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Research paper

Existence and uniqueness of solutions for nonlinear hyperbolic fractional differential equation with integral boundary conditions

Brahim Tellab [1]*, Kamel Haouam [2]

[1] *Department of Mathematics, Ouargla University 30000 Ouargla, Algeria*
[2] *Mathematics and Informatics Department, LAMIS Laboratory, Tebessa University, 12000 Tebessa, Algeria*
**Corresponding author E-mail: brahimtel@yahoo.fr*

Abstract

In this paper, we investigate the existence and uniqueness of solutions for second order nonlinear fractional differential equation with integral boundary conditions. Our result is an application of the Banach contraction principle and the Krasnoselskii fixed point theorem.

Keywords: *Fractional Derivatives; Contraction Principle; Fixed Point Theorem; Integral Equation.*

1. Introduction

Fractional differential equations have been of great interest and attracted many researchers in recent years; this is due to the development of the above cited concept. It have found applications in several different disciplines as physics, engineering, economics, electrochemistry, electromagnetism etc. (See [3, 4, 9, 13, 20]).
Such equation have recently proved to be valuable tools in modelling of many phenomena. (See papers [2, 7, 12, 14, 16, 20]).
In [10], Benchohra and Ouaar discussed the existence of solutions to the boundary value problem:

$$^{C}D^{\alpha}y(t)=f(t,y(t)),\ t\in J=\left[0,T\right],\ \alpha\in\left(0,1\right], \tag{1}$$

$$y(0)=\mu\int_0^T y(s)ds=y(T), \tag{2}$$

$^{C}D^{\alpha}$ is the Caputo fractional derivative $f:J\times R\to R$ is continuous function and $\mu\in R^{*}$.
In [8], Sotoris K. Ntouyas investigated the existence and uniqueness of solution of the following problem:

$$^{C}D^{q}x(t)=f(t,x(t)),\ 0\prec t\prec 1,\ 0\prec q\le 1, \tag{3}$$

$$x(0)=\alpha I^{p}x(\eta),\ 0\prec\eta\prec 1, \tag{4}$$

$^{C}D^{q}$ denotes always the Caputo fractional derivative of order q, $f:\left[0,1\right]\times R\to R$ is continuous function, $\alpha\in R$ such that $\alpha\neq\Gamma(p+1)/\eta^{p}$, Γ is the Euler function and I^{p}, $0\prec p\prec 1$ is the Riemann-Liouville fractional integral of order p.
In this paper, we consider the following nonlinear fractional differential equation with integral boundary conditions:

$$^{C}D^{\alpha}y(t)=f(t,y(t)),\ t\in J=\left[0,1\right] \tag{5}$$

$$y(0)=\int_0^1 y(s)ds \tag{6}$$

$$y(1)=\frac{1}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}y(s)ds \tag{7}$$

Where $^{C}D^{\alpha}$ is the Caputo fractional derivative of order α, $1\prec\alpha\le 2$, $0\prec\beta\le 1$ and $f:\left[0,1\right]\times R\to R$ is continuous function

2. Preliminaries

Now, we present some basic definitions and lemmas of fractional calculus which will be used in our theorems [1], [4], and [18].

Definition 2.1: *For a differentiable function h: [0, +∞) → R, the Caputo derivative of fractional order α is defined by*

$$^{C}D^{\alpha}h(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}h^{(n)}(s)ds,$$

$$n-1\prec\alpha\prec n,\quad n=\left[\alpha\right]+1,$$

Where $\left[\alpha\right]$ denotes the integer part of α and Γ is the gamma function.

Definition 2.2: *The Riemann-Liouville fractional integral of order α is given by*

$$I^{\alpha}h(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(s)ds,$$

Where $h: R^+ \to R$ is a Lebesgue measurable function, provided the integral exists?

Lemma 2.1: *[19] Let* $\alpha \succ 0$, *then the differential equation* ${}^C D^\alpha h(t) = 0$ *has solutions*

$$h(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1},\ c_i \in R, i = 0,1,2,\dots,n-1, n = [\alpha]+1.$$

Lemma 2.2: *[19] Let* $\alpha \succ 0$, *then*

$$I^\alpha\, {}^C D^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1},$$

Where $c_i \in R, i = 0,1,2,\dots,n-1, n = [\alpha]+1$.

