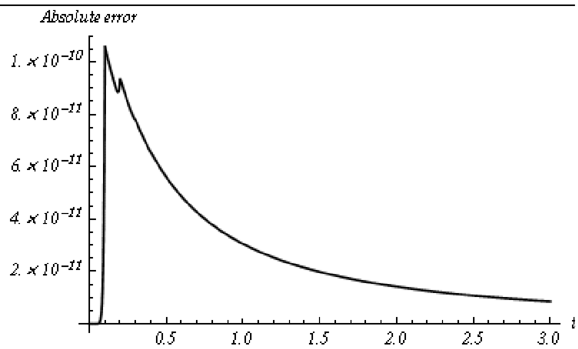
11


Absolute error
1.5 × 10^−12
1. × 10^−12
5. × 10^−13
0.5 1.0 1.5 2.0 2.5 3.0
t

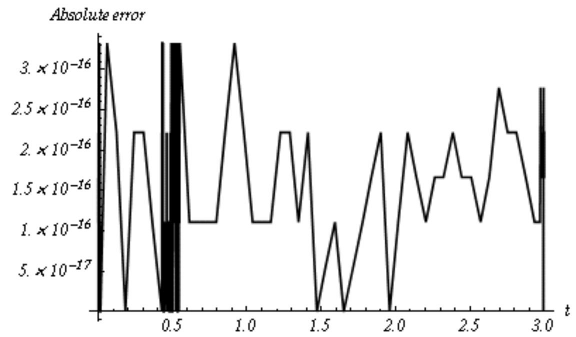

Absolute error
2.5 × 10^−9
2. × 10^−9
1.5 × 10^−9
1. × 10^−9
5. × 10^−10
0.5 1.0 1.5 2.0 2.5 3.0
t

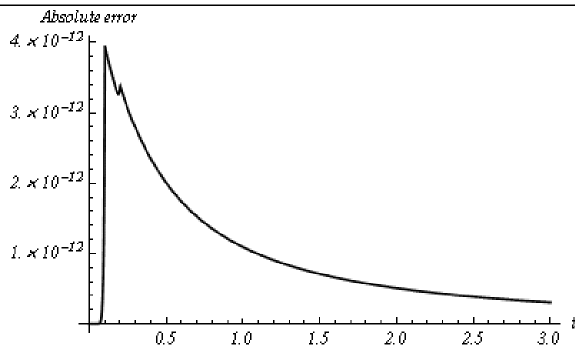
13

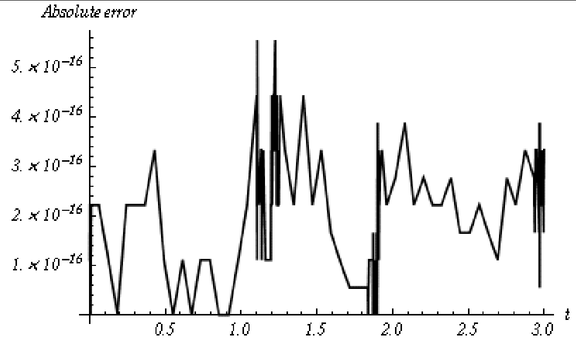
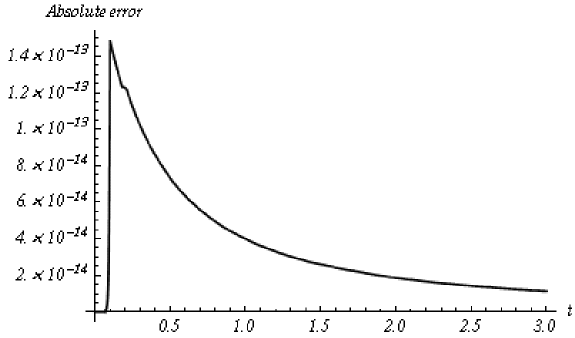

Absolute error
4. × 10^−16
3. × 10^−16
2. × 10^−16
1. × 10^−16
0.5 1.0 1.5 2.0 2.5 3.0
t


Absolute error
1. × 10^−10
8. × 10^−11
6. × 10^−11
4. × 10^−11
2. × 10^−11
0.5 1.0 1.5 2.0 2.5 3.0
t

15


Absolute error
3. × 10^−16
2.5 × 10^−16
2. × 10^−16
1.5 × 10^−16
1. × 10^−16
5. × 10^−17
0.5 1.0 1.5 2.0 2.5 3.0
t


Absolute error
4. × 10^−12
3. × 10^−12
2. × 10^−12
1. × 10^−12
0.5 1.0 1.5 2.0 2.5 3.0
t

17		

3.2 Case study 2:

Consider the differential equation

$$u'(t) = -u - t - 2, \qquad u(0) = 1. \tag{23}$$

This is a nonlinear problem which has the exact solution

$$u(t) = 2e^{-t} - t - 1. \tag{24}$$

Defining a differential equation for each subinterval m from (23)

$$\frac{dU_m}{dt} = -U_m - t - 2, \qquad U_m(t_m) = f_m, \qquad t \in [t_m, t_{m+1}]. \tag{25}$$

where $f_m = U_{m-1}(t_m)$, $U_{-1}(t_0) = u(0) = 1$, $m = 0,1,2,\ldots,n-1$

Substituting by $U_m(t) = \sum_{n=0}^{s} c_n \left(t - t_m\right)^n$ and its derivatives into (16) leads to

$$\begin{gathered}\sum_{n=1}^{s} n\, c_n \left(t - t_m\right)^{n-1} = -\left(\sum_{n=0}^{s} c_n \left(t - t_m\right)^n\right) - \left(t_m + \left(t - t_m\right)\right) - 2, \\ c_0 = f_m,\ t \in [t_m, t_{m+1}].\end{gathered} \tag{26}$$

Solving (26) leads to:

$$\begin{aligned}
c_0 &= f_m, & c_1 &= -2 - t_m - f_m, \\
c_2 &= \frac{1}{2}\left(1 + t_m + f_m\right), & c_3 &= -\frac{1}{6}\left(1 + t_m + f_m\right), \\
c_4 &= \frac{1}{24}\left(1 + t_m + f_m\right), & c_5 &= -\frac{1}{120}\left(1 + t_m + f_m\right), \\
c_6 &= \frac{1}{720}\left(1 + t_m + f_m\right), & c_7 &= -\frac{1}{5040}\left(1 + t_m + f_m\right), \\
& & c_8 &= \frac{1}{40320}\left(1 + t_m + f_m\right), \\
& & &\vdots
\end{aligned} \tag{27}$$

If we need accuracy $O(h^5)$, we'll substitute by (27) into(3), we obtain the approximate analytic Taylor series solution.

$$U_m(t) \simeq f_m - (2+t_m+f_m)(t-t_m) + \frac{1}{2}\left(1+t_m+f_m\right)(t-t_m)^2 - \frac{1}{6}\left(1+t_m+f_m\right)(t-t_m)^3 + \frac{1}{24}\left(1+t_m+f_m\right)(t-t_m)^4, \qquad t \in [t_m, t_{m-1}] \tag{28}$$

