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Research paper

An improvement of H. Wang preconditioner for L-matrices

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Abstract

In this paper, we improve the preconditioner, that introduced by H. Wang et al [6]. The H. Wang preconditioner $P \in \mathbb{R}^{n\times n}$ has only one non-zero, non-diagonal element in $P(n,1)$ or $P(1,n)$, when $a_(1,n)a_(n,1) \neq 0$. But the new preconditioner has only one non-zero, non-diagonal element in $P(i,j)$ or $P(j,i)$ if $a_(i,j)a_(j,i) \neq 0$, so the H. Wang preconditioner is a spacial case of the new preconditioner for L-matrices. Also we present two models to construct a better $I+S$ type preconditioner for the *AOR* iterative method. Convergence analysis are given, numerical results are presented which show the effectiveness of the new preconditioners.

Keywords: *Linear system; AOR method; Jacobi method; Gauss-Seidel method; Spectral radius; M-matrix; L-matrix; Preconditioner.*

1. Introduction

Consider the following linear system

$$Ax = b, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ are given and $x \in \mathbb{R}^n$ is unknown. For simplicity, suppose that

$$A = I - L - U, \tag{2}$$

where I is identity matrix and $-L$ and $-U$ are strictly lower and upper triangular parts of matrix A, respectively. The accelerated overrelaxation(*AOR*) iterative method [3] is given by,

$$x^{(i+1)} = L_{\gamma,\omega}x^{(i)} + (I-\gamma L)^{-1}\omega b, \qquad i = 0,1,2,..., \tag{3}$$

whose iteration matrix is

$$L_{\gamma,\omega} = (I-\gamma L)^{-1}[(1-\omega)I + (\omega-\gamma)L + \omega U], \tag{4}$$

where ω and γ are real parameters with $\omega \neq 0$.
Now, let us consider the preconditioned linear system,

$$PAx = Pb, \tag{5}$$

where $P = I+S$ is a nonsingular matrix and $S \in \mathbb{R}^{n\times n}$. For L-matrices linear systems, first, Evans et al [2] proposed the preconditioned matrix $P = I+S$, where

$$S = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ -a_{n1} & 0 & \dots & 0 \end{pmatrix} \in \mathbb{R}^{n\times n}, \quad \text{or} \quad S = \begin{pmatrix} 0 & 0 & \dots & -a_{1n} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \in \mathbb{R}^{n\times n}. \tag{6}$$

This preconditioners has been studied by Yun [9] and Li et al [4]. recently H. wang et al [6] provided a preconditioner and improved the convergence rate of the *AOR* iterative method. They considered

$$S_{\alpha\beta} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \frac{-a_{n1}}{\alpha}-\beta & 0 & \dots & 0 \end{pmatrix} \in \mathbb{R}^{n\times n} \quad \text{or} \quad S_{\alpha\beta} = \begin{pmatrix} 0 & 0 & \dots & \frac{-a_{1n}}{\alpha}-\beta \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \in \mathbb{R}^{n\times n}. \tag{7}$$

But if, $a_{1n}a_{n1} = 0$ these preconditioners are invalid. For solve this problem, we suggest the new preconditioner as follow.

2. Improvement of the H. Wang preconditioner

Consider the following linear system

$$\tilde{A}x = \tilde{b}, \tag{8}$$

where $\tilde{A} = (I+\tilde{S}_{\alpha\beta rt})A$ and $\tilde{b} = (I+\tilde{S}_{\alpha\beta rt})b$, with $\tilde{S}_{\alpha\beta rt} \in \mathbb{R}^{n\times n}$ and for $i,j = 1,...,n$

$$(\tilde{S}_{\alpha\beta rt})_{ij} = \begin{cases} \frac{-a_{rt}}{\alpha}-\beta, & i=r,\ j=t, \\ 0, & otherwise. \end{cases} \tag{9}$$

Here, $\alpha,\beta \in \mathbb{R}$ and $r,t \in N = \{1,2,...,n\}$, $r \neq t$. Clearly $(I+\tilde{S}_{\alpha\beta rt})$ is an nonsingular matrix.