Lemma 2.3: *Let* $1 \prec \alpha \le 2$ *and let* $h: J \times R \to R$ *be a given continuous function. Then, the boundary-value problem*

$${}^C D^\alpha y(t) = h(t),\quad t \in J \tag{8}$$

$$y(0) = \int_0^1 y(s)ds \tag{9}$$

$$y(1) = \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} y(s)ds \tag{10}$$

has a unique solution defined by:

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds +$$

$$\int_0^1 \left[\frac{1}{\gamma_1 \Gamma(\alpha)\Gamma(\beta)} \int_s^1 (1-r)^{\beta-1}(r-s)^{\alpha-1} dr - \right.$$

$$\left. \frac{(1-s)^{\alpha-1}}{\gamma_1 \Gamma(\alpha)} + \left(\frac{2\gamma_2}{\gamma_1 \alpha \Gamma(\alpha)} - \frac{2t}{\alpha\Gamma(\alpha)} \right)(1-s)^\alpha \right] h(s)ds$$

Where

$$\gamma_1 = 1 - \frac{1}{\Gamma(\beta+1)},\quad \gamma_2 = 1 - \frac{1}{\Gamma(\beta+2)}$$

Proof of lemma 2.3 Applying lemma 2.2, we can reduce the problem (8)-(10) to an equivalent integral equation

$$y(t) = I_0^\alpha h(t) + c_0 + c_1 t$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds + c_0 + c_1 t, \tag{11}$$

for some constants $c_1, c_2 \in R$. By integrating and using Fubini's theorem, we obtain

$$\int_0^1 y(s)ds = \int_0^1 \frac{(1-\tau)^\alpha}{\alpha\Gamma(\alpha)} h(\tau)d\tau + c_0 + \frac{c_1}{2}. \tag{12}$$

Applying (9), we find $y(0) = c_0$,
and with (12), we arrive to

$$y(0) = \int_0^1 \frac{(1-\tau)^\alpha}{\alpha\Gamma(\alpha)} h(\tau)d\tau + c_0 + \frac{c_1}{2},$$

then,

$$c_1 = -2\int_0^1 \frac{(1-\tau)^\alpha}{\alpha\Gamma(\alpha)} h(\tau)d\tau. \tag{13}$$

Now, we operate (10) and (11), we get

$$\frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} y(s)ds$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \int_0^s (1-s)^{\beta-1}(s-r)^{\alpha-1} h(r)drds$$

$$+ \frac{c_0}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} ds + \frac{c_1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} s ds,$$

that is to say:

$$\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s)ds + c_0 + c_1$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \int_0^s (1-s)^{\beta-1}(s-r)^{\alpha-1} h(r)drds$$

$$+ \frac{c_0}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} ds + \frac{c_1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} s ds,$$

after the simplification we obtain:

$$\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s)ds + c_0 + c_1$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \int_s^1 (1-r)^{\beta-1}(r-s)^{\alpha-1} h(s)drds$$

$$+ \frac{c_0}{\Gamma(\beta+1)} + \frac{c_1}{\Gamma(\beta+2)},$$

which may be written,

$$\left(1 - \frac{1}{\Gamma(\beta+1)}\right) c_0 + \left(1 - \frac{1}{\Gamma(\beta+2)}\right) c_1$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \int_s^1 (1-r)^{\beta-1}(r-s)^{\alpha-1} h(s)drds$$

$$- \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s)ds \tag{14}$$

In putting $\gamma_1 = 1 - \frac{1}{\Gamma(\beta+1)}$, $\gamma_2 = 1 - \frac{1}{\Gamma(\beta+2)}$,
(14) becomes:

$$\gamma_1 c_0 + \gamma_2 c_1$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \int_s^1 (1-r)^{\beta-1}(r-s)^{\alpha-1} h(s)drds - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s)ds.$$

Using (13), we obtain

$$c_0 = \frac{1}{\gamma_1 \Gamma(\alpha)\Gamma(\beta)} \int_0^1 \int_s^1 (1-r)^{\beta-1}(r-s)^{\alpha-1} h(s)drds$$

$$- \frac{1}{\gamma_1 \Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s)ds + \frac{2\gamma_2}{\gamma_1 \alpha\Gamma(\alpha)} \int_0^1 (1-s)^\alpha h(s)ds. \tag{15}$$