Substituting by (27) into (12) for obtaining $p's$ and $q's$ then substituting by them into(4), we obtain the Padé approximants

$$\begin{aligned} U_m(t) \simeq {} & (24(t-t_m) - 12t_m(t-t_m) - 12t_m^{\,2}(t-t_m)^4 - 6(t-t_m)^2 - 12t_m(t-t_m)^2 - \\ & 12f_m + 12t_m f_m - 12(t-t_m)f_m - 18t_m(t-t_m)f_m - 11(t-t_m)^2 f_m + t_m(t-t_m)^2 f_m + \\ & 12f_m^{\,2} - 6(t-t_m)f_m^{\,2} + (t-t_m)^2 f_m^{\,2}) / (-12 + 12t_m + 6t_m(t-t_m) + (t-t_m)^2 + \\ & t_m(t-t_m)^2 + 12f_m + 6(t-t_m)f_m + (t-t_m)^2 f_m), \qquad t \in [t_m, t_{m-1}] \end{aligned} \tag{29}$$

if we need accuracy $O(h^9)$, we can obtain the approximate analytic Taylor series solution

$$\begin{aligned} U_m(t) \simeq {} & f_m - (2+t_m+f_m)(t-t_m) + \frac{1}{2}\left(1+t_m+f_m\right)(t-t_m)^2 - \\ & \frac{1}{6}\left(1+t_m+f_m\right)(t-t_m)^3 + \frac{1}{24}\left(1+t_m+f_m\right)(t-t_m)^4 - \\ & \frac{1}{120}\left(1+t_m+f_m\right)(t-t_m)^5 + \frac{1}{720}\left(1+t_m+f_m\right)(t-t_m)^6 - \\ & \frac{1}{5040}\left(1+t_m+f_m\right)(t-t_m)^7 + \frac{1}{40320}\left(1+t_m+f_m\right)(t-t_m)^8, \qquad t \in [t_m, t_{m-1}] \end{aligned} \tag{30}$$

and Padé approximants

$$\begin{aligned} U_m(t) \simeq {} & (10080(t-t_m) + 1680t_m(t-t_m) - 1680t_m^{\,2}(t-t_m) + 840(t-t_m)^2 - \\ & 1680t_m(t-t_m)^2 + 360(t-t_m)^3 - 40t_m(t-t_m)^3 - 40t_m^{\,2}(t-t_m)^3 - 20(t-t_m)^4 - \\ & -40t_m(t-t_m)^4 - 5040f_m + 1680t_m f_m - 2520t_m(t-t_m)f_m - 1860(t-t_m)^2 f_m + \\ & 180t_m(t-t_m)^2 f_m - 40(t-t_m)^3 f_m - 60t_m(t-t_m)^3 f_m - 39(t-t_m)^4 f_m + \\ & t_m(t-t_m)^4 f_m + 1680f_m^{\,2}) / (-5040 + 1680t_m - 1680(t-t_m) + 840t_m(t-t_m) - \\ & 180(t-t_m)^2 + 180t_m(t-t_m)^2 + 20t_m(t-t_m)^3 + (t-t_m)^4 + t_m(t-t_m)^4 + \\ & 1680f_m + 840(t-t_m)f_m + 180(t-t_m)^2 f_m + 20(t-t_m)^3 f_m + (t-t_m)^4 f_m), \\ & \qquad t \in [t_m, t_{m-1}]. \end{aligned} \tag{31}$$

Fig. **3** shows the exact solution of (23). Table 3 shows the absolute error between (23) exact solution and Padé approximants and Taylor series for different values of h.

Table 4 shows the absolute error between (23) exact solution and different forms of Padé approximants and Taylor series for $h = 1$.

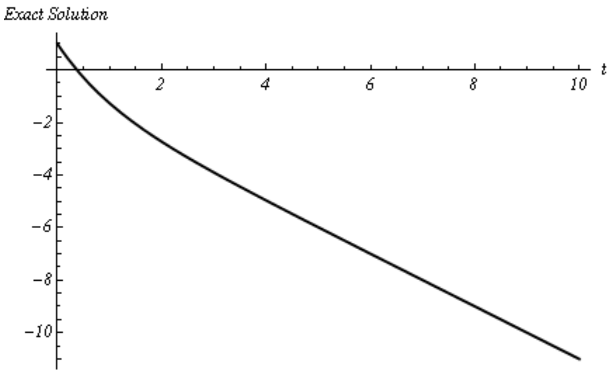
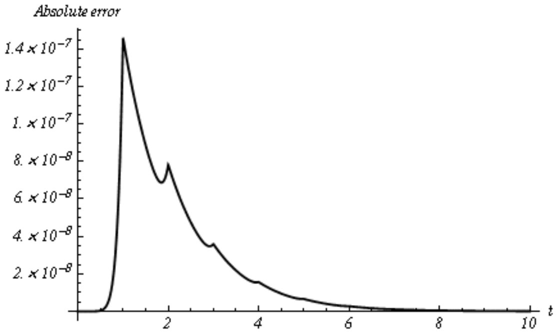
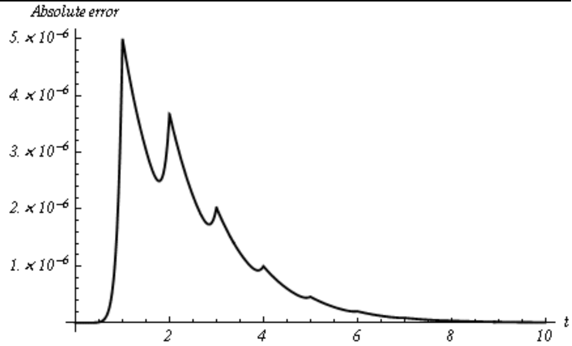
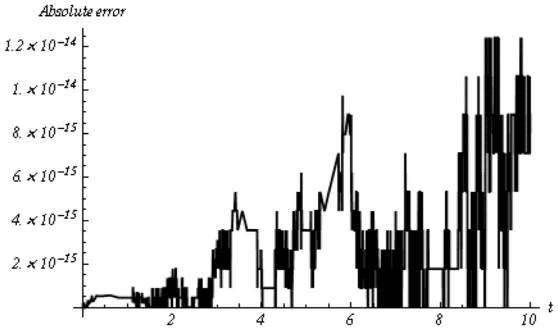
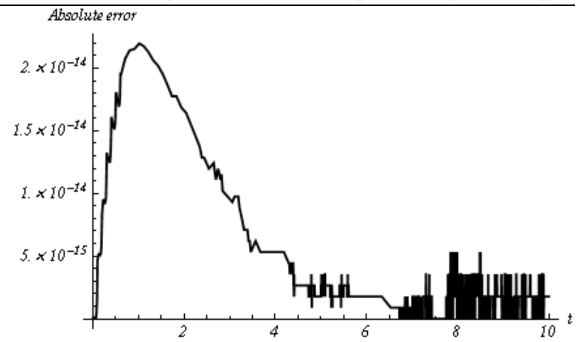
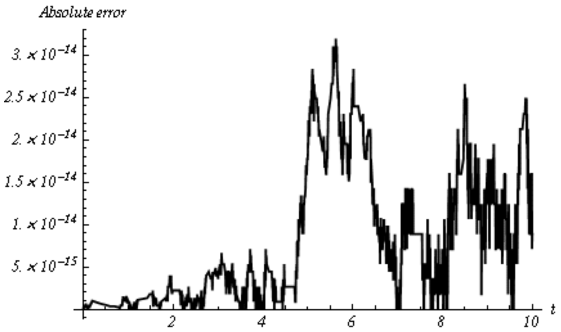
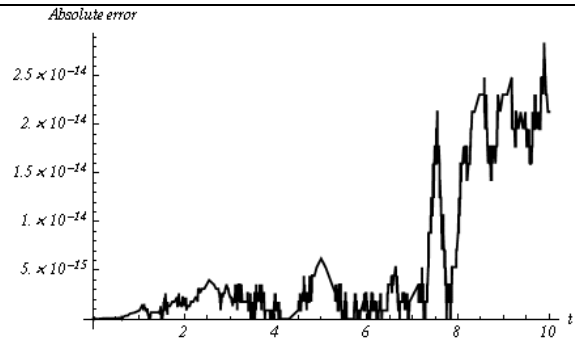
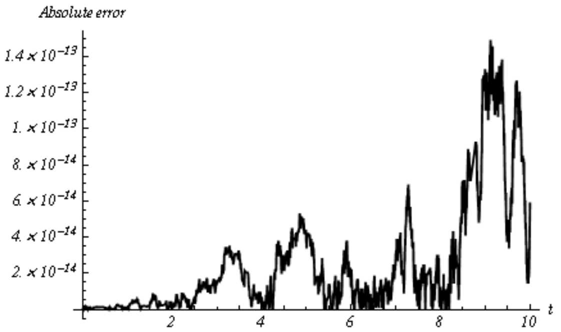
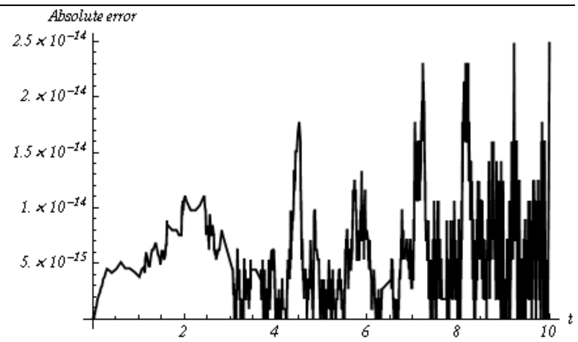


Fig. 3: The exact solution of (23).