The elements $\tilde{a}_{ij}$ of $\tilde{A}$ are given by the expression

$$\tilde{a}_{ij} = \begin{cases} a_{ij}, & i \neq r, \\ a_{rt}(1-\frac{1}{\alpha})-\beta & i=r,\ j=t, \qquad i,j=1,...,n \\ a_{rj}-(\frac{a_{rt}}{\alpha}+\beta)a_{tj} & i=r,\ j\neq t. \end{cases} \tag{10}$$

Since $A = I - L - U$, we have, $\tilde{a}_{rr} = 1-(\frac{a_{rt}}{\alpha}+\beta)a_{tr}$ and also, if we put

$$\tilde{A} = \tilde{D} - \tilde{L} - \tilde{U}, \tag{11}$$

where $\tilde{D}$ is diagonal matrix and $-\tilde{L}$ and $-\tilde{U}$ are strictly lower and upper triangular parts of matrix $\tilde{A}$, respectively. We have

$$\begin{aligned} \tilde{A} &= (I+\tilde{S}_{\alpha\beta rt})A \\ &= (I+\tilde{S}_{\alpha\beta rt})(I-L-U) \\ &= I+\tilde{S}_{\alpha\beta rt}-L-U-\tilde{S}_{\alpha\beta rt}L-\tilde{S}_{\alpha\beta rt}U. \end{aligned} \tag{12}$$

If $r > t$, put

$$\tilde{S}_{\alpha\beta rt}U = \acute{D}+\acute{L}+\acute{U}, \tag{13}$$

where $\acute{D}$ is diagonal matrix and $\acute{L}$ and $\acute{U}$ are strictly lower and upper triangular parts of matrix $\tilde{S}_{\alpha\beta rt}U$, respectively. So

$$\tilde{A} = (I-\acute{D})-(L+\tilde{S}_{\alpha\beta rt}L-\tilde{S}_{\alpha\beta rt}+\acute{L})-(U+\acute{U}) = \tilde{D}-\tilde{L}-\tilde{U}$$

where,

$$\tilde{D} = I-\acute{D},\ \tilde{L} = L+\tilde{S}_{\alpha\beta rt}L-\tilde{S}_{\alpha\beta rt}+\acute{L} \text{ and } \tilde{U} = U+\acute{U}. \tag{14}$$

If $r < t$, put

$$\tilde{S}_{\alpha\beta rt}L = \acute{D}+\acute{L}+\acute{U}, \tag{15}$$

so,

$$\tilde{A} = (I-\acute{D})-(L+\acute{L})-(U+\tilde{S}_{\alpha\beta rt}U-\tilde{S}_{\alpha\beta rt}+\acute{U}) = \tilde{D}-\tilde{L}-\tilde{U}$$

where,

$$\tilde{D} = I-\acute{D},\ \tilde{L} = L+\acute{L} \text{ and } \tilde{U} = U+\tilde{S}_{\alpha\beta rt}U-\tilde{S}_{\alpha\beta rt}+\acute{U} \tag{16}$$

The *AOR* iterative method for the preconditioned system (8) is given by

$$x^{(i+1)} = \tilde{L}^{r,t}_{\gamma,\omega}x^{(i)}+(\tilde{D}-\gamma\tilde{L})^{-1}\omega(I+\tilde{S}_{\alpha\beta rt})b, \qquad i=0,1,2,..., \tag{17}$$

whose iteration matrix is

$$\tilde{L}^{r,t}_{\gamma,\omega} = (\tilde{D}-\gamma\tilde{L})^{-1}[(1-\omega)\tilde{D}+(\omega-\gamma)\tilde{L}+\omega\tilde{U}], \tag{18}$$

where ω and γ are real parameters with $\omega \neq 0$.

3. Convergence analysis

In the sequel, we need the following. Let $A, B \in \mathbb{R}^{n\times n}$. If $a_{ij} \geq b_{ij}$ ($a_{ij} > b_{ij}$), $i,j = 1,2,\ldots,n$, we write $A \geq B$ ($A > B$). The same notation applies to vectors $x, y \in \mathbb{R}^n$. If $A \in \mathbb{R}^{n\times n}$ satisfies $A \geq 0 (> 0)$ then it is said to be *nonnegative (positive)*. The same terminology applies to vectors $x \in \mathbb{R}^n$. (see [8].) A matrix $A \in \mathbb{R}^{n\times n}$ is said to be an L-matrix if $a_{ii} > 0, i = 1,2,\ldots,n$, and $a_{ij} \leq 0, i \neq j = 1,2,\ldots,n$. (see [5].) A matrix $A \in \mathbb{R}^{n\times n}$ is said to be an M-matrix if $a_{ij} \leq 0$, $i \neq j = 1,2,\ldots,n$, A is nonsingular and $A^{-1} \geq 0$. (see [5].) A matrix A is said to be irreducible if the directed graph associated with A is strongly connected. (see [5].) Let $A \geq 0$ then:

1. A has positive real eigenvalue equal to its spectral radius $\rho(A)$;
2. A has an eigenvector $x \geq 0$, with at least a positive entry, corresponding to $\rho(A)$;
3. If A is irreducible, then $\rho(A)$ is a simple eigenvalue of A and A has an eigenvector $x > 0$ corresponding to $\rho(A)$.