A combinaison of (11), (13) and (15) leads to

$$y(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s)ds$$

$$+\frac{1}{\gamma_1\Gamma(\alpha)\Gamma(\beta)}\int_0^1\int_s^1 (1-r)^{\beta-1}(r-s)^{\alpha-1}h(s)drds$$

$$-\frac{1}{\gamma_1\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} h(s)ds$$

$$+\left[\frac{2\gamma_2}{\gamma_1\alpha\Gamma(\alpha)}-\frac{2t}{\alpha\Gamma(\alpha)}\right]\int_0^1 (1-s)^{\alpha} h(s)ds,$$

i.e.,

$$y(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds+$$

$$\int_0^1\left[\frac{1}{\gamma_1\Gamma(\alpha)\Gamma(\beta)}\int_s^1 (1-r)^{\beta-1}(r-s)^{\alpha-1}dr-\right.$$

$$\left.\frac{(1-s)^{\alpha-1}}{\gamma_1\Gamma(\alpha)}+\left(\frac{2\gamma_2}{\gamma_1\alpha\Gamma(\alpha)}-\frac{2t}{\alpha\Gamma(\alpha)}\right)(1-s)^{\alpha}\right]h(s)ds \quad \blacksquare$$

3. Existence and uniqueness result

Theorem 3.1: *(Fixed point theorem of Banach) [6] Let X a Banach space and $T: X \to X$ a contracting mapping. Then T has a unique fixed point i.e.*

$\exists! x \in X: \; Tx = x.$

Our first result of existence is based on theorem of Banach contracting application

Theorem 3.2: *Suppose that the function $f: [0,1]\times R \to R$ is continuous and there is a constant $L \succ 0$ such that:*

$(H_1): |f(t,x)-f(t,y)| \le L|x-y|, t\in[0,1], \; x,y\in R.$

If $LA \prec 1$, then the boundary value problem (5)-(7) has a unique solution, where

$$A=\frac{1}{\Gamma(\alpha+1)}+\frac{\mathrm{B}(\beta,\alpha)}{|\gamma_1|(\alpha+\beta)\Gamma(\alpha)\Gamma(\beta)}+\frac{1}{|\gamma_1|\Gamma(\alpha+1)}$$

$$+\frac{2|\gamma_2|}{|\gamma_1|\Gamma(\alpha+2)}+\frac{2}{\Gamma(\alpha+2)} \tag{16}$$

Proof of theorem 3.2

We define the operator F, by:

$$(Fy)(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,y(s))ds$$

$$+\int_0^1\left[\frac{1}{\gamma_1\Gamma(\alpha)\Gamma(\beta)}\int_s^1 (1-r)^{\beta-1}(r-s)^{\alpha-1}dr-\frac{(1-s)^{\alpha-1}}{\gamma_1\Gamma(\alpha)}\right.$$

$$\left.+\left(\frac{2\gamma_2}{\gamma_1\alpha\Gamma(\alpha)}-\frac{2t}{\alpha\Gamma(\alpha)}\right)(1-s)^{\alpha}\right] f(s,y(s))ds, \quad t\in[0,1]. \tag{17}$$

If we put $\sup_{t\in[0,1]} |f(t,0)| = M$ we show that $FB_\rho \subset B_\rho$, where B

$B_\rho = \{y \in C([0,1],R): \|y\| \le \rho\}$ and $\rho \ge \frac{MA}{1-LA}$.