Table 3: The absolute error between the exact solution (24) and Padé approximants and Taylor series for different values of h.

h	Padé $\left[\frac{4}{4}\right] O(h^9)$	Truncated Series $O(h^9)$
1		
0.1		
0.01		
.001		

3.3 Case study 3:

Consider the differential equation

$$u'(t) = u - \frac{2+t}{(1+t)^2}, \qquad u(0) = 1. \tag{32}$$

This is a nonlinear problem which has the exact solution

$$u(t)=\frac{1}{1+t}. \tag{33}$$

Defining a differential equation for each subinterval m from (32)

$$\frac{dU_m}{dt}=U_m-\frac{2+t}{(1+t)^2}, \qquad U_m(t_m)=f_m, \qquad t\in[t_m,t_{m+1}]. \tag{34}$$

where $f_m=U_{m-1}(t_m)$, $U_{-1}(t_0)=u(0)=1$, $m=0,1,2,\ldots,n-1$

Substituting by $U_m(t)=\sum_{n=0}^{s}c_n\,(t-t_m)^n$ and its derivatives into (34) leads to

$$\sum_{n=1}^{s}n\,c_n\,(t-t_m)^{n-1}=\left(\sum_{n=0}^{s}c_n\,(t-t_m)^n\right)-\left(\sum_{n=0}^{s}\left(\frac{d^n}{dt^n}\frac{2+t}{(1+t)^2}\right)\Bigg|_{t=t_m}(t-t_m)^n\right),$$
$$c_0=f_m,\ t\in[t_m,t_{m+1}]. \tag{35}$$

Solving (35) leads to:

$$\begin{aligned}
c_0&=f_m,\\
c_1&=-\frac{2}{(1+(t-t_m))^2}-\frac{(t-t_m)}{(1+(t-t_m))^2}+f_m,\\
c_2&=\frac{3}{2(1+(t-t_m))^3}+\frac{(t-t_m)}{2(1+(t-t_m))^3}-\frac{1}{(1+(t-t_m))^2}-\frac{(t-t_m)}{2(1+(t-t_m))^2}+\frac{f_m}{2},\\
c_3&=-\frac{4}{3(1+(t-t_m))^4}-\frac{(t-t_m)}{3(1+(t-t_m))^4}+\frac{1}{2(1+(t-t_m))^3}+\frac{(t-t_m)}{6(1+(t-t_m))^3}-\\
&\quad\frac{1}{3(1+(t-t_m))^2}-\frac{(t-t_m)}{6(1+(t-t_m))^2}+\frac{f_m}{6},\\
c_4&=\frac{5}{4(1+(t-t_m))^5}+\frac{(t-t_m)}{4(1+(t-t_m))^5}-\frac{1}{3(1+(t-t_m))^4}-\frac{(t-t_m)}{12(1+(t-t_m))^4}+\\
&\quad\frac{1}{8(1+(t-t_m))^3}+\frac{(t-t_m)}{24(1+(t-t_m))^3}-\frac{1}{12(1+(t-t_m))^2}-\frac{(t-t_m)}{24(1+(t-t_m))^2}+\frac{f_m}{24},\\
c_5&=-\frac{1}{(1+(t-t_m))^6}-\frac{(t-t_m)}{12(1+(t-t_m))^6}+\frac{21}{20(1+(t-t_m))^5}+\frac{(t-t_m)}{20(1+(t-t_m))^5}-\\
&\quad\frac{1}{15(1+(t-t_m))^4}-\frac{(t-t_m)}{60(1+(t-t_m))^4}+\frac{1}{40(1+(t-t_m))^3}+\frac{(t-t_m)}{120(1+(t-t_m))^3}-\\
&\quad\frac{1}{60(1+(t-t_m))^2}-\frac{(t-t_m)}{120(1+(t-t_m))^2}+\frac{f_m}{120},\\
&\ \ \vdots
\end{aligned} \tag{36}$$

Substituting by (36) into(3) for obtaining the needed approximate analytic Taylor series and substituting by (36) into one of (9)-(13) for obtaining the appropriate p's and q's which are used to obtain Padé approximants. The Padé approximants gives the exact solution in this example because the exact solution of this problem is rational function. Figure 4 shows the exact solution of (33) which is identical with PAM solution using Padé. If we use the PAM series solution, the result is not accepted especially near the pole of the exact solution.

