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Research paper

A modified Adomian decomposition method for singular initial value Emden-Fowler type equations

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Abstract

Traditional Adomian decomposition method (ADM) usually fails to solve singular initial value problems of Emden-Fowler type. To overcome this shortcoming, a new and effective modification of ADM that only requires calculation of the first Adomian polynomial is formally proposed in the present paper. Three singular initial value problems of Emden-Fowler type with $\alpha = 1,\ 2,$ and $> 2,$ and have been selected to demonstrate the efficiency of the method.

Keywords: *Singular Initial Value Problems; Emden-Fowler Type; Adomian Decomposition Method; New Modification.*

1. Introduction

Singular initial value problems play a fundamental role in a wide range of scientific disciplines. In this article, a special type of singular initial value problems which can be expressed as the following form is investigated.

$$u'' + \frac{\alpha}{x}u' + g(x)h(u) = k(x), \quad u(0) = a, \quad u'(0) = b.$$

Singularity behaviour that occurs at $x = 0$ is the main difficulty of this type of initial value problems. In recent years, a variety of methods have been adopted to handle this type of initial value problems. For example, Wazwaz employed a general approach for constructing the exact and series solution of this problem by means of the variational iteration method [1]. Parand et al. applied an approximation algorithm for the solution of this problem using Hermite functions, as basis functions, and collocation method [2]. Parand et al. also adopted a pseudospectral technique based on the rational Legendre functions and Gauss-Radau integration to handle this problem [3]. For further methods, the reader is referred to the references [4-19]. In the present article, ADM is modified effectively to solve presented singular initial value problem. The rest of this article is arranged as follows:
In Section 2, the basic ideas of the method are expressed with details. In Section 3, the proposed method is implemented to solve three singular initial value problems of Emden-Fowler type with $\alpha = 1,\ 2,$ and $> 2.$ Finally, Section 4 is devoted to presenting conclusion.

2. Modified ADM

Let's consider the following nonlinear equation

$$u'' + R(u) + N(u) = f(x), \tag{1}$$

with the following initial conditions

$$u(0) = \xi_1, \quad u'(0) = \xi_2,$$

where R is a linear operator, N is a nonlinear operator, and $f(x)$ is a known function. It is assumed that the unknown function $u(x)$ can be presented by an infinite series, say

$$u(x) = \sum_{n=0}^{+\infty} u_n(x), \tag{2}$$

and the nonlinear term $N(u)$ can be expressed as an infinite series of polynomials given by

$$N(u) = \sum_{n=0}^{+\infty} A_n(u_0, u_1, \dots, u_n), \tag{3}$$

where $A_n,\ n = 0,1,\dots$ are called the Adomian polynomials and are defined by [20]

$$A_n = \begin{cases} N(u_0) & n = 0, \\ \frac{1}{n}\sum_{i=0}^{n-1}(i+1)u_{i+1}\frac{dA_{n-1-i}}{du_0}, & n = 1,2,\dots. \end{cases}$$

Applying the inverse operator $L^{-1} = \int_0^x\int_0^x(.)dxdx,$ on both sides of Eq. (1) and considering (2) and (3), leads to

$$\sum_{n=0}^{+\infty} u_n = u(0) + u'(0)x + L^{-1}[f(x)]$$
$$-L^{-1}[R(\sum_{n=0}^{+\infty} u_n) + \sum_{n=0}^{+\infty} A_n(u_0, u_1, \dots, u_n)]. \tag{4}$$

We rewrite (4) as follows [21], [22]

$$\sum_{n=0}^{+\infty} u_n = u(0) + u'(0)x$$

$$+L^{-1}[\sum_{n=0}^{+\infty} a_n x^n] - pL^{-1}[\sum_{n=0}^{+\infty} a_n x^n] + L^{-1}[f(x)]$$

$$-L^{-1}[R(\sum_{n=0}^{+\infty} u_n) + \sum_{n=0}^{+\infty} A_n(u_0, u_1, ..., u_n)],$$

where p is an artificial parameter and $a_i,\ i=0,1,...$ are unknown coefficients. We now define