Let $y\in B_\rho$, $t\in[0,1]$, this leads to write:

$$\|(Fy)(t)\| \le \sup_{t\in[0,1]}\left\{\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,y(s))ds\right.$$

$$+\frac{1}{|\gamma_1|\Gamma(\alpha)\Gamma(\beta)}\int_0^1\int_s^1 (1-r)^{\beta-1}(r-s)^{\alpha-1} f(s,y(s))drds$$

$$+\frac{1}{|\gamma_1|\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f(s,y(s))ds$$

$$+\frac{2|\gamma_2|}{|\gamma_1|\alpha\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha} f(s,y(s))ds$$

$$\left.+\frac{2t}{\alpha\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha} f(s,y(s))ds\right\} \le$$

$$\sup_{t\in[0,1]}\left\{\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(|f(s,y(s))-f(s,0)|+|f(s,0)|\right)ds\right.$$

$$+\frac{1}{|\gamma_1|\Gamma(\alpha)\Gamma(\beta)}\int_0^1\int_s^1 (1-r)^{\beta-1}(r-s)^{\alpha-1}\times$$

$$\left(|f(s,y(s))-f(s,0)|+|f(s,0)|\right)drds$$

$$+\frac{1}{|\gamma_1|\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left(|f(s,y(s))-f(s,0)|\right.$$

$$\left.+|f(s,0)|\right)ds+\frac{2|\gamma_2|}{|\gamma_1|\alpha\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha}\times$$

$$\left(|f(s,y(s))-f(s,0)|+|f(s,0)|\right)ds+\frac{2}{\alpha\Gamma(\alpha)}\times$$

$$\left.\int_0^1 (1-s)^{\alpha}\left(|f(s,y(s))-f(s,0)|+|f(s,0)|\right)ds\right\}. \tag{18}$$

To put $u=\frac{r-s}{1-r}$, i.e.,

$1-r=(1-u)(1-s), \quad dr=(1-s)du$

we obtain:

$$\int_0^1\int_s^1 (1-r)^{\beta-1}(r-s)^{\alpha-1}drds$$

$$=\int_0^1 (1-s)^{\alpha+\beta-1} ds\int_0^1 (1-u)^{\beta-1}u^{\alpha-1}du$$

$$=\frac{\mathrm{B}(\beta,\alpha)}{\alpha+\beta}. \tag{19}$$

By substitution in (18), and after simplification, we get

$$\|(Fy)(t)\| \le (L\rho+M)\left\{\frac{1}{\Gamma(\alpha+1)}+\right.$$

$$\frac{B(\beta,\alpha)}{|\gamma_1|(\alpha+\beta)\Gamma(\alpha)\Gamma(\beta)}+\frac{1}{|\gamma_1|\Gamma(\alpha+1)}+$$

$$\left.\frac{2|\gamma_2|}{|\gamma_1|\Gamma(\alpha+2)}+\frac{2}{\Gamma(\alpha+2)}\right\}$$

$$\le (L\rho+M)A \le \rho, \tag{20}$$

which means that $FB_\rho \subset B_\rho$. (21)

Now, suppose that $x, y \in C([0,1], R)$ and $t \in [0,1]$, then we have:

$$\|(Fx)-(Fy)\|$$

$$\le \sup_{t\in[0,1]}\left\{\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f(s,x(s))-f(s,y(s))\right|ds\right.$$

$$+\frac{1}{|\gamma_1|\Gamma(\alpha)\Gamma(\beta)}\int_0^1\int_s^1 (1-r)^{\beta-1}(r-s)^{\alpha-1}\times$$

$$|f(s,x(s))-f(s,y(s))|drds$$

$$+\frac{1}{|\gamma_1|\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}|f(s,x(s))-f(s,y(s))|ds$$

$$+\frac{2|\gamma_2|}{|\gamma_1|\alpha\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha}|f(s,x(s))-f(s,y(s))|ds$$

$$\left.+\frac{2}{\alpha\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha}|f(s,x(s))-f(s,y(s))|ds\right\}$$

$$\le L\|x-y\|\left\{\frac{1}{\Gamma(\alpha+1)}+\frac{B(\beta,\alpha)}{|\gamma_1|(\alpha+\beta)\Gamma(\alpha)\Gamma(\beta)}+\right.$$

$$\left.\frac{1}{|\gamma_1|\Gamma(\alpha+1)}+\frac{2|\gamma_2|}{|\gamma_1|\Gamma(\alpha+2)}+\frac{2}{\Gamma(\alpha+2)}\right\}$$

$$= LA\|x-y\|.$$

Since we assumed $0 \prec LA \prec 1$, then F is a contraction. Using the principle of Banach fixed point; we deduce that the problem (5)-(7) has a unique solution. ■