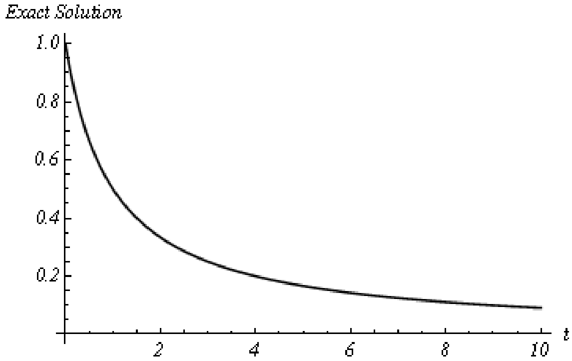
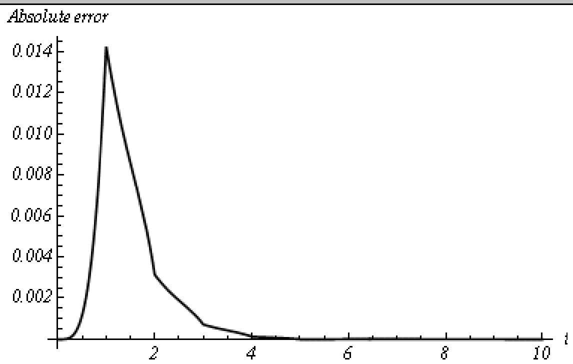
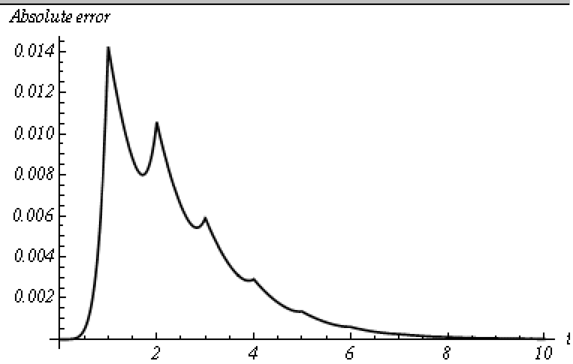
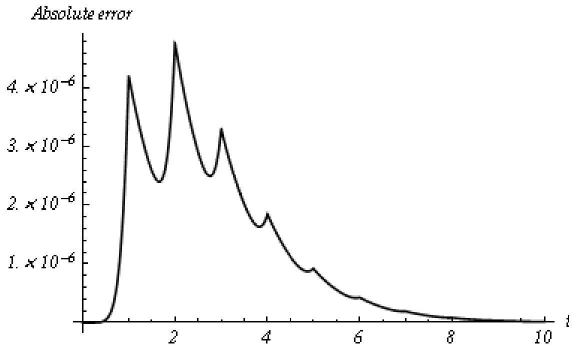
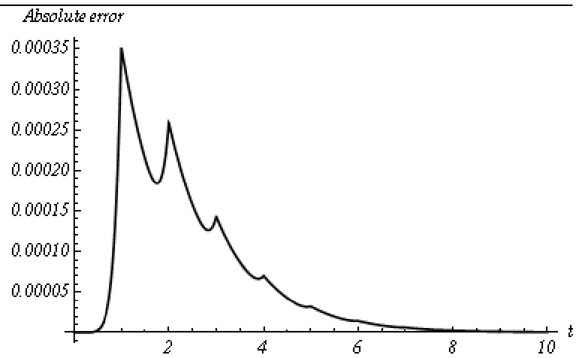
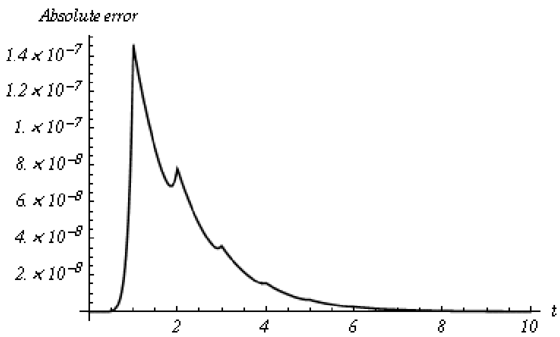
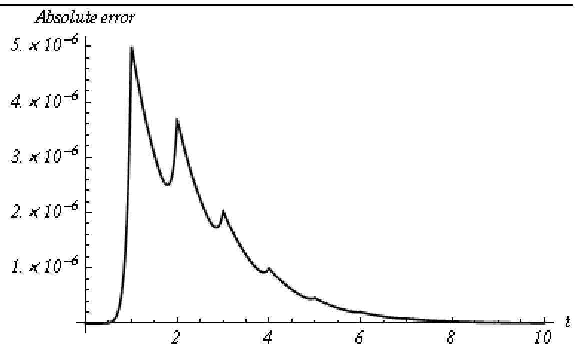
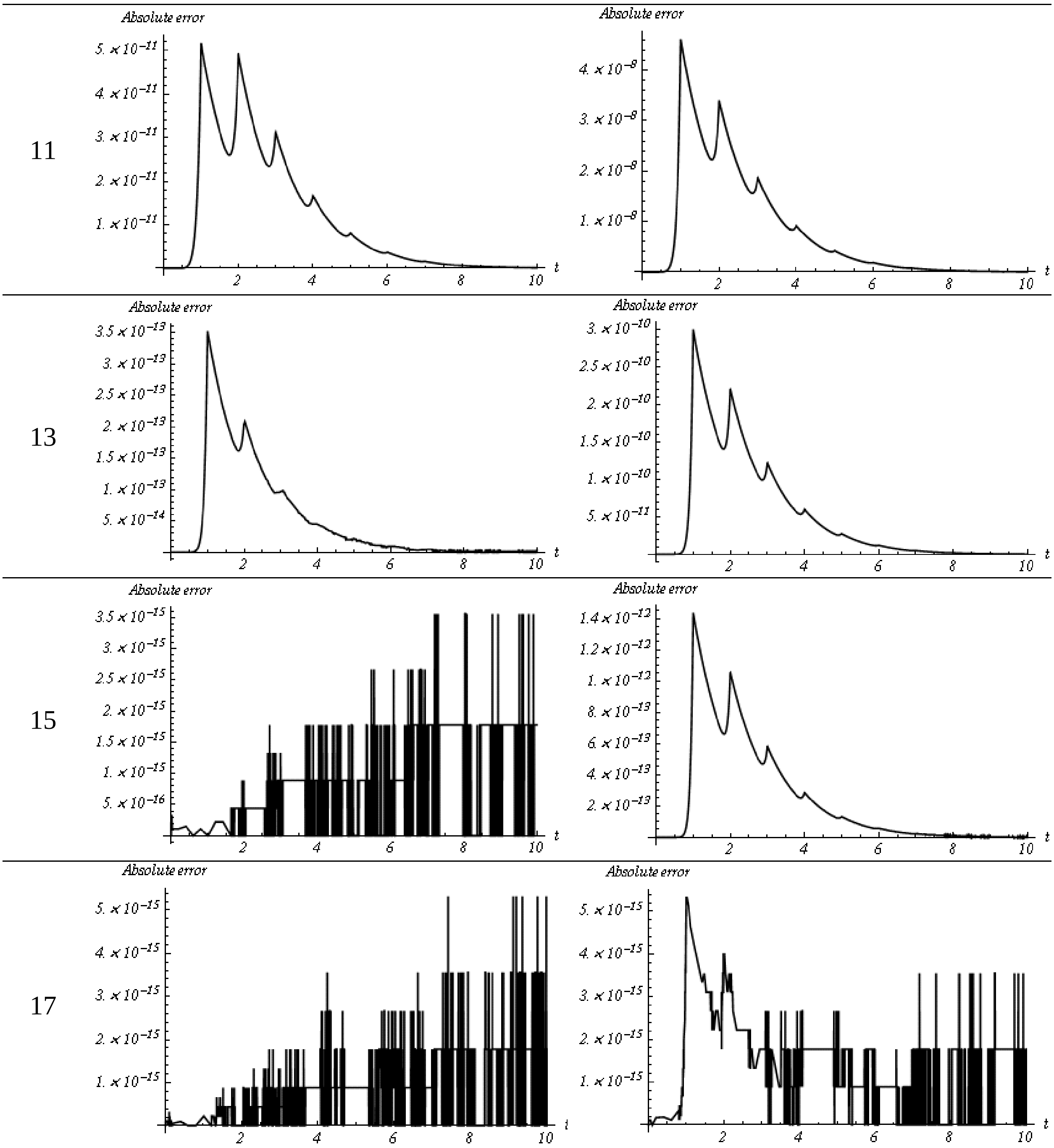


Figure 4: The exact solution of (33).

Table 4: The absolute error between the exact solution (24) and different forms of Padé approximants and Taylor series for $h=1$.

Order of h	Padé [*n/n*]	Truncated Series
5		
7		
9		



3.4 Case study 4 :

Consider the differential equation

$$u'(t) = e^{t}u^{2} - u + e^{-t}, \qquad u(0) = \frac{-1}{a}. \tag{37}$$

This case study is very good, which shows the power of PAM. It is a nonlinear problem which has the exact solution

$$u(t) = -e^{-t}\left(\frac{\cos t - a\sin t}{\sin t + a\cos t}\right). \tag{38}$$

This solution has infinite number of singular points.
Defining a differential equation for each subinterval m from (37)

$$\frac{dU_m}{dt} = e^t U_m{}^2 - U_m + e^{-t}, \qquad U_m(t_m) = f_m, \qquad t \in [t_m, t_{m+1}]. \tag{39}$$

where $f_m = U_{m-1}(t_m)$, $U_{-1}(t_0) = u(0) = \frac{-1}{a}$, $m = 0,1,2,\ldots,n-1$

Substituting by $U_m(t) = \sum_{n=0}^{s} c_n (t - t_m)^n$ and its derivatives into (39) leads to

$$\sum_{n=1}^{s} n c_n (t - t_m)^{n-1} = \left(\sum_{n=0}^{s} \left(\frac{d^n}{dt^n} e^t\right)\Bigg|_{t=t_m} (t - t_m)^n\right)\left(\sum_{n=0}^{s} c_n (t - t_m)^n\right)^2 - \left(\sum_{n=0}^{s} c_n (t - t_m)^n\right) + \left(\sum_{n=0}^{s} \left(\frac{d^n}{dt^n} e^{-t}\right)\Bigg|_{t=t_m} (t - t_m)^n\right), \qquad c_0 = f_m, \quad t \in [t_m, t_{m+1}]. \tag{40}$$

Solving (40) leads to:

$$\begin{aligned}
c_0 &= f_m, \\
c_1 &= e^{-t_m} - f_m + e^{t_m} f_m{}^2, \\
c_2 &= \frac{1}{2}(-2e^{-t_m} + 3f_m - 2e^{t_m} f_m{}^2 + 2e^{2t_m} f_m{}^3), \\
c_3 &= \frac{1}{6}(5e^{-t_m} - 7f_m + 11e^{t_m} f_m{}^2 - 6e^{2t_m} f_m{}^3 + 6e^{3t_m} f_m{}^4), \\
c_4 &= \frac{1}{24}(-12e^{-t_m} + 29f_m - 36e^{t_m} f_m{}^2 + 52e^{2t_m} f_m{}^3 - 24e^{3t_m} f_m{}^4 + 24e^{4t_m} f_m{}^5), \\
c_5 &= \frac{1}{120}(41e^{-t_m} - 101f_m + 221e^{t_m} f_m{}^2 - 220e^{2t_m} f_m{}^3 + 300e^{3t_m} f_m{}^4 - \\
&\quad 120e^{4t_m} f_m{}^5 + 120e^{5t_m} f_m{}^6), \\
c_6 &= \frac{1}{720}(-142e^{-t_m} + 543f_m - 982e^{t_m} f_m{}^2 + 1862e^{2t_m} f_m{}^3 - 1560e^{3t_m} f_m{}^4 + \\
&\quad 2040e^{4t_m} f_m{}^5 - 720e^{5t_m} f_m{}^6 + 720e^{6t_m} f_m{}^7), \\
&\vdots
\end{aligned} \tag{41}$$

Substituting by (41) into(3) for obtaining the needed approximate analytic Taylor series and substituting by (41) into one of (9)-(13) or others for obtaining the appropriate $p's$ and $q's$ which are used to obtain Padé approximants.