(see [5].) Let $A \geq 0$ then:

1. If $\alpha x \leq Ax$ for some $x \geq 0$, with at least a positive entry, then $\alpha \leq \rho(A)$;
2. If $Ax \leq \beta x$ for some $x > 0$, then $\rho(A) \leq \beta$. Moreover, if A is irreducible and if $Ax \leq \beta x$ for some $x \geq 0$, then $\rho(A) \leq \beta$ and $x > 0$.
3. If A is irreducible and if $\alpha x \leq Ax \leq \beta x$ for some $x > 0$, then $\alpha \leq \rho(A) \leq \beta$.

Let A and $\tilde{A}$ be the coefficient matrices of the linear systems (1) and (8), respectively. If $0 \leq \gamma \leq \omega \leq 1$ ($\omega \neq 0$ and $\gamma \neq 1$) and A is an irreducible L-matrix with $0 < a_{rt}a_{tr} < \alpha(\alpha > 1)$, $\beta \in (\frac{-a_{rt}}{\alpha}+\frac{1}{a_{tr}}, \frac{-a_{rt}}{\alpha}) \cap ((1-\frac{1}{\alpha})a_{rt}, \frac{-a_{rt}}{\alpha})$ then the iterative matrices $L_{\gamma,\omega}$ and $\tilde{L}^{r,t}_{\gamma,\omega}$ associated to the *AOR* method applied to the linear systems (1) and (8), respectively, are nonnegative and irreducible. Moreover $\tilde{A}$ is an irreducible L-matrix.

Proof. It is easy to see that when $a_{rt}a_{tr} < \alpha$ ($\alpha > 1$) and $\beta \in (\frac{-a_{rt}}{\alpha}+\frac{1}{a_{tr}}, \frac{-a_{rt}}{\alpha}) \cap ((1-\frac{1}{\alpha})a_{rt}, \frac{-a_{rt}}{\alpha})$, we have $\tilde{a}_{rr} = 1-(\frac{a_{rt}}{\alpha}+\beta)a_{tr} > 0$ and $\frac{a_{rt}}{\alpha}+\beta < 0$ so, $\tilde{a}_{ij} = a_{rj}-(\frac{a_{rt}}{\alpha}+\beta)a_{tj} < 0$ (for $\mathrm{i = r}$ and $\mathrm{j \neq t}$), and also $\tilde{a}_{rt} = a_{rt}(1-\frac{1}{\alpha})-\beta < 0$, so $\tilde{A}$ is an L-matrix and the directed graph associated to A is a subgraph of the directed graph associated to $\tilde{A}$, then Since A is irreducible $\tilde{A}$ is irreducible too. Also from (11), we have $\tilde{D} > 0$, $\tilde{L} \geq 0$ and $\tilde{U} \geq 0$. The rest of proof is similar to the Lemma 3 in [6]. □

Note1:
When A is an L-matrix then under the assumptions of Lemma 3, $\tilde{S}_{\alpha\beta rt} \geq 0$.
Let $L_{\gamma,\omega}$ and $\tilde{L}^{r,t}_{\gamma,\omega}$ be the iteration matrices of the *AOR* method applied to the linear systems (1) and (8), respectively. If $0 \leq \gamma \leq \omega \leq 1$ ($\omega \neq 0$ and $\gamma \neq 1$) and A is an irreducible L-matrix with $0 < a_{rt}a_{tr} < \alpha(\alpha > 1)$, $\beta \in (\frac{-a_{rt}}{\alpha}+\frac{1}{a_{tr}}, \frac{-a_{rt}}{\alpha}) \cap ((1-\frac{1}{\alpha})a_{rt}, \frac{-a_{rt}}{\alpha})$, then:

1. $\rho(\tilde{L}^{r,t}_{\gamma,\omega}) \leq \rho(L_{\gamma,\omega})$, if $\rho(L_{\gamma,\omega}) < 1$;
2. $\rho(\tilde{L}^{r,t}_{\gamma,\omega}) = \rho(L_{\gamma,\omega}) = 1$;
3. $\rho(\tilde{L}^{r,t}_{\gamma,\omega}) \geq \rho(L_{\gamma,\omega})$, if $\rho(L_{\gamma,\omega}) > 1$.

