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An extension of the paper "Cone metric spaces and fixed point theorems of contractive mappings"

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Abstract

Huang and Zhang [1] established the results of fixed point theorems using contractive method in cone metric space. Later on, Rezapour and Hamlbarani [2] generalize the results [1] with non-normal cones and omitted the assumption of normality. In the present paper we improve these results.

Keywords: *Complete cone metric space, common fixed point, contractive condition.*

1 Introduction and Preliminaries

Very recently, Huang and Zhang [1] introduce the notion of cone metric spaces. He replaced real number system by ordered Banach space. He also gave the condition in the setting of cone metric spaces. These authors also described the convergence of sequences in the cone metric spaces and introduce the corresponding notion of completeness. The results in [1] were generalized by Sh. Rezapour and R Hamlbarani in [2] omitting the assumption of normality on the cone. Subsequently, many authors have generalized the results of Huang and Zhang and have studied fixed point theorems for normal and non-normal cone.

The purpose of this paper is to prove common fixed point theorems for contractive mappings in the context of complete cone metric space omitting the condition of normality of the cone. Our results generalize and improve the results of [2].

First, we recall some standard notations and definitions in cone metric spaces with some of their properties (see [1] and [2]).

Definition 1.1. Let E be a real Banach space and P a subset of E. P is called a cone if and only if:

(i) P is closed, non-empty and $P \neq \{0\}$,
(ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b,
(iii) $x \in P$ and $-x \in P \Rightarrow x = 0 \Leftrightarrow P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leqslant$ on E with respect to P by $x \leqslant y$ if and only if $y - x \in P$. We shall write $x \ll y$ if $y - x \in \text{int}P$, $\text{int}P$ denotes the interior of P. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$0 \leqslant x \leqslant y$ implies $\| x \| \leqslant K \| y \|$.

The least positive number K satisfying the above is called the normal constant of P[1].

The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}_{n\geqslant 1}$ is a sequence such that $x_1 \leqslant x_2 \leqslant - - - \leqslant y$ for some $y \in E$, then there is $x \in E$ such that $\lim_{n\to\infty} \| x_n - x \| = 0$. Equivalently, the cone P is called regular if and only if every decreasing sequence which is bounded from below is convergent.

It is well known that every regular cone is normal.

In the following, we always suppose E is a Banach space, P is a cone in E with $IntP \neq \phi$ and $\leqslant$ is partial ordering with respect to P.

Definition 1.2.([1]). Let X be a non-empty set. Suppose the mapping $d : X \times X \to E$ satisfies

(i) $0 \leqslant d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(iii) $d(x, y) \leqslant d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Example 1.3. Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geqslant 0\}, X = \mathbb{R}$ and $d : X \times X \to E$ defined by $d(x, y) = (| x - y |, \propto | x - y |)$, where $\propto \geqslant 0$ is a constant. Then (X, d) is is a cone metric space [1].

Definition 1.4.([1]) Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}_{n\geqslant 1}$ a sequence in X. Then

(i) $\{x_n)_{n\geqslant 1}$ converges to x whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geqslant N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

(ii) $\{x_n)_{n\geqslant 1}$ is a Cauchy sequence whenever for every$c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geqslant N$.

(iii) (X, d) is a complete cone metric space, if every Cauchy sequence is convergent.

Lemma 1.5([2]) There are no normal cones with normal constant $M < 1$, and for each $k > 1$ there are cones with normal constant $M > k$.

Finally, note that the relations $intP + intP \subseteq intP$ and $\lambda\, intP \subseteq intP (\lambda > 0)$ hold.

2 Main Results

In this section, we shall prove some fixed point theorems for pair of contractive maps by using non normality of the cone (by omitting the assumption of normality). We improve some results of [2].

Theorem 2.1. Let (X, d) be a complete cone metric space and the mappings $T_1, T_2 : X \to X$ satisfy the contractive condition,

$d(T_1x, T_2y) \leqslant kd(x, y)$ for all $x, y \in X$

where $k \in [0, 1)$ is a constant. Then T_1 and T_2 have a unique common fixed point in X. For each $x \in X$, the iterative sequences $\{T_1^{2n+1}x\}_{n\geqslant 1}$ and $\{T_2^{2n+2}x\}_{n\geqslant 1}$ converge to the common fixed point.