The PAM truncated series solution $O(h^5)$ is

$$\begin{aligned}
U_m(t) &\simeq f_m + (e^{-t_m} - f_m + e^{t_m} f_m{}^2)(t - t_m) + \frac{1}{2}(-2e^{-t_m} + 3f_m - 2e^{t_m} f_m{}^2 + \\
&2e^{2t_m} f_m{}^3)(t - t_m)^2 + \frac{1}{6}(5e^{-t_m} - 7f_m + 11e^{t_m} f_m{}^2 - 6e^{2t_m} f_m{}^3 + 6e^{3t_m} f_m{}^4) \\
&(t - t_m)^3 + \frac{1}{24}(-12e^{-t_m} + 29f_m - 36e^{t_m} f_m{}^2 + 52e^{2t_m} f_m{}^3 - 24e^{3t_m} f_m{}^4 + \\
&24e^{4t_m} f_m{}^5)(t - t_m)^4, \qquad t \in [t_m, t_{m-1}].
\end{aligned} \tag{42}$$

The PAM Padé solution $O(h^5)$ is

$$U_m(t) \simeq (24(t-t_m)+24(t-t_m)^2+24e^{t_m}f_m-120(t-t_m)e^{t_m}f_m+58(t-t_m)^2e^{t_m}f_m- 144e^{2t_m}f_m{}^2+126(t-t_m)e^{2t_m}f_m{}^2+26(t-t_m)^2e^{2t_m}f_m{}^2+60e^{3t_m}f_m{}^3-174(t-t_m)e^{3t_m} f_m{}^3+87(t-t_m)^2e^{3t_m}f_m{}^3-24e^{4t_m}f_m{}^4+102(t-t_m)e^{4t_m}f_m{}^4+2(t-t_m)^2e^{4t_m}f_m{}^4+ 24e^{5t_m}f_m{}^5-48(t-t_m)e^{5t_m}f_m{}^5+28(t-t_m)^2e^{5t_m}f_m{}^5)/(24e^{t_m}+48(t-t_m)e^{t_m}+ 28(t-t_m)^2e^{t_m}-144e^{2t_m}f_m-102(t-t_m)e^{2t_m}f_m+2(t-t_m)^2e^{2t_m}f_m+60e^{3t_m}f_m{}^2+ 174(t-t_m)e^{3t_m}f_m{}^2+87(t-t_m)^2e^{3t_m}f_m{}^2-144e^{4t_m}f_m{}^3-126(t-t_m)e^{4t_m}f_m{}^3+26(t- t_m)^2e^{4t_m}f_m{}^3+24e^{5t_m}f_m{}^4+120(t-t_m)e^{5t_m}f_m{}^4+58(t-t_m)^2e^{5t_m}f_m{}^4-24(t- t_m)e^{6t_m}f_m{}^5+24(t-t_m)^2e^{6t_m}f_m{}^5), \qquad t\in[t_m,t_{m-1}].$$

It is massive to write the obtained solution $O(h^9)$ but the results are summarized in the following figures. Figure 5 shows the exact solution of (37). Table 5 shows the absolute error between (37) exact solution and Padé approximants and Taylor series for different values of h . Table 6 shows the absolute error between (37) exact solution and different forms of Padé approximants and Taylor series for $h = 0.1$.

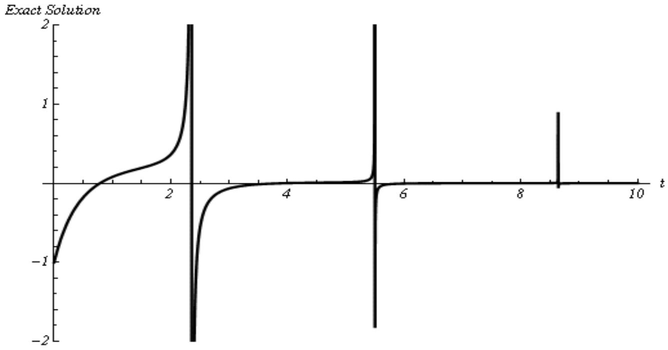
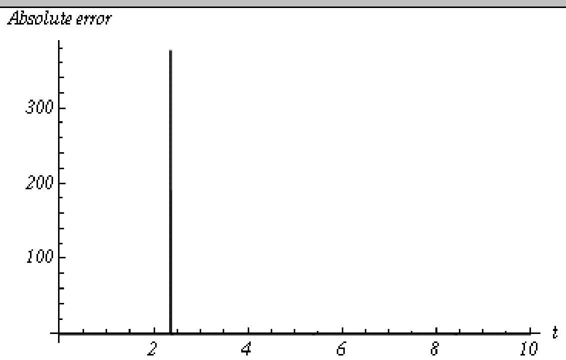
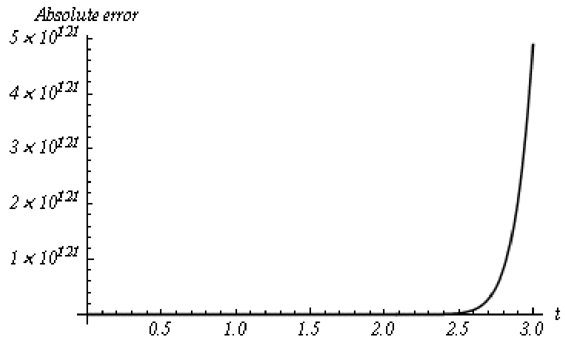
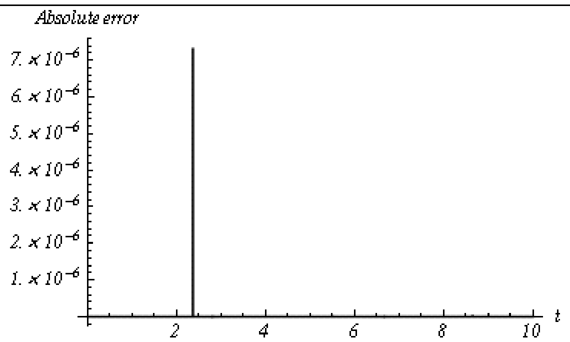
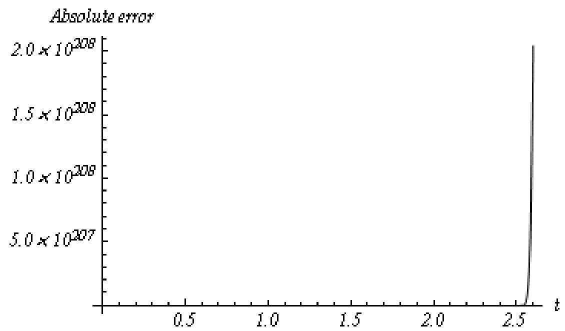
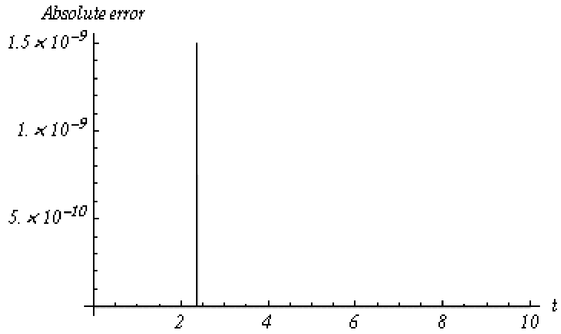


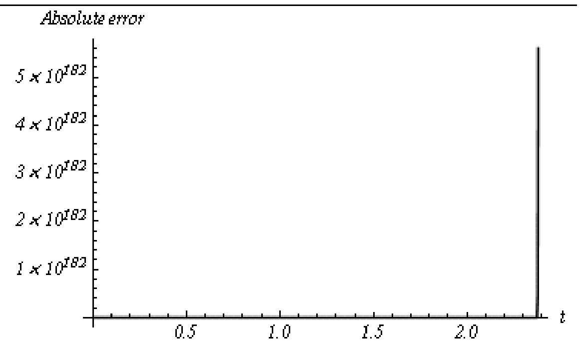
Figure 5: The exact solution of (37).

