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Research paper

Estimation of parameters in stochastic differential equations with two random effects

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Abstract

In this paper we investigate consistency and asymptotic normality of the posterior distribution of the parameters in the stochastic differential equations (SDE's) with diffusion coefficients depending nonlinearly on a random variables $\emptyset_i$ and μ_i (the random effects).The distributions of the random effects $\emptyset_i$ and μ_i depends on unknown parameters which are to be estimated from the continuous observations of the independent processes $(X_i(t), t \in [0, T_i], i = 1, \dots, n)$. We propose the Gaussian distribution for the random effect $\emptyset_i$ and the exponential distribution for the random effect μ_i , we obtained an explicit formula for the likelihood function and find the estimators of the unknown parameters in the random effects.

Keywords: *Stochastic Differential Equations; Maximum Likelihood Estimator; Nonlinear Random Effects; Posterior Consistency; Posterior Normality.*

1. Introduction

Stochastic differential equations play an important role in many areas of science fields as physics, engineering, chemistry, neuroscience, biology, finance (Gugushvili and P. Spreij (2012)[8]). Statistical estimation of parameters in the diffusion processes has been studied for a long time. In the recent years, the stochastic differential equations with random effects have been the subject of diverse applications such as pharmacokinetic/pharmacodynamics, neuronal modeling (Delattre and Lavelle, 2013[4], Donnet and Samson, 2013[7], Picchini et al. 2010[13]).Maximum likelihood estimator of the parameters of the random effect, is generally not possible, because of the likelihood function is not available in most cases. Many references proposed approximations for the unknown likelihood function, for general mixed SDEs an approximations of the likelihood have been proposed (Picchini and Ditlevsen, 2011[12]), linearization (Beal and Sheiner (1982)[3]), or approximating the conditional transition density of the diffusion process given the random effects by a Hermit expansion‹ (Aït-Sahalia (2002)[1]).Maitra et al. (2015) [11]) studied consistency and asymptotic normality of the posterior distribution of the parameters in the SDE's with one random effect in the drift term, Delattre et al. (2012) [6] and alkreemawi et al. (2015) [2] are studied the maximum likelihood estimator for random effects in more generally for fixed T and n tending to infinity (for non i.i.d. sample paths, see Maitra et al. (2014) [10]) and they found an explicit expression for likelihood function and exact likelihood estimator by investigate the linear random effect in the drift (multiple and additive case respectively) together with a specific distribution for the random effect. Almost researcher studied the random effect in the drift not in diffusion except Delattre and Lavelle, 2013[4] who incorporate measurement error and propose an approximation of the likelihood with the extended Kalman filter, and Delattre et al. (2014) [5] who used one random effect in the diffusion coefficient We study the consistency and asymptotic normality of the Bayesian posterior distribution when the random effects are Gaussian with a specific distributions and focus on discretely observed SDEs.

In the present work we focus on stochastic differential equation with two random effects in diffusion coefficient and suppose that the drift coefficient without random effect. We consider n real valued stochastic processes$(X_i(t), t \in [0, T_i], i = 1, \dots, n)$, with dynamics ruled by the following SDEs:

$$dX_i(t) = b\big(X_i(t)\big)dt + \sigma(X_i(t), \emptyset_i, \mu_i)dW_i(t)\,, \quad X_i(0) = x^i, \quad i = 1, \dots, n \qquad (1)$$

Where $W_1, \dots, W_n$ are n independent wiener processes, $\emptyset_1, \dots, \emptyset_n$ and $\mu_1, \dots, \mu_n$ are n $i.i.d.$ random variables taking values in ($\mathbb{R}$ and $\mathbb{R}^+$) respectively, $\emptyset_1, \dots, \emptyset_n$, $\mu_1, \dots, \mu_n$ and $W_1, \dots, W_n$ are independent and x^i, $i = 1, \dots, n$ are known real values. The functions $b(x)$(drift term) and $\sigma(x)$(diffusion term) are known real valued functions. Each process $X_i(t)$ represents an individual, the variables $\emptyset_i$ and μ_i represents the random effects of individual i , the random variables $\emptyset_1, \dots, \emptyset_n$ have a common distribution $g(\varphi, \theta)dv(\varphi)$ on $\mathbb{R}$ and the random variables $\mu_1, \dots, \mu_n$ have a common distribution $h(\mu, \beta)du(\mu)$ on $\mathbb{R}^+$ where θ and β are an unknown parameters belonging to a set $\boldsymbol{\Theta} \subset \mathbb{R}^p$ where v and u are a dominating measures.