Proof. From Lemmas 3, 3 and 3, since $L_{\gamma,\omega}$ and $\tilde{L}^{r,t}_{\gamma,\omega}$ are nonnegative and irreducible matrices, there is a positive vector $x > 0$, such that

$$L_{\gamma,\omega}x = \lambda x, \tag{19}$$

where $\rho(L_{\gamma,\omega}) = \lambda$. Equivalently, we can write

$$[(1-\omega)I+(\omega-\gamma)L+\omega U]x = \lambda(I-\gamma L)x, \tag{20}$$

and also, we have

$$\omega Ux = (\lambda-1+\omega)x+(\gamma-\omega-\lambda\gamma)Lx. \tag{21}$$

On the other hand, for the positive vector x we have,

$$\tilde{L}^{r,t}_{\gamma,\omega}x-\lambda x = (\tilde{D}-\gamma\tilde{L})^{-1}[(1-\omega)\tilde{D}+(\omega-\gamma)\tilde{L}+\omega\tilde{U}-\lambda(\tilde{D}-\gamma\tilde{L})]x. \tag{22}$$

Case(1): If $r > t$, from (14), since $\tilde{U} = U+\acute{U}$ and from (21) we have,

$$\omega\tilde{U}x = \omega(U+\acute{U})x = (\lambda-1+\omega)x+(\gamma-\omega-\lambda\gamma)Lx+\omega\acute{U}x, \tag{23}$$

and also

$$\lambda(\tilde{D}-\gamma\tilde{L})x = \lambda(1-\gamma)\tilde{D}x+\lambda\gamma(\tilde{D}-\tilde{L})x = \lambda(1-\gamma)\tilde{D}x+\lambda\gamma(I+\tilde{S}_{\alpha\beta rt}-L-\tilde{S}_{\alpha\beta rt}U+\acute{U}-\tilde{S}_{\alpha\beta rt}L)x. \quad (24)$$

From (22), (23) and (24), we have

$$\tilde{L}^{r,t}_{\gamma,\omega}x-\lambda x = (\tilde{D}-\gamma\tilde{L})^{-1}[(1-\gamma)(1-\lambda)(\tilde{D}-I)+(\omega-\gamma+\lambda\gamma)(\tilde{S}_{\alpha\beta rt}U-\tilde{S}_{\alpha\beta rt})x$$
$$+(\omega-\gamma+\lambda\gamma)\tilde{S}_{\alpha\beta rt}L-(\lambda\gamma-\gamma)\acute{U}]x,$$

again from (21) we have

$$\tilde{L}^{r,t}_{\gamma,\omega}x-\lambda x=(1-\lambda)(\tilde{D}-\gamma\tilde{L})^{-1}[(\gamma-1)\acute{D}-(1-\gamma)\tilde{S}_{\alpha\beta rt}-\gamma(\acute{D}+\acute{L})]x.$$

Put $B=(\gamma-1)\acute{D}-(1-\gamma)\tilde{S}_{\alpha\beta rt}-\gamma(\acute{D}+\acute{L})$, from (13) and Note1, we conclude that, $B\leq 0$. So if $\lambda<1$ then $z=(1-\lambda)(\tilde{D}-\gamma\tilde{L})^{-1}Bx$ is nonpositive vector, and $\tilde{L}^{r,t}_{\gamma,\omega}x\leq\lambda x$, so from Lemma 3 we obtain

$$\rho(\tilde{L}^{r,t}_{\gamma,\omega})\leq\lambda=\rho(L_{\gamma,\omega})<1.$$

And if $\lambda=1$, then $z=0$ and $\tilde{L}^{r,t}_{\gamma,\omega}x=\lambda x$, from Lemma 3 we obtain,

$$\rho(\tilde{L}^{r,t}_{\gamma,\omega})=\lambda=\rho(L_{\gamma,\omega})=1,$$

Finally if, $\lambda>1$, then z will be nonnegative vector and $\tilde{L}_{\gamma,\omega}x\geq\lambda x$, again from Lemma 3 we obtain,