Proof. For each $x_o \in X$ and and $n \geqslant 1$, set $x_1 = T_1x_o, x_3 = T_1x_2 = T_1^3x_o, ---- x_{2n+1} = T_1x_{2n} = T_1^{2n+1}x_o, ------$

Similarly, we can have $x_2 = T_2x_1 = T_2^2x_0$, $x_4 = T_2x_3 = T_2^4x_o, ------$

$$----------- x_{2n+2} = T_2x_{2n+1} = T_2^{2n+2}x_o, ---- .$$

Then

$$d(x_{2n+1},x_{2n}) = d(T_1x_{2n},T_2x_{2n-1}) \leqslant kd(x_{2n},x_{2n-1})$$

$$\leqslant k^2d(x_{2n-1},x_{2n-2}) \leqslant ---------- \leqslant k^{2n}d(x_1,x_0).$$

So for $n > m$,

$$d(x_{2n},x_{2m}) \leqslant d(x_{2n},x_{2n-1}) + d(x_{2n-1},x_{2n-2}) + ------ + d(x_{2m+1},x_{2m})$$

$$\leqslant (k^{2n-1} + k^{2n-2} + ------ + k^{2m})d(x_1,x_o) \leqslant \tfrac{k^{2m}}{1-k}d(x_1,x_o)$$

Let $0 \ll c$ be given. Choose $\delta > o$ such that $c + N_\delta(o) \subseteq P$, where $N_\delta(o) = \{y \in E : \| y \| < \delta\}$. Also choose a natural number N_1 such that $\frac{k^{2m}}{1-k}d(x_1,x_o) \in N_\delta(o)$, for all $m \geqslant N_1$. Then $\frac{k^{2m}}{1-k}d(x_1,x_o) \ll c,\ for\ all\ m \geqslant N_1$.

Thus $$d(x_{2n},x_{2m}) \ll \tfrac{k^{2m}}{1-k}d(x_1,x_o) \ll c.$$

for all $n > m$. Therefore $\{x_{2n}\}_{n\geqslant 1}$ is a Cauchy sequence in (X,d). Since (X,d) is a complete cone metric space, there exists $x^* \in X$ such that $x_{2n} \to x^*$. Choose a natural number N_2 such that

$d(x_{2n},x^*) \ll \frac{c}{2}$ for all $n \geqslant N_2$. Hence

$$d(T_1x^*,x^*) \leqslant d(T_1x_{2n},T_1x^*) + d(T_1x_{2n},x^*)$$

$$\leqslant kd(x_{2n},x^*) + d(x_{2n+1},x^*)$$

$$\leqslant d(x_{2n},x^*) + d(x_{2n+1},x^*) \ll \tfrac{c}{2} + \tfrac{c}{2} = c.$$

for all $n \geqslant N_2$. Thus, $d(T_1x^*,x^*) \ll \frac{c}{2m}$, for all $m \geqslant 1$. So, $\frac{c}{2m} - d(T_1x^*,x^*) \in P$, for all $m \geqslant 1$. Since $\frac{c}{2m} \to 0$, (as $m \to \infty$) and P is closed, $-d(T_1x^*,x^*) \in P$. But $d(T_1x^*,x^*) \in P$. Therefore $d(T_1x^*,x^*) = 0$, and so $T_1x^* = x^*$.

Similarly it can be established that $T_2x^* = x^*$. Hence $T_1x^* = x^* = T_2x^*$. Thus x^* is the common fixed point of pair of maps T_1 and T_2.

Corollary 2.2 Let (X,d) be a complete cone metric space. Suppose the mappings $T_1,T_2 : X \to X$ satisfy for some positive integer n,

$$d(T_1^{2n+1}x, T_2^{2n+2}y), \leqslant kd(x,y),$$

for all $x,y \in X$, where $k \in [0,1)$ is a constant. Then T_1 and T_2 have a unique common fixed point in X.

Theorem 2.3 Let (X,d) be a complete cone metric space and the mappings $T_1,T_2 : X \to X$ satisfy the contractive condition

$$d(T_1x,T_2y), \leqslant k(d(T_1x,x) + d(T_2y,y)),$$

for all $x,y \in X$, where $k \in [0,\frac{1}{2})$ is a constant.

Then, T_1 and T_2 have a unique common fixed point in X. For each $x \in X$, the iterative sequences $\{T_1^{2n+1}x\}_{n\geqslant 1}$and $\{T_2^{2n+2}x\}_{n\geqslant 1}$ converge to the common fixed point.