$$u_0 = \xi_1 + \xi_2 x + L^{-1}[\sum_{n=0}^{+\infty} a_n x^n],$$

$$u_1 = L^{-1}[f(x)] - pL^{-1}[\sum_{n=0}^{+\infty} a_n x^n] - L^{-1}[R(u_0) + A_0(u_0)],$$

$$u_2 = -L^{-1}[R(u_1) + A_1(u_0, u_1)],$$

$$u_3 = -L^{-1}[R(u_2) + A_2(u_0, u_1, u_2)],$$

$$\vdots$$

To avoid calculation of $A_n,\ n=1,2,...$, let determine $a_i,\ i=0,1,...$ such that $u_1 = 0$. This implies that

$$u_2 = u_3 = \cdots = 0.$$

Setting $p=1$, yields the solution of Eq. (1) with the initial conditions as follows

$$u(x) = \xi_1 + \xi_2 x + L^{-1}[\sum_{n=0}^{+\infty} a_n x^n].$$

3. Application

In this section, three singular initial value Emden-Fowler type equations, including a homogeneous nonlinear Emden-Fowler equation with $\alpha=1$, and two inhomogeneous Emden-Fowler equations, with $\alpha=2$ and $\alpha>2$ will be solved to illustrate the efficiency of the method. The computations associated with these examples have been performed by Maple package.

Example 3.1: Consider the homogeneous nonlinear Emden-Fowler equation with $\alpha=1$ [1]

$$u'' + \frac{1}{x}u' - u^3 + 3u^5 = 0,$$

with the following initial conditions

$$u(0)=1, \quad u'(0)=0.$$

Traditional ADM. As we know in the traditional ADM, we will reach the following expression

$$\sum_{n=0}^{+\infty} u_n = u(0) + u'(0)x$$

$$-L^{-1}[R(\sum_{n=0}^{+\infty} u_n) + \sum_{n=0}^{+\infty} A_n(u_0, u_1, ..., u_n)], \quad (5)$$

where $L^{-1} = \int_0^x \int_0^x (.)dxdx,\ R(u) = (1/x)u'$, and $A_n, n=0,1,...$ are as the following

$$A_0 = -u_0^3 + 3u_0^5,$$

$$A_1 = -3u_0^2 u_1(x) + 15u_0^4 u_1(x),$$

$$\vdots$$

We can define

$$u_0 = u(0) + u'(0)x,$$

$$u_n = -L^{-1}[R(u_{n-1}) + A_{n-1}(u_0, u_1, ..., u_{n-1})], \quad n=1,2,... .$$

Therefore

$$u_0(x) = 1, \quad u_1(x) = -x^2,$$

$$u_2(x) = x^2 + x^4, \quad u_3(x) = -x^2 - \frac{4}{3}x^4 - \frac{13}{10}x^6,$$

$$\vdots$$

Now, the series solution derived by the traditional ADM can be written as follows

$$u(x) = 1 - x^2 - \frac{1}{3}x^4 - \frac{13}{10}x^6 + \cdots.$$

Modified ADM. To solve the problem by the modified ADM, let us rewrite (5) as follows

$$\sum_{n=0}^{+\infty} u_n = u(0) + u'(0)x$$

$$+L^{-1}[\sum_{n=0}^{+\infty} a_n x^n] - pL^{-1}[\sum_{n=0}^{+\infty} a_n x^n]$$

$$-L^{-1}[R(\sum_{n=0}^{+\infty} u_n) + \sum_{n=0}^{+\infty} A_n(u_0, u_1, ..., u_n)],$$

where p is an artificial parameter and $a_i,\ i=0,1,...$ are unknown coefficients. We now define

$$u_0 = u(0) + u'(0)x + L^{-1}[\sum_{n=0}^{+\infty} a_n x^n] = 1 + \frac{1}{2}a_0 x^2$$

$$+\frac{1}{6}a_1 x^3 + \frac{1}{12}a_2 x^4 + \frac{1}{20}a_3 x^5 + \cdots,$$

$$u_1 = -pL^{-1}[\sum_{n=0}^{+\infty} a_n x^n] - L^{-1}[R(u_0) + A_0(u_0)],$$

$$u_2 = -L^{-1}[R(u_1) + A_1(u_0, u_1)],$$

$$\vdots$$

To avoid calculation of $A_n,\ n=1,2,...$, let determine $a_i,\ i=0,1,...$ such that $u_1 = 0$. Thus