$$\rho(\tilde{L}^{r,t}_{\gamma,\omega})\geq\lambda=\rho(L_{\gamma,\omega})>1.$$

Case(2): If $r<t$, from (16) and (22) we have

$$\tilde{L}^{r,t}_{\gamma,\omega}x-\lambda x = (\tilde{D}-\gamma\tilde{L})^{-1}[(1-\lambda)\tilde{D}-\gamma(1-\lambda)\tilde{L}-\omega(\tilde{D}-\tilde{U})+\omega\tilde{L}]x =$$
$$(\tilde{D}-\gamma\tilde{L})^{-1}[(1-\lambda)\tilde{D}-\gamma(1-\lambda)L-\omega(I+\tilde{S}_{\alpha\beta rt}-\tilde{S}_{\alpha\beta rt}L-U)$$
$$+\omega L-\gamma(1-\lambda)\acute{L}-\omega(\acute{L}-\tilde{S}_{\alpha\beta rt}U)+\omega\acute{L}]x =$$
$$(\tilde{D}-\gamma\tilde{L})^{-1}[(1-\lambda)(\tilde{D}-I)-\omega(\tilde{S}_{\alpha\beta rt}-\tilde{S}_{\alpha\beta rt}L)-\gamma(1-\lambda)\acute{L}+ w\tilde{S}_{\alpha\beta rt}U]x =$$
$$(\tilde{D}-\gamma\tilde{L})^{-1}[(1-\lambda)(\tilde{D}-I)+(\lambda-1)\tilde{S}_{\alpha\beta rt}(I-\gamma L)-\gamma(1-\lambda)\acute{L}]x$$

from (20) we have

$$\tilde{L}^{r,t}_{\gamma,\omega}x-\lambda x = (\lambda-1)(\tilde{D}-\gamma\tilde{L})^{-1}[\acute{D}+\tfrac{1}{\lambda}\tilde{S}_{\alpha\beta rt}[(1-\omega)I+(\omega-\gamma)L+\omega U]+\gamma\acute{L}]x.$$

Put $E=\acute{D}+\frac{1}{\lambda}\tilde{S}_{\alpha\beta rt}[(1-\omega)I+(\omega-\gamma)L+\omega U]+\gamma\acute{L}$, Clearly, E is nonnegative matrix, so if $\lambda<1$, $g=(\lambda-1)(\tilde{D}-\gamma\tilde{L})^{-1}Ex\leq 0$, and $\tilde{L}^{r,t}_{\gamma,\omega}x\leq\lambda x$ and from Lemma 3 we have

$$\rho(\tilde{L}^{r,t}_{\gamma,\omega})\leq\lambda=\rho(L_{\gamma,\omega})<1.$$

The rest of proof is in similar way with case (1). □

Let L_{GS} and $\tilde{L}^{r,t}_{GS}$ be the iteration matrices of the Gauss-Seidel method applied to the linear systems (1) and (8), respectively. If A is an nonsingular irreducible M-matrix with $0<a_{rt}a_{tr}<\alpha(\alpha>1)$, $\beta\in(\frac{-a_{rt}}{\alpha}+\frac{1}{a_{tr}},\frac{-a_{rt}}{\alpha})\cap((1-\frac{1}{\alpha})a_{rt},\frac{-a_{rt}}{\alpha})$, then $\tilde{A}$ is an irreducible M-matrix and :

1. $\rho(\tilde{L}^{r,t}_{GS})\leq\rho(L_{GS})$, if $\rho(L_{GS})<1$;
2. $\rho(\tilde{L}^{r,t}_{GS})=\rho(L_{GS})=1$;
3. $\rho(\tilde{L}^{r,t}_{GS})\geq\rho(L_{GS})$, if $\rho(L_{GS})>1$.

Proof. Same as Lemma 3, it is clear that $\tilde{A}$ is an irreducible L-matrix. For a L-matrix A the statement " A is a nonsingular M-matrix " is equivalent to the statement " there exists a positive vector $y\in\mathbb{R}^n$ $(y>0)$ such that $Ay>0$" (see Theorem 6.2.3. Condition I_{27} of [1]). But $P=I+\tilde{S}_{\alpha\beta rt}\geq 0$ implies that $\tilde{A}y=PAy>0$ so $\tilde{A}$ is an M-matrix too. From Theorem 2.6. in [7] the rest of proof is trivial. □

4. Models for Selecting r and t

Consider how to select r and t to construct a better $I+S$ type preconditioner. Now we state the two following Lemmas, we use these Lemmas to construct a better $I+S$ preconditioners. (see [5].) If $A=(a_{i,j})\geq 0$, is an irreducible $n\times n$ matrix the either

$$\sum_{j=1}^{n}a_{i,j}=\rho(A)\ \text{ for all }\ 1\leq \mathrm{i}\leq \mathrm{n}$$

$$\min_{1\leq i\leq n}(\sum_{j=1}^{n}a_{i,j})<\rho(A)<\max_{1\leq i\leq n}(\sum_{j=1}^{n}a_{i,j})$$