Proof. For each $x_o \in X$, and $n \geqslant 1$, set $x_1 = T_1x_o, x_3 = T_1x_2 = T_1^3x_o, - - - - - x_{2n+1} = T_1x_{2n} = T_1^{2n+1}x_o, - - - -$

Similarly, we can have $x_2 = T_2x_1 = T_2^2x_o, x_4 = T_2x_3 = T_2^4x_o,$

$$- - - - - x_{2n+2} = T_2x_{2n+1} = T_2^{2n+2}x_o, - - - - .$$

Then,

$$d(x_{2n+1},x_{2n}) = d(T_1x_{2n},T_2x_{2n-1})$$

$$\leqslant k(d(T_1x_{2n},x_{2n}) + d(T_2x_{2n-1},x_{2n-1}))$$

$$= k(d(x_{2n+1},x_{2n}) + d(x_{2n},x_{2n-1}))$$

So,

$$d(x_{2n+1},x_{2n}) \leqslant \tfrac{k}{1-k}d(x_{2n},x_{2n-1}) = hd(x_{2n},x_{2n-1})$$

Where $h = \frac{k}{1-k}$. For $n > m$,

$$d(x_{2n},x_{2m}) \leqslant d(x_{2n},x_{2n-1}) + d(x_{2n-1},x_{2n-2}) + - - - - - - + d(x_{2m+1},x_{2m})$$

$$\leqslant (h^{2n-1} + h^{2n-2} + - - - - - + h^{2m})d(x_1, x_o) \leqslant \tfrac{h^{2m}}{1-h}d(x_1, x_o)$$

Let $0 \ll c$ be given.Choose a natural number N_1 such that $\frac{h^{2m}}{1-h}d(x_1, x_o) \ll c$,

for all $m \geqslant N_1$. Thus $d(x_{2n},x_{2m}) \ll c$,

for $n > m$. Therefore $\{x_{2n}\}_{n\geqslant 1}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete cone metric space, there exists $x^* \in X$ such that $x_{2n} \to x^*$. Choose a natural number N_2 such that

$d(x_{2n+1},x_{2n}) \ll \frac{c(1-k)}{2k}$ and

$d(x_{2n+1},x^*) \ll \frac{c(1-k)}{2}$, for all $n \geqslant N_2$. Hence

for $n \geqslant N_2$ we have

$$d(T_1x^*, x^*) \leqslant d(T_1x_{2n}, T_1x^*) + d(T_1x_{2n},x^*)$$

$$\leqslant k(d(T_1x_{2n}, x_{2n}) + d(T_1x^*, x^*))$$

$$+d(x_{2n+1},x^*).$$

Thus,

$$d(T_1x^*, x^*) \leqslant \tfrac{1}{1-k}(kd(x_{2n+1}, x_{2n}) + d(x_{2n+1},x^*))$$

$$\ll \tfrac{c}{2} + \tfrac{c}{2} = c$$

Thus, $d(T_1x^*, x^*) \ll \frac{c}{2m}$, for all $m \geqslant 1$. So $\frac{c}{2m} - d(T_1x^*, x^*) \in P$, for all $m \geqslant 1$. Since $\frac{c}{2m} \to 0$ $(as\, m \to \infty)$ and P is closed, $-d(T_1x^*, x^*) \in P$. But $d(T_1x^*, x^*) \in P$. Therefore, $d(T_1x^*, x^*) = 0$, and so $T_1x^* = x^*$. Now, if $y^*$is another fixed point of T_1 then $d(x^*, y^*) = d(T_1x^*, T_1y^*) \leqslant k(d(T_1x^*, x^*) + d(T_1y^*, y^*)) = 0$.

Hence $x^* = y^*$. Therefore, the fixed point of T_1 is unique. Similarly, it can be established that $T_2x^* = x^*$. Hence $T_1x^* = x^* = T_2x^*$. Thus x^* is the common fixed point of pair of maps $T_1$and T_2.

Theorem 2.4 Let (X, d) be a complete cone metric space and the mappings $T_1, T_2 : X \to X$ satisfy the contractive condition

$d(T_1x, T_2y), \leqslant k(d(T_1x, y) + d(x, T_2y)),$

for all $x, y \in X$, where $k \in [0, \frac{1}{2})$ is a constant.