$$(-1 - \frac{1}{2}a_0 - \frac{1}{2}pa_0)x^2 + (-\frac{1}{12}a_1 - \frac{1}{6}pa_1)x^3$$

$$+(-\frac{1}{2}a_0 - \frac{1}{36}a_2 - \frac{1}{12}pa_2)x^4 + \cdots = 0.$$

It can be easily shown that

$$a_0 = -\frac{2}{1+p}, \quad a_1 = 0, \quad a_2 = \frac{36}{(1+p)(1+3p)}, \quad a_3 = 0, \quad \cdots.$$

Setting $p=1$, results in

$$u(x)=1-\frac{1}{2}x^2+\frac{3}{8}x^4-\frac{5}{16}x^6+\cdots=\frac{1}{\sqrt{1+x^2}},$$

which is the exact solution of the problem.

Example 3.2: Consider the inhomogeneous Emden-Fowler equation with $\alpha=2$ [1]

$$u''+\frac{2}{x}u'-(6+4x^2)u=6-6x^2-4x^4,$$

subject to the following initial conditions

$$u(0)=1,\quad u'(0)=0.$$

Traditional ADM. Applying the traditional ADM yields

$$u_0(x)=1+3x^2-\frac{1}{2}x^4-\frac{2}{15}x^6,$$

$$u_1(x)=-3x^2+\frac{13}{6}x^4+\frac{53}{150}x^6-\frac{1}{20}x^8-\frac{4}{675}x^{10},$$

$$u_2(x)=6x^2-\frac{53}{18}x^4-\frac{27}{250}x^6+\frac{869}{4200}x^8+\frac{1663}{121500}x^{10}$$

$$-\frac{53}{29700}x^{12}-\frac{8}{61425}x^{14},$$

$\vdots$

Consequently, the series solution obtained by the traditional ADM is as follows

$$u(x)=1+6x^2-\frac{23}{18}x^4+\frac{14}{125}x^6+\frac{659}{4200}x^8$$

$$+\frac{943}{121500}x^{10}-\frac{53}{29700}x^{12}-\frac{8}{61425}x^{14}+\cdots.$$

Modified ADM. A similar procedure, described in previous example, leads to

$$u_0=u(0)+u'(0)x+L^{-1}[\textstyle\sum_{n=0}^{+\infty}a_nx^n]=1+\frac{1}{2}a_0x^2$$

$$+\frac{1}{6}a_1x^3+\frac{1}{12}a_2x^4+\frac{1}{20}a_3x^5+\cdots,$$

$$u_1=L^{-1}[6-6x^2-4x^4]-pL^{-1}[\textstyle\sum_{n=0}^{+\infty}a_nx^n]-L^{-1}[R(u_0)],$$

where $L^{-1}=\int_0^x\int_0^x(.)dxdx$ and $R(u)=(2/x)u'-(6+4x^2)u$. Now, by setting $u_1(x)=0$, we find

$$(6-a_0-\frac{1}{2}pa_0)x^2+(-\frac{1}{6}a_1-\frac{1}{6}pa_1)x^3$$

$$+(-\frac{1}{6}+\frac{1}{4}a_0-\frac{1}{18}a_2-\frac{1}{12}pa_2)x^4+\cdots=0.$$

It can be shown that

$$a_0=\frac{12}{2+p},\ a_1=0,\ a_2=-\frac{6(p-16)}{(2+p)(2+3p)},\ a_3=0,\ \cdots.$$