Table 5: The absolute error between the exact solution (38) and Padé approximants and Taylor series for different values of h .

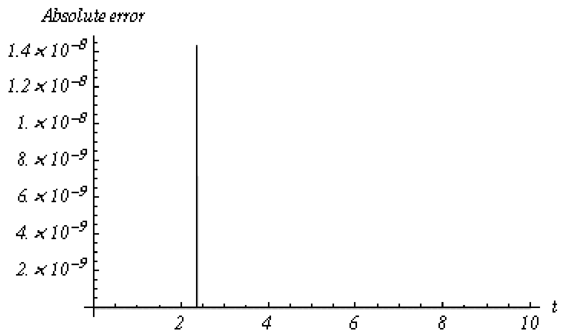
h	Padé $\left[4/4\right] O(h^9)$	Truncated Series $O(h^9)$
1		
0.1		

0.01





.001



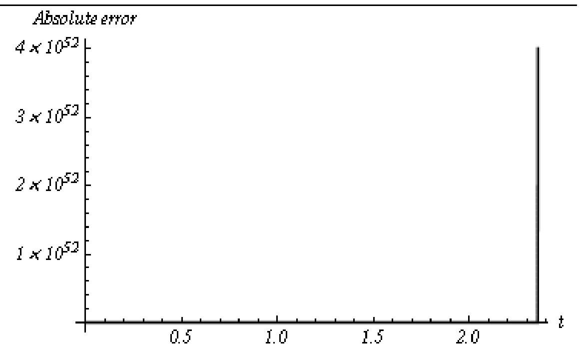
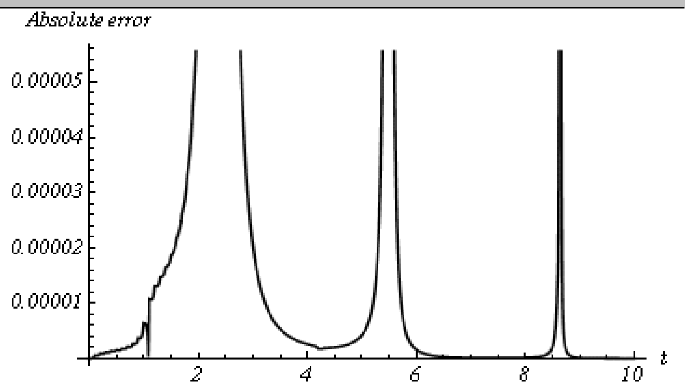
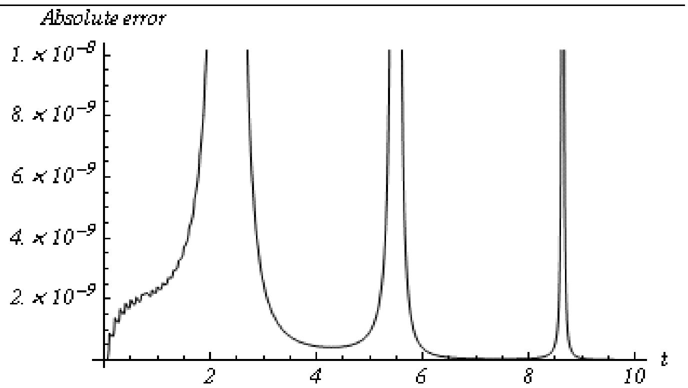
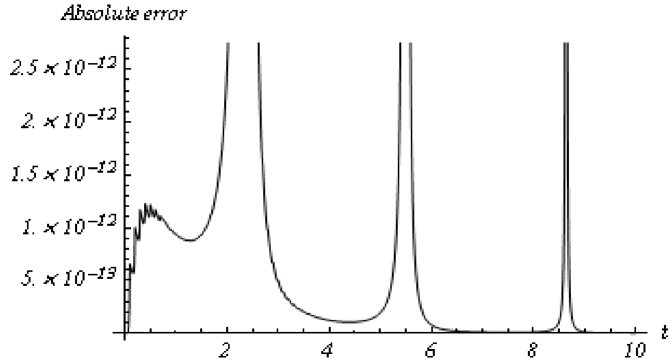


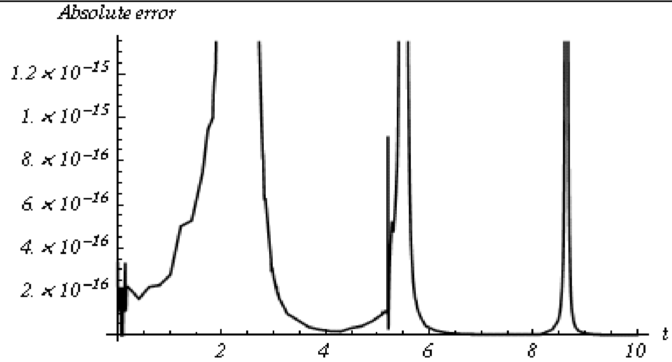
Table 6: The absolute error between the exact solution (38) and different forms of Padé approximants and Taylor series for $h = 0.1$.

Order of h	Padé [n/n]
5	
7	

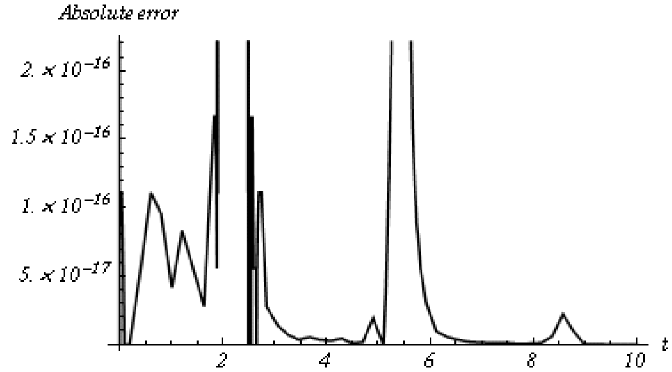
9



11



13



4 Error estimation and changing the initial condition

Now, unlike the above case-studies, PAM will be applied only when the exact solution is not known. Thus, in practice the error at each interval points will not be known. It is essential then to know, a priori, that the PAM error at each interval points decrease to zero as h decreases to zero or the accuracy order increases to infinity. Of course, it does not make sense to apply a zero interval size or infinity order of accuracy to PAM, but the point is that we can make the error as small as we wish by selecting h sufficiently small or the order of accuracy sufficiently high. That this is correct when all calculations are exact will be established next. In this case the truncation error in computing $U_m(t)$ using PAM is bounded in direct proportion with respect to the interval size h and order of accuracy which takes us to PAM is convergent.

Turning now to the computation of PAM solution as described by the algorithm, we first specify values for the parameters of the problem and the initial data. Because, in any practical computing device, the number of digits allocated to a number is limited, it will probably be necessary to chop or round these numbers before they are stored. The error committed by doing this is called inherent roundoff. Also, during the computation, arithmetic operations are performed that produce results with more digits than the operands, and these results must be chopped or rounded before they are stored. This error is called arithmetic roundoff [21]. At the present time there is no universally accepted method to analyze roundoff error after a large number of time steps. The three main methods for analyzing roundoff accumulation are the analytical method, the probabilistic method [22] and the interval arithmetic method [23, 24], each of which has both advantages and disadvantages.

If we follow the results in Tables 1-6, we can observe that the absolute error is reduced as h is decreased or the order of convergence accuracy is increased. The absolute error is not reduced indefinitely as a result of the roundoff error.