If $A=(a_{i,j})\geq 0$, is an irreducible $n\times n$ matrix the either

$$\sum_{i=1}^{n}a_{i,j}=\rho(A)\ \text{ for all }\ 1\leq \mathrm{j}\leq \mathrm{n}$$

$$\min_{1\leq j\leq n}(\sum_{i=1}^{n}a_{i,j})<\rho(A)<\max_{1\leq j\leq n}(\sum_{i=1}^{n}a_{i,j})$$

Proof. Since $\rho(A)=\rho(A^T)$ and $A^T\geq 0$ is an irreducible $n\times n$ matrix, so from Lemma 4, the proof is trivial. □

If $L_{\gamma,\omega}$ and $\tilde{L}^{r,t}_{\gamma,\omega}$ be the iteration matrices of the *AOR* method applied to the linear systems (1) and (8), respectively, we write $(L_{\gamma,\omega})_{i,j}=(l_{i,j})$ and $(\tilde{L}^{r,t}_{\gamma,\omega})_{i,j}=(l^{r,t}_{i,j})$ for $i,j=1,\ldots,n$ when $P=(I+\tilde{S}_{\alpha\beta rt})$. Now, suppose that γ and $\omega\in\mathbb{R}$, $\omega\neq 0$ be two fixed parameters, from Lemma 4, we select r such that

$$\sum_{j=1}^{n}l_{r,j}=\max_{1\leq i\leq n}(\sum_{j=1}^{n}l_{i,j}).$$

For selecting t, we present two models.

Model1:

$$\sum_{j=1}^{n}l^{r,t}_{r,j}=\min_{1\leq k\leq n}(\sum_{j=1}^{n}l^{r,k}_{r,j}),\quad k\neq r.$$

Model2:

$$\sum_{i=1}^{n}l_{i,t}=\max_{1\leq j\leq n}(\sum_{i=1}^{n}l_{i,j}).$$

Note2:

Selecting r and t in Model2 are not depend on α and β, but not Model1. So here we suppose that α and β are two arbitrary parameters that satisfy in conditions of Lemma 3.

Now for computing r and t put $e=(1,1,\ldots,1)^T\in\mathbb{R}^n$, it is easy to see that

$$(L_{\gamma,\omega}e)_i=\sum_{j=1}^{n}l_{i,j}, \quad (25)$$

and

$$\max_{1\leq i\leq n}(\sum_{j=1}^{n}l_{i,j})=(L_{\gamma,\omega}e)_r,$$

so we should compute $u_1=L_{\gamma,\omega}e$, but from (4), we have

$$u_1=(I-\gamma L)^{-1}[(1-\omega)I+(\omega-\gamma)L+\omega U]e,$$

so if we put $b_1=[(1-\omega)I+(\omega-\gamma)L+\omega U]e$, computing u_1 is equivalent to solving the lower triangular system

$$(I-\gamma L)u_1=b_1. \quad (26)$$

Model1:

Same as (25) we have

$$(\tilde{L}^{r,k}_{\gamma,\omega}e)_r=\sum_{j=1}^{n}l^{r,k}_{r,j},$$

so for $k = 1,2,...,n$ we should compute $(\tilde{L}^{r,k}_{\gamma,\omega}e)_r$, but from (18) we put

$$u_2 = (\tilde{D} - \gamma\tilde{L})^{-1}[(1-\omega)\tilde{D} + (\omega-\gamma)\tilde{L} + \omega\tilde{U}]e,$$

and

$$b_2 = [(1-\omega)\tilde{D} + (\omega-\gamma)\tilde{L} + \omega\tilde{U}]e,$$

clearly, since only, $(\tilde{L}^{r,k}_{\gamma,\omega}e)_r$ is needed, so computing $b_2(r+1:n)$ is not necessary. Also since $(1-\omega)\tilde{D} + (\omega-\gamma)\tilde{L} + \omega\tilde{U}$ differs with $(1-\omega)I + (\omega-\gamma)L + \omega U$ in rth row, so $b_2(1:r-1) = b_1(1:r-1)$, and only we should compute $b_2(r)$. Since $\tilde{D} - \gamma\tilde{L}$ is different with $I - \gamma L$ in rth row and $b_2(1:r-1) = b_1(1:r-1)$ in the lower triangular system

$$(\tilde{D} - \gamma\tilde{L})u_2 = b_2,$$

only need to compute $u_2(r)$, so

$$u_2(r) = [b_2(r) - (\tilde{D} - \gamma\tilde{L})(r,1:r-1)u_1(1:r-1)]/(\tilde{D} - \gamma\tilde{L})(r,r).$$

Modle2:
Here we should compute $u_3 = L^T_{\gamma,\omega}e$, from (4) we have

$$u_3 = [(1-\omega)I + (\omega-\gamma)L + \omega U]^T((I-\gamma L)^T)^{-1}e,$$

if we put $b_3 = ((I-\gamma L)^T)^{-1}e$, computing b_3 is equivalent to solving the upper triangular system $(I-\gamma L)^T b_3 = e$. Finally

$$u_3 = [(1-\omega)I + (\omega-\gamma)L + \omega U]^T b_3.$$

These models when $\gamma = 0$ and $\omega = 1$ reduced to simpler models, for preconditioned Jacobi method (see[10]).