Example 3.1 Consider the following boundary problem:

$${}^{C}D^{3/2}y(t)=\frac{1}{t^2+4}\times\frac{|x|}{1+|x|}+t\cos^2 t, \quad t\in[0,1] \tag{22}$$

$$y(0)=\int_0^1 y(s)ds \tag{23}$$

$$y(1)=\frac{1}{\Gamma(1/2)}\int_0^1 (1-s)^{-1/2}y(s)ds. \tag{24}$$

In this example, $\alpha = 3/2$, $\beta = 1/2$ and

$$f(t,x)=\frac{1}{t^2+4}\times\frac{|x|}{1+|x|}+t\cos^2 t.$$

We have

$$|f(t,x)-f(t,y)|=\frac{1}{t^2+4}\times\frac{\left||x|-|y|\right|}{(1+|x|)(1+|y|)}$$

$$\le 1/4|x-y|,$$

then, $L = 1/4$.

By a simple calculation, we find: $LA = 0.3715... \prec 1$, and by theorem 3.1 we deduce that the problem (22)-(24) has a unique solution.

Theorem 3.3: *(Arzela-Ascoli's theorem) [5]*

Let $A \subset C(K, R^n)$, $(K = [a,b] \subset R)$. A is relatively compact (i.e, $\overline{A}$ is compact) if and if:

1) A is uniformly bounded.
2) A is equicontinuous .

- Recall that a function f is uniformly bounded in A if there exists a constant $M \succ 0$ such that:

$$\|f\| = \sup_{x\in K}|f(x)| \le M, \quad \forall f \in A$$

Theorem 3.4: *(Fixed point theorem of Krasnoselskii) [15]*

Let F a non-empty set, closed and convex in a Banach space X. T_1 and T_2 are two applications of F in X such that

1) $T_1(x)+T_2(y) \in F, \quad \forall x, y \in F,$
2) T_1 is a contraction,
3) T_2 is compact and continuous.

Then, T_1+T_2 has a fixed point in F, i.e, there exists $x \in F$ such that $T_1(x)+T_2(x)=x.$

The second existence result is obtained by using the fixed point theorem of Krasnoselskii.

Theorem 3.5: *Let $f:[0,1]\times R \to R$ be a continuous function which that satisfies the conditions (H_1) and (H_2): $|f(t,x)| \le \mu(t)$, $\forall (t,x) \in [0,1]$ and $\mu \in C([0,1], R^+)$. Suppose that*

$$L\left\{\frac{B(\beta,\alpha)}{|\gamma_1|(\alpha+\beta)\Gamma(\alpha)\Gamma(\beta)}+\frac{1}{|\gamma_1|\Gamma(\alpha+1)}+\right.$$

$$\left.\frac{2|\gamma_2|}{|\gamma_1|\Gamma(\alpha+2)}+\frac{2}{\Gamma(\alpha+2)}\right\}\prec 1. \tag{25}$$

Then, the value problem (5)-(7) has a unique solution.

Proof of theorem 3.5

Let $\sup_{x\in[0,1]}|\mu(t)| = \|\mu\|.$

We set

$$\rho^* \geq \|\mu\| \left\{ \frac{1}{\Gamma(\alpha+1)} + \frac{B(\beta,\alpha)}{|\gamma_1|(\alpha+\beta)\Gamma(\alpha)\Gamma(\beta)} + \right.$$

$$\left. + \frac{1}{|\gamma_1|\Gamma(\alpha+1)} + \frac{2|\gamma_2|}{|\gamma_1|\Gamma(\alpha+2)} + \frac{2}{\Gamma(\alpha+2)} \right\}$$

and we consider the set

$$B_{\rho^*} = \left\{ y \in C\left([0,1], R\right) : \|y\| \leq \rho^* \right\}.$$

We define two operators P and Q on $B_{\rho*}$ by:

$$(Py)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, y(s)\right) ds, \quad t \in [0,1]$$

$$(Qy)(t)$$
$$= \frac{1}{\gamma_1\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \int_s^1 (1-r)^{\beta-1}(r-s)^{\alpha-1} drds$$
$$+ \frac{1}{\gamma_1\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f\left(s, y(s)\right) ds$$
$$+ \frac{2\gamma_2}{\gamma_1\Gamma(\alpha+1)} \int_0^1 (1-s)^{\alpha} f\left(s, y(s)\right) ds$$
$$+ \frac{2t}{\Gamma(\alpha+1)} \int_0^1 (1-s)^{\alpha} f\left(s, y(s)\right) ds,$$