In practice, what will we do if we solve problems and don't know its exact solution or needs to change its parameters or initial conditions? If one wishes arbitrarily high accuracy, one need only choose h sufficiently small or large order of accuracy. If one has a prescribed accuracy, it is often estimated in an a posteriori manner as follows. One calculates for both h and smaller h and takes those figures which are in agreement for the two calculations. For example, if at a point t and for h = 0.1 one finds U = 0.876 532 while for h = 0.01 one finds at the same point that U = 0.876 513, then one assumes that the result U = 0.8765 is an accurate result.

In the following, I'll use the notation $U_m^{[h,m]}(t)$ for denoting the PAM solution with step size h and order of accuracy m $o(h^m)$.

If we take case study 2 for example, Table 7 shows the difference between two PAM solutions for two different values of h, fixing the accuracy order, which indicates that the accuracy is increased as the step size h is reduced. Table **8** and **Table 9** show the difference between two PAM solutions for two different order of accuracy, fixing the step size, which indicates that the accuracy is increased as the order is increased.

Table 7: The difference between two PAM solution as h is changed.

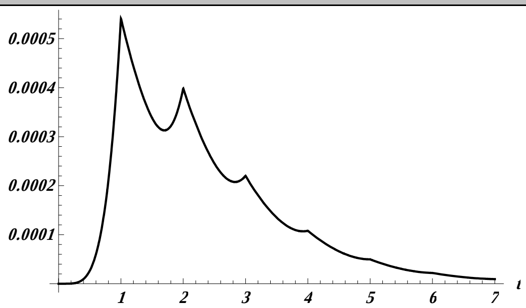
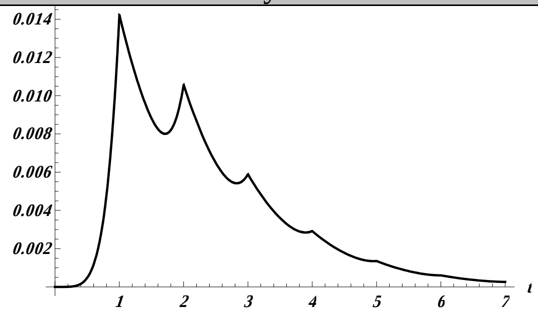
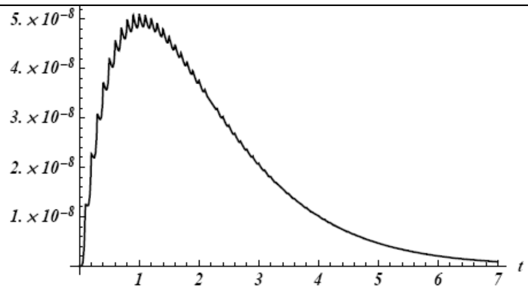
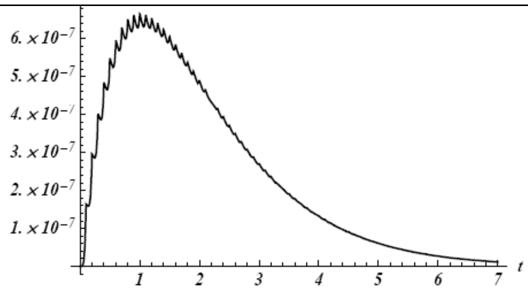
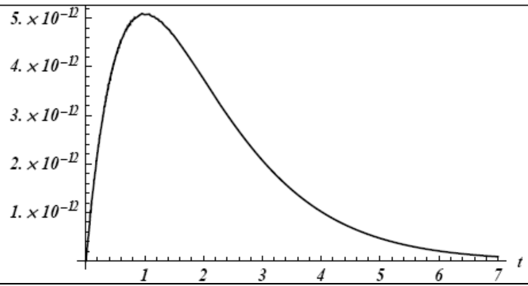
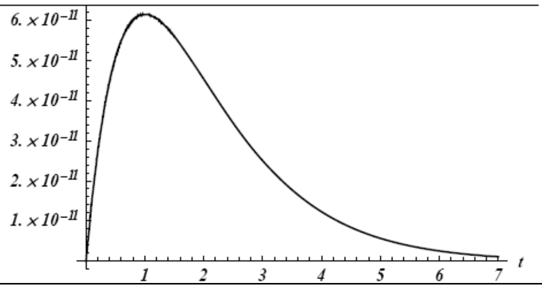
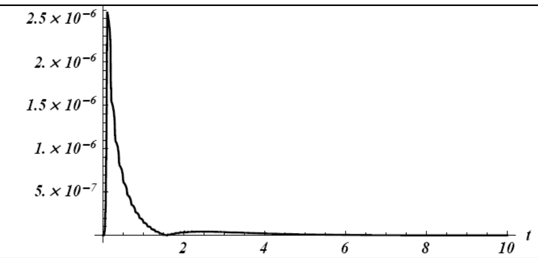
	Padé	Taylor
$\left\|U_m^{[1,5]} - U_m^{[0.1,5]}\right\|$		
$\left\|U_m^{[0.1,5]} - U_m^{[0.01,5]}\right\|$		
$\left\|U_m^{[0.01,5]} - U_m^{[0.001,5]}\right\|$		

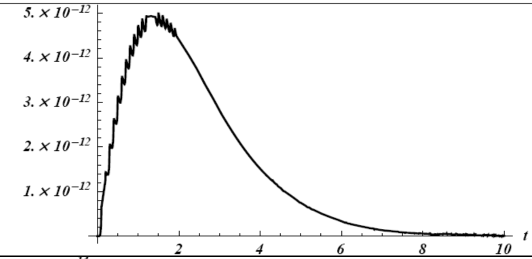
Table 8: The difference between two PAM Padé solution as order of accuracy is changed.

Padé

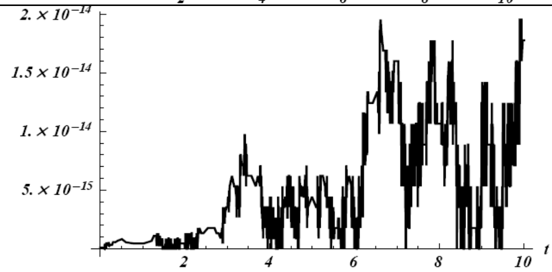
$\left|U_m^{[0.1,5]} - U_m^{[0.1,7]}\right|$



$\left|U_m^{[0.1,7]} - U_m^{[0.1,9]}\right|$



$\left|U_m^{[0.1,9]} - U_m^{[0.1,11]}\right|$



$\left|U_m^{[0.1,11]} - U_m^{[0.1,13]}\right|$

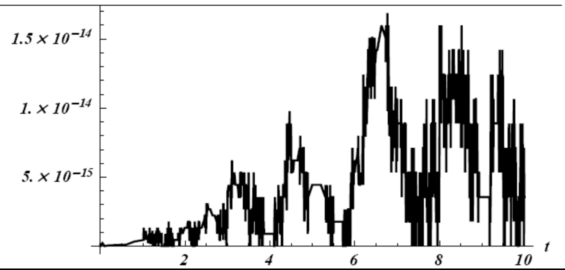
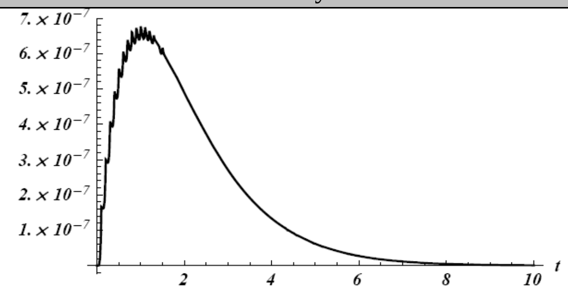


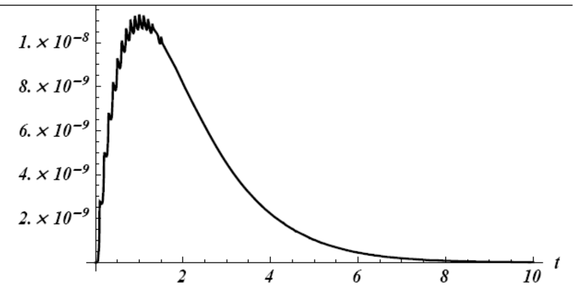
Table 9: The difference between two PAM Taylor series solution as order of accuracy is changed.