Setting $p=1$, yields

$$u(x)=1+2x^2+\frac{1}{2}x^4+\frac{1}{6}x^6+\cdots=x^2$$

$$+(1+x^2+\frac{1}{2}x^4+\frac{1}{6}x^6+\cdots),$$

which converges to the exact solution

$$u(x)=x^2+e^{x^2}.$$

Example 3.3: Consider the inhomogeneous Emden-Fowler equation with $\alpha>2$ [1]

$$u''+\frac{4}{x}u'-(18x+9x^4)u=20-36x^3-18x^6,$$

with the following initial conditions

$$u(0)=1,\quad u'(0)=0.$$

Traditional ADM. Using the traditional ADM leads to

$$u_0(x)=1+10x^2-\frac{9}{5}x^5-\frac{9}{28}x^8,$$

$$u_1(x)=-40x^2+3x^3+\frac{54}{5}x^5+\frac{3}{10}x^6$$

$$+\frac{297}{245}x^8-\frac{1539}{7700}x^{11}-\frac{81}{5096}x^{14},$$

$$u_2(x)=160x^2-6x^3-\frac{234}{5}x^5+\frac{39}{25}x^6$$

$$-\frac{12519}{3430}x^8+\frac{9}{20}x^9+\frac{156573}{134750}x^{11}+\frac{9}{440}x^{12}$$

$$+\frac{2873799}{63763700}x^{14}-\frac{365229}{47647600}x^{17}-\frac{729}{1936480}x^{20},$$

$\vdots$

Therefore, the series solution resulted from the traditional ADM is as follows

$$u(x)=1+130x^2-3x^3-\frac{189}{5}x^5+\frac{93}{50}x^6$$

$$-\frac{18927}{6860}x^8+\frac{9}{20}x^9+\frac{23571}{24500}x^{11}+\frac{9}{440}x^{12}$$

$$+\frac{3720573}{127527400}x^{14}-\frac{365229}{47647600}x^{17}-\frac{729}{1936480}x^{20}+\cdots.$$

Modified ADM. In a manner similar to that described in previous examples, we can define

$$u_0=u(0)+u'(0)x+L^{-1}[\textstyle\sum_{n=0}^{+\infty}a_nx^n]=1+\frac{1}{2}a_0x^2$$

$$+\frac{1}{6}a_1x^3+\frac{1}{12}a_2x^4+\frac{1}{20}a_3x^5+\cdots,$$

$$u_1=L^{-1}[20-36x^3-18x^6]-pL^{-1}[\textstyle\sum_{n=0}^{+\infty}a_nx^n]-L^{-1}[R(u_0)],$$

where $L^{-1}=\int_0^x\int_0^x(.)dxdx$ and $R(u)=(4/x)u'-(18x+9x^4)u$. Now, if we set $u_1(x)$ equal to zero, then we obtain

$$(10-2a_0-\frac{1}{2}pa_0)x^2+(3-\frac{1}{3}a_1-\frac{1}{6}pa_1)x^3$$

$$+(-\frac{1}{9}a_2-\frac{1}{12}pa_2)x^4+\cdots=0.$$

It can be easily shown that

$$a_0=\frac{20}{4+p},\ a_1=\frac{18}{2+p},\ a_2=0,\ a_3=-\frac{36(p-1)}{(1+p)(4+p)},\ \cdots.$$

Setting $p=1$, results in

$$u(x)=1+2x^2+x^3+\frac{1}{2}x^6+\cdots=2x^2$$

$$+(1+x^3+\frac{1}{2}x^6+\cdots)=2x^2+e^{x^3},$$

which is the exact solution of the problem.

4. Conclusion

In this article, a new and effective modification of Adomian decomposition method was proposed to solve singular, Emden-Fowler type, initial value problems with $\alpha=1,\ 2,$ and >2. As it was observed

- The method only requires the calculation of the first Adomian polynomial.
- The method provides the solution of the problems in the form of a convergent series, whereas the traditional ADM fails.
- The method overcomes the singularity at $x=0$.

It is worth mentioning that the proposed method can be extended for solving systems of ordinary differential equations of Emden-Fowler type as follows [23]

$$u_1''+\frac{\alpha_1}{x}u_1'+f_1(u_1,u_2)=g_1(x),\ \ u_1(0)=a_1,\ \ u_1'(0)=0,$$

$$u_2''+\frac{\alpha_2}{x}u_2'+f_2(u_1,u_2)=g_2(x),\ \ u_2(0)=a_2,\ \ u_2'(0)=0.$$

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