5. Numerical Example

In this section we give the numerical examples to illustrate the results obtained in Sections 3 and 4. In all tables, we report the spectral radii of the corresponding iteration matrix. In these tables n represents the dimension of matrix and also, the meaning of notations J and GS are the Jacobi and Gauss-Seidel iterative methods and $M_i(r,t)$ is the vector (r,t) where r and t are obtained by Model i, i=1,3 in [10], $M_{m_i}(r,t)$ is the vector (r,t) where r and t are obtained by Model i, i=1,2. ρ_1, ρ_3, ρ_{m_1} and ρ_{m_2} are the spectral radii of iteration matrices when the preconditioned to (1) obtained by Model1 ,Model3 in [10], Model1 and Model2, respectively. The numerical results in the following tables are computed using MATLAB 7.9. (See [10].) Let

$$A_1 = \begin{pmatrix} 1.00000 & -0.00580 & -0.19350 & -0.25471 & -0.03885 \\ -0.28424 & 1.00000 & -0.16748 & -0.21780 & -0.21577 \\ -0.24764 & -0.26973 & 1.00000 & -0.18723 & -0.08949 \\ -0.13880 & -0.01165 & -0.25120 & 1.00000 & -0.13236 \\ -0.25809 & -0.08162 & -0.13940 & -0.04890 & 1.00000 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 1.00000 & -0.23661 & -0.37369 & -0.25833 & -0.05480 \\ -0.13602 & 1.00000 & -0.10578 & -0.38675 & -0.32750 \\ -0.12569 & -0.01525 & 1.00000 & -0.26597 & -0.17207 \\ -0.14603 & -0.18344 & -0.34914 & 1.00000 & -0.35613 \\ -0.15730 & -0.34795 & -0.09515 & -0.00397 & 1.00000 \end{pmatrix}$$

Note that A_1 is a strictly diagonally dominant matrix, but A_2 is not. Since, for the A_1 and A_2, the module of the off diagonal of elements are less than one so, we consider $\alpha = -\frac{1}{a_{rt}} > 1$ and $\beta = 0$, clearly $(\tilde{S}_{\alpha\beta rt})_{r,t} = 1$. The numerical results are given in Tables 1 and 2. (see [7].)
For A_3, we report the spectral radii of the corresponding preconditioned iteration matrix that obtained by Model2. The numerical results are given in Table3.

$$A_3 = \begin{pmatrix} 1 & -\frac{1}{n\times 1100} & -\frac{1}{(n-1)\times 1100} & \cdots & -\frac{1}{3\times 1100} & -\frac{1}{22} \\ -\frac{1}{n\times 10+1} & 1 & -\frac{1}{3\times 10+2} & \cdots & -\frac{1}{(n-1)\times 10+2} & -\frac{1}{n\times 10+2} \\ -\frac{1}{(n-1)\times 10+1} & -\frac{1}{2\times 10+3} & 1 & \cdots & -\frac{1}{(n-1)\times 10+3} & -\frac{1}{n\times 10+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\frac{1}{3\times 10+1} & -\frac{1}{(n-2)\times 10+(n-1)} & -\frac{1}{(n-3)\times 10+(n-1)} & \cdots & 1 & -\frac{1}{n\times 10+(n-1)} \\ -\frac{100}{7} & -\frac{1}{(n-1)\times 10+n} & -\frac{1}{(n-2)\times 10+n} & \cdots & -\frac{1}{2\times 10+n} & 1 \end{pmatrix}.$$

$$A_4 = \begin{pmatrix} 1 & -\frac{100}{7} & -\frac{1}{(n-1)\times 1100} & \cdots & -\frac{1}{3\times 1100} & 0 \\ -\frac{1}{22} & 1 & -\frac{1}{3\times 10+2} & \cdots & -\frac{1}{(n-1)\times 10+2} & -\frac{1}{n\times 10+2} \\ -\frac{1}{(n-1)\times 10+1} & -\frac{1}{2\times 10+3} & 1 & \cdots & -\frac{1}{(n-1)\times 10+3} & -\frac{1}{n\times 10+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\frac{1}{3\times 10+1} & -\frac{1}{(n-2)\times 10+(n-1)} & -\frac{1}{(n-3)\times 10+(n-1)} & \cdots & 1 & -\frac{1}{n\times 10+(n-1)} \\ 0 & -\frac{1}{(n-1)\times 10+n} & -\frac{1}{(n-2)\times 10+n} & \cdots & -\frac{1}{2\times 10+n} & 1 \end{pmatrix}.$$