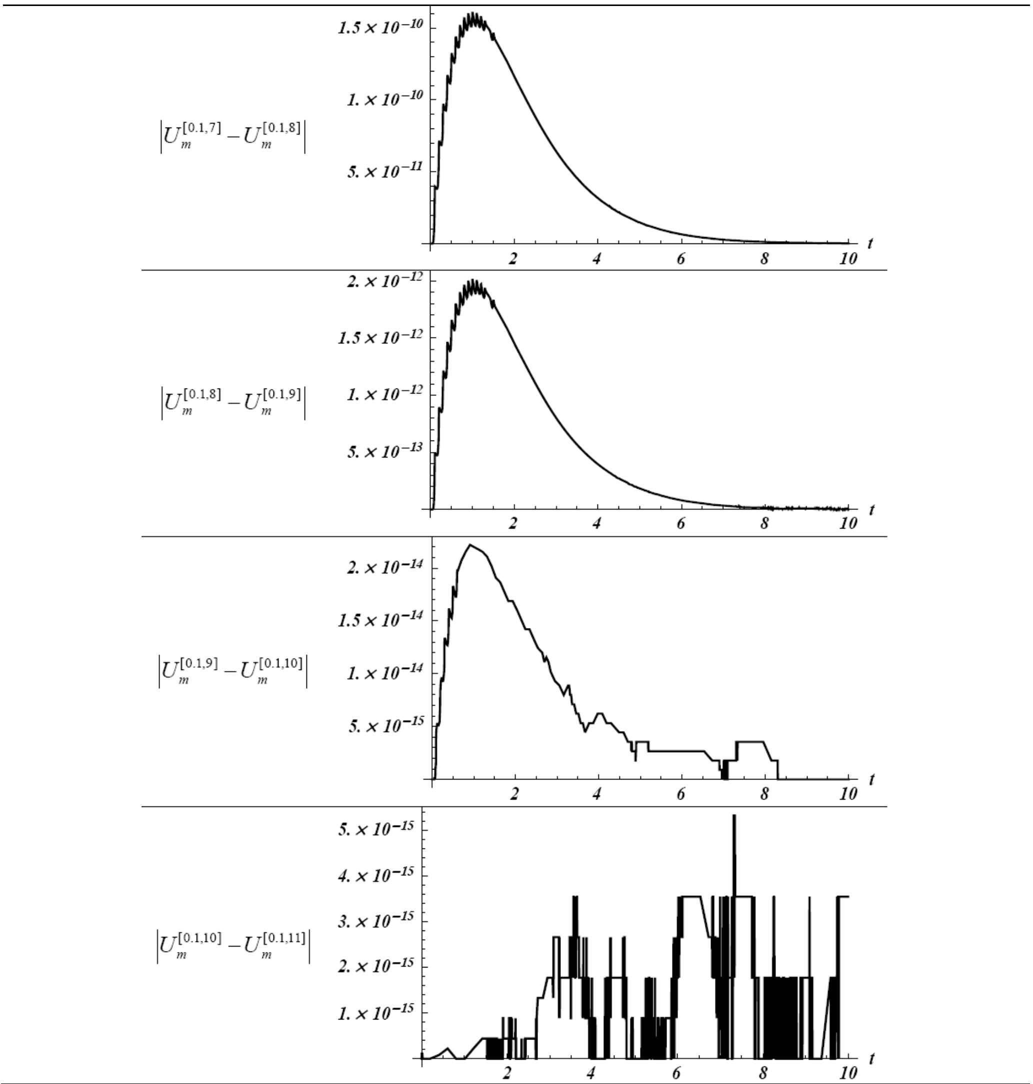
Taylor

$\left|U_m^{[0.1,5]} - U_m^{[0.1,6]}\right|$



$\left|U_m^{[0.1,6]} - U_m^{[0.1,7]}\right|$





5 Comparing PAM with Runge-Kutta method

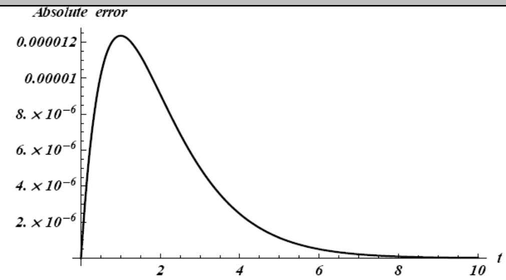
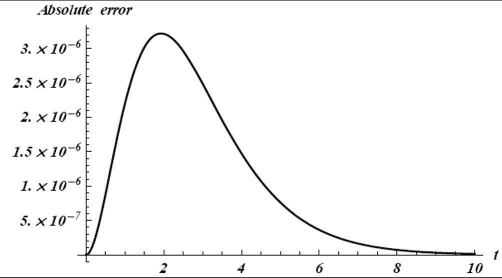
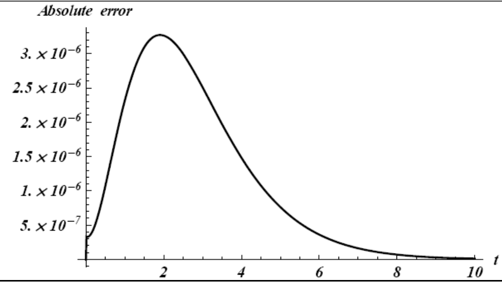
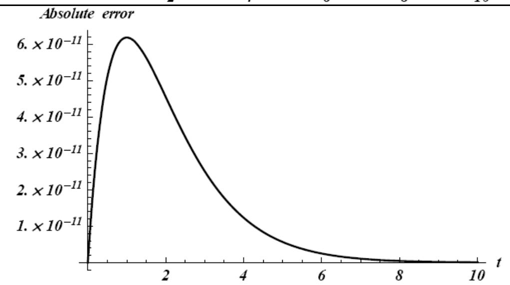
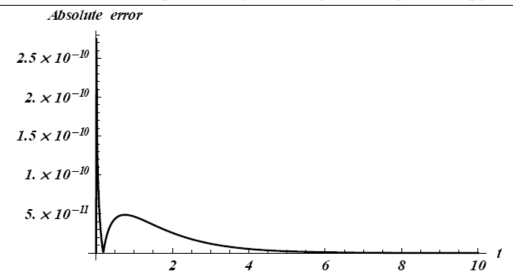
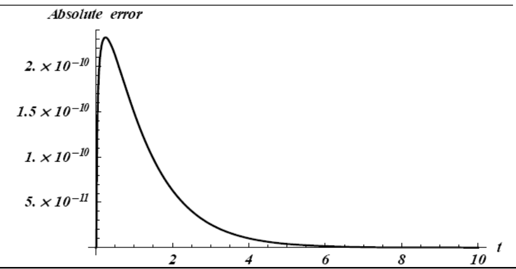
If we try to solve the same case studies with Runge-Kutta, it is founded that Runge-Kutta doesn't give accepted results in the case studies 1, 3 and 4 because of the singular points in the solution. Case study 2 can be solved by Runge-Kutta. Table 10 shows figures of the absolute error using Runge-Kutta and PAM with different order of accuracy.

PAM has no order limit. PAM gives an analytic solution form which can be used for analytic differentiation and integration. In the other side, Runge-Kutta gives only numerical values at limited points of the interval.

6 Conclusion

The piecewise analytic method is promising method. PAM can be used for solving any ordinary differential equation with any order of accuracy. PAM is very easy to use, the main effort in PAM in calculating $U_m(t)$, which is now very easy to calculate because of the symbolic mathematical software. For non-mathematician, they can now test their equations with any initial condition.

Table 10: The absolute error between the exact solution and different methods with different order of accuracy.

Method	Absolute Error
RK2-Euler-Cauchy	
PAM Padé[1/1] $o(h^2)$	
PAM series $o(h^2)$	
RK4-Classical	
PAM Padé [2/2] $o(h^4)$	
PAM series $o(h^4)$	

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