Then, T_1 and T_2 have a unique common fixed point in X. For each $x \in X$, the iterative sequences $\{T_1^{2n+1}x\}_{n\geqslant1}$and $\{T_2^{2n+2}x\}_{n\geqslant1}$ converge to the common fixed point.

Proof. For each $x_o \in X$, and $n \geqslant 1$, set $x_1 = T_1x_o, x_3 = T_1x_2 = T_1^3x_o, - - - - - x_{2n+1} = T_1x_{2n} = T_1^{2n+1}x_o, - - - -$

Similarly, we can have $x_2 = T_2x_1 = T_2^2x_o, x_4 = T_2x_3 = T_2^4x_o,$

$$- - - - - x_{2n+2} = T_2x_{2n+1} = T_2^{2n+2}x_o, - - - -$$

Then,

$$\begin{aligned}
d(x_{2n+1}, x_{2n}) &= d(T_1x_{2n}, T_2x_{2n-1}) \\
&\leqslant k(d(T_1x_{2n}, x_{2n-1}) + d(T_2x_{2n-1}, x_{2n})) \\
&\leqslant k(d(x_{2n+1}, x_{2n}) + d(x_{2n}, x_{2n-1}))
\end{aligned}$$

So,

$d(x_{2n+1}, x_{2n}) \leqslant \frac{k}{1-k}d(x_{2n}, x_{2n-1}) = hd(x_{2n}, x_{2n-1})$

Where $h = \frac{k}{1-k}$. For $n > m$,

$$\begin{aligned}
d(x_{2n}, x_{2m}) &\leqslant d(x_{2n}, x_{2n-1}) + d(x_{2n-1}, x_{2n-2}) + - - - - - - + d(x_{2m+1}, x_{2m}) \\
&\leqslant (h^{2n-1} + h^{2n-2} + - - - - - - + h^{2m})d(x_1, x_o) \\
&\leqslant \frac{h^{2m}}{1-h}d(x_1, x_o)
\end{aligned}$$

Let $0 \ll c$ be given.Choose a natural number N_1 such that $\frac{h^{2m}}{1-h}d(x_1, x_o) \ll c$, for all $m \geqslant N_1$. Thus $d(x_{2n}, x_{2m}) \ll c$,

for $n > m$. Therefore $\{x_{2n}\}_{n\geqslant1}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete cone metric space, there exists $x^* \in X$ such that $x_{2n} \to x^*$. Choose a natural number N_2 such that

$d(x_{2n}, x^*) \ll \frac{c(1-k)}{3}$ for all $n \geqslant N_2$

Hence for $n \geqslant N_2$ we have

$$\begin{aligned}
d(T_1x^*, x^*) &\leqslant d(T_1x_{2n}, T_1x^*) + d(T_1x_{2n}, x^*) \\
&\leqslant k(d(T_1x^*, x_{2n}) + d(T_1x_{2n}, x^*)) + d(x_{2n+1}, x^*) \\
&\leqslant k(d(T_1x^*x^*) + d(x_{2n}, x^*) + d(x_{2n+1}, x^*))
\end{aligned}$$

$$+d(x_{2n+1,}x^*).$$

Thus,

$$d(T_1x^*, x^*) \leqslant \frac{1}{1-k}(kd(x_{2n}, x^*) + d(x_{2n+1,}x^*))$$

$$+d(x_{2n+1,}x^*) \ll \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c.$$

Thus, $d(T_1x^*, x^*) \ll \frac{c}{2m}$, for all $m \geqslant 1$. So $\frac{c}{2m} - d(T_1x^*, x^*) \in P$, for all $m \geqslant 1$. Since $\frac{c}{2m} \to 0$ $(as\, m \to \infty)$ and P is closed, $-d(T_1x^*, x^*) \in P$. But $d(T_1x^*, x^*) \in P$. $Therefore$ $d(T_1x^*, x^*) = 0$ and so $T_1x^* = x^*$. Now, if $y^*$is another fixed point of T_1, then .

$$d(x^*, y^*) = d(T_1x^*, T_1y^*) \leqslant k(d(T_1x^*, y^*) + d(T_1y^*, x^*)) = 2kd(x^*, y^*)$$

Hence $d(x^*, y^*) = 0$ and so $x^* = y^*$. Therefore, the fixed point of T_1 is unique. Similarly, it can be established that $T_2x^* = x^*$. Hence $T_1x^* = x^* = T_2x^*$. Thus x^* is the common fixed point of pair of maps $T_1$and T_2.