For A_4, it is clear, since $a_{1n}a_{n1} = 0$ the H. Wang preconditioner is invalid but by our new preconditioner we have the new results that are given in Table 4. Let,

$$A_5 = \begin{pmatrix} 1 & -0.2 & -0.2 & -0.1 & -0.25 & -0.4 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ -0.3 & -0.5 & 1 & -0.05 & -0.25 & -0.1 \\ -0.25 & -0.1 & -0.55 & 1 & -0.3 & -0.1 \\ -0.2 & -0.15 & -0.3 & -0.05 & 1 & -0.5 \\ -0.3 & -0.25 & -0.25 & -0.1 & -0.3 & 1 \end{pmatrix}$$

For A_5,the numerical results are given in Table 5.

Table 1. Comparison of the spectral radii of the Jacobi, method for Example, 5

	$M_1(r,t)$	ρ_1	$M_3(r,t)$	ρ_3	
A_1	$(4,5)$	0.490685	$(2,1)$	0.579796	
A_2	$(2,3)$	0.769261	$(4,3)$	0.751899	
	$M_{m_1}(r,t)$	ρ_{m_1}	$M_{m_2}(r,t)$	ρ_{m_2}	$\rho(J)$
A_1	$(2,3)$	0.550251	$(2,1)$	0.599796	0.629054
A_2	$(4,2)$	0.709061	$(4,3)$	0.751899	0.767901

Table 2. Comparison of the spectral radii of the Gauss-Seidel, method for Example, 5

	$M_1(r,t)$	ρ_1	$M_3(r,t)$	ρ_3	
A_1	$(4,5)$	0.364181	$(2,1)$	0.383960	
A_2	$(2,3)$	0.534910	$(4,3)$	0.646546	
	$M_{m_1}(r,t)$	ρ_{m_1}	$M_{m_2}(r,t)$	ρ_{m_2}	$\rho(GS)$
A_1	$(2,1)$	0.383960	$(2,4)$	0.333417	0.384956
A_2	$(2,4)$	0.574424	$(2,4)$	0.574424	0.684891

Table 3. Numerical results for Example, 5

	$n=10$	$n=20$	$n=30$
(γ,ω)	$(0.85,0.9)$	$(0.7,0.95)$	$(0.85,0.95)$
(α,β)	$(100,-14.14286)$	$(50,-13.99999)$	$(200,-14.21428)$
$M_{m_2}(r,t)$	$(10,1)$	$(20,1)$	$(30,1)$
ρ_{m_2}	0.169754	0.172622	0.158450
$\rho(L_{\gamma,\omega})$	0.725002	0.738723	0.708978

Table 4. Numerical results for Example, 5

	(γ,ω)	(α,β)	$M_{m_2}(r,t)$	ρ_{m_2}	$\rho(L_{\gamma,\omega})$
$n=10$	$(0.85,0.9)$	$(100,-14)$	$(1,2)$	0.31933	0.76464
$n=20$	$(0.7,0.95)$	$(100,-14)$	$(1,2)$	0.34243	0.77516
$n=30$	$(0.85,0.95)$	$(200,-14)$	$(1,2)$	0.30978	0.74675

Table 5. Numerical results for Example, 5

(γ,ω)	(α,β)	(r,t)	$\rho(L^{r,t}_{\gamma,\omega})$	$\rho(L_{\gamma,\omega})$
$(0.85,0.9)$	$(10,0)$	$(1,4)$	1.30360	1.30301
$(0.7,0.95)$	$(5,0)$	$(5,3)$	1.28053	1.27812
$(0.85,0.95)$	$(40,0)$	$(6,5)$	1.32026	1.31985

6. Conclusion

This paper presents new preconditioned *AOR* iterative method that is valid even $a_{1n}a_{n1}=0$, and from the above numerical experiments, we get that the results are in concord with Theorems in Section3. Also we introduced two models to construct a better $I+S$ type preconditioned *AOR* iterative method. The Model2 is independent of choosing α and β, but a natural problem is, how to choose the optimal parameters α and β. Further research is required.

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