$t \in [0,1]$

Let $x, y \in B_{\rho^*}$, we have

$$\|Px + Qy\| \leq \frac{\|\mu\|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds$$
$$+ \frac{\|\mu\|}{|\gamma_1|\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \int_s^1 (1-r)^{\beta-1}(r-s)^{\alpha-1} drds$$
$$+ \frac{\|\mu\|}{|\gamma_1|\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} ds + \frac{2|\gamma_2|\|\mu\|}{|\gamma_1|\Gamma(\alpha+1)} \int_0^1 (1-s)^{\alpha} ds$$
$$+ \frac{2\|\mu\|}{\Gamma(\alpha+1)} \int_0^1 (t-s)^{\alpha} ds$$
$$\leq \|\mu\| \left\{ \frac{1}{\Gamma(\alpha+1)} + \frac{B(\beta,\alpha)}{|\gamma_1|(\alpha+\beta)\Gamma(\alpha)\Gamma(\beta)} + \right.$$
$$\left. + \frac{1}{|\gamma_1|\Gamma(\alpha+1)} + \frac{2|\gamma_2|}{|\gamma_1|\Gamma(\alpha+2)} + \frac{2}{\Gamma(\alpha+2)} \right\}$$
$$\leq \rho^*.$$

Then, $Px + Qy \in B_{\rho^*}$, we have

$$\|Qx - Qy\| \leq L\|x-y\| \times$$
$$\left\{ \frac{B(\beta,\alpha)}{|\gamma_1|(\alpha+\beta)\Gamma(\alpha)\Gamma(\beta)} + \frac{1}{|\gamma_1|\Gamma(\alpha+1)} + \right.$$
$$\left. \frac{2|\gamma_2|}{|\gamma_1|\Gamma(\alpha+2)} + \frac{2}{\Gamma(\alpha+2)} \right\}$$

By exploiting (25), we deduce that Q is a contraction. According to the definition of the operator P, we deduce that the continuity of f implies that of P. In addition, we have:

$$\|Px\| \leq \|\mu\| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds$$

$$\leq \frac{\|\mu\|}{\Gamma(\alpha+1)}, \tag{26}$$

which implies that P is uniformly bounded.
Now we show that P is compact. We have

$$(Py)(t_1) - (Py)(t_2)$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} f\left(s, y(s)\right) ds$$

$$- \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} f\left(s, y(s)\right) ds$$

$$= \frac{1}{\Gamma(\alpha)} \left(\int_0^{t_1} (t_1 - s)^{\alpha-1} f\left(s, y(s)\right) ds \right.$$

$$- \int_0^{t_1} (t_2 - s)^{\alpha-1} f\left(s, y(s)\right) ds$$

$$\left. - \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} f\left(s, y(s)\right) ds \right).$$

Taking into account the condition (H_1), we set

$$f^* = \sup_{(t,x)\in[0,1]\times B_{\rho^*}} |f(t,x)|.$$

Then we can write

$$\left|(Py)(t_1) - (Py)(t_2)\right|$$

$$= \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] f(s, y(s)) ds \right.$$

$$\left. + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} f(s, y(s)) ds \right|$$

$$\leq \frac{f^*}{\Gamma(\alpha)} \left| \int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] ds \right.$$

$$\left. + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds \right|,$$

a simple calculation leads to:

$$\left|(Py)(t_1) - (Py)(t_2)\right| \leq \frac{f^*}{\Gamma(\alpha+1)} \left| t_2^{\alpha} - t_1^{\alpha} \right|. \tag{27}$$

The second member of (27) is independent of y and tends to zero when $t_2 - t_1 \to 0$, so P is equicontinuous. Using the Arzela-Ascoli theorem, we deduce that P is compact in B_{ρ^*}. Thus all the assumptions of the fixed point theorem of Krasnoselskii are satisfied. Which implies that the boundary value problem (5)-(7) has a unique solution on $[0,1]$.

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