Theorem 2.5 Let (X, d) be a complete cone metric space and the mappings $T_1, T_2 : X \to X$ satisfy the contractive condition

$$d(T_1x, T_2y), \leqslant kd(x, y) + ld(x, T_2y),$$

for all $x, y \in X$, where $k, l \in [0, 1)$ is a constant.

Then, T_1 and T_2 have a unique common fixed point in X. whenever $k + l < 1$.

Proof. For each $x_o \in X$, and $n \geqslant 1$, set $x_1 = T_1x_o, x_3 = T_1x_2 = T_1^3x_o, - - - - - x_{2n+1} = T_1x_{2n} = T_1^{2n+1}x_o, - - - -$

Similarly, we can have $x_2 = T_2x_1 = T_2^2x_o, x_4 = T_2x_3 = T_2^4x_o, - - - - - - - - - - x_{2n+2} = T_2x_{2n+1} = T_2^{2n+2}x_o, - - - -$

Then,

$$d(x_{2n+1,}x_{2n}) = d(T_1x_{2n,}T_2x_{2n-1})$$

$$\leqslant k(d(x_{2n,}x_{2n-1}) + ld(T_2x_{2n-1,}x_{2n}))$$

$$= kd(x_{2n,}x_{2n-1}) \leqslant k^{2n}d(x_{1,}x_o)$$

Thus for $n > m$, we have

$$d(x_{2n,}x_{2m}) \leqslant d(x_{2n,}x_{2n-1}) + d(x_{2n-1,}x_{2n-2}) + - - - - - - + d(x_{2m+1,}x_{2m})$$

$$\leqslant (k^{2n-1} + k^{2n-2} + - - - - - - - - - + k^{2m})d(x_1, x_o)$$

$$\leqslant \frac{k^{2m}}{1-k}d(x_{1,}x_o)$$

Let $0 \ll c$ be given. Choose a natural number N_1 such that $\frac{k^{2m}}{1-k}d(x_1, x_o) \ll c$, for all $m \geqslant N_1$. Thus $d(x_{2n,}x_{2m}) \ll c$,

for $n > m$. Therefore $\{x_{2n}\}_{n \geqslant 1}$ is a Cauchy sequence in (X, d) is a complete cone metric space, there exists $x^* \in X$ such that $x_{2n} \to x^*$. Choose a natural number N_2 such that

$$d(x_{2n,}x^*) \ll \frac{c}{3} \text{ for all } n \geqslant N_2$$

Hence, for $n \geqslant N_2$ we have

$$d(T_1x^*, x^*) \leqslant d(x_{2n}, T_1x^*) + d(x_{2n,}x^*)$$

$$= d(T_1x_{2n-1}, T_1x^*) + d(x_{2n}, x^*)$$

$$\leqslant kd(x_{2n-1,}x^*) + ld(T_1x_{2n-1,}x^*) + d(x_{2n,}x^*)$$

$$\leqslant d(x_{2n-1,}x^*) + d(x_{2n,}x^*) + d(x_{2n,}x^*).$$

So,

$$d(T_1x^*, x^*) \leqslant \tfrac{c}{3} + \tfrac{c}{3} + \tfrac{c}{3} = c.$$

Thus,

$$d(T_1x^*, x^*) \leqslant \tfrac{c}{2m}, \text{ for all } m \geqslant 1.$$

Hence $\frac{c}{2m} - d(T_1x^*, x^*) \in P$, for all $m \geqslant 1$. Since $\frac{c}{2m} \to 0$ $(as\, m \to \infty)$ and P is closed, $-d(T_1x^*, x^*) \in P$. But, $d(T_1x^*, x^*) \in P$. Therefore $d(T_1x^*, x^*) = 0$, and so $T_1x^* = x^*$. Now, if y^* is another fixed point of T_1 and $k + l < 1$, then

$$d(x^*, y^*) = d(T_1x^*, y^*) \leqslant kd(x^*, y^*) + ld(T_1x^*, y^*)$$

$$= (k + l)d(x^*, y^*).$$

Hence $d(x^*, y^*) = 0$ and so $x^* = y^*$. Therefore, the fixed point of T_1 is unique whenever $k + l < 1$. Similarly, it can be established that $T_2x^* = x^*$. Hence $T_1x^* = x^* = T_2x^*$. Thus x^* is the unique common fixed point of pair of maps $T_1$and T_2, .whenever $k + l < 1$.

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