Our aim is to estimate $\psi = (\theta, \beta)$ from the continuous observations $(X_i(t), t \in [0, T_i], i = 1, \dots, n)$ and prove consistency and asymptotic normality of the Bayesian posterior distribution of $\psi = (\theta, \beta)$. We focus on a special case of nonlinear random effect in the diffusion coefficient in the model (1), i.e. $\sigma(x, \emptyset_i, \mu_i) = (\emptyset_i + \mu_i)^{-1}\,\sigma(x)$, where σ is a known real function and $\emptyset_i$ is a Gaussian and μ_i is an exponential, an explicit likelihood formula and the maximum likelihood estimator of ψ are obtained.

The rest of the paper is organized as follows. Section 2 contains the notation and assumptions. The general results of the estimation of the parameters are introduced in section 3. In section 4 and exponential distribution respectively. Conclusion is given in section 5.

2. Notations and assumptions

Consider n real valued stochastic processes $(X_i(t), t \geq 0)$, $i = 1, \dots, n$ with dynamics ruled by (1). The processes $W_1, \dots, W_n$ and the random variables $\emptyset_1, \dots, \emptyset_n$ and $\mu_1, \dots, \mu_n$ are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ .Consider the filtration $(\mathcal{F}_t, t \geq 0)$ defined by $\mathcal{F}_t = \sigma(\emptyset_i, \mu_i, W_i(s), s \leq t, i = 1, \dots, n)$. As $\mathcal{F}_t = \sigma(\emptyset_i, \mu_i, W_i(s), s \leq t) \vee \mathcal{F}_t^i$, with $\mathcal{F}_t^i = \sigma(\emptyset_i, \emptyset_j, \mu_i, \mu_j, W_j(s), s \leq t, j \neq i)$ independent of W_i, each process W_i is a $(\mathcal{F}_t, t \geq 0)$-Brownian motion. Moreover, the random variables $\emptyset_i$, μ_i are $\mathcal{F}_0$ – measurable. We assume that:

H1

i) The function $b(x)$ is C^1 on $\mathbb{R}$, and such that: $\exists K > 0, \forall x \in \mathbb{R}, b^2(x) \leq K(1 + x^2)$,

ii) The function $\sigma(x, \varphi, \mu)$ is C^1 on $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^+$ and $\forall (x, \varphi, \mu) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^+$, $\sigma^2(x, \varphi, \mu) \leq K(1 + x^2 + |\varphi|^2 + |\mu|^2)$.

From **H1**, the process $(X_i(t))$ is well define and $(\emptyset_i, \mu_i, X_i(t))$ adapted to filtration $(\mathcal{F}_t, t \geq 0)$.

The n processes $(\emptyset_i, \mu_i, X_i(t), i = 1, \dots, n)$ are independent. For all φ, μ and all $x^i \in \mathbb{R}$, the stochastic differential equation

$$dX_i^{\varphi,\mu}(t) = b\left(X_i^{\varphi,\mu}(t)\right)dt + \sigma\left(X_i^{\varphi,\mu}(t), \varphi, \mu\right)dW_i(t), \quad X_i^{\varphi,\mu}(0) = x^i \quad (2).$$

Admits a unique strong solution process $(X_i^{\varphi,\mu}(t), t \geq 0)$ adapted to filtration $(\mathcal{F}_t, t \geq 0)$. We deduce that the conditional distribution of X_i given $\emptyset_i = \varphi$ and $\mu_i = \mu$ identical to the distribution of $X_i^{\varphi,\mu}$.

3. A general results of estimation of the parameters

3.1. Exact likelihood

We introduce the distribution $Q_{\varphi,\mu}^{x^i,T_i}$ of $(X_i^{\varphi,\mu}(t), t \in [0, T_i])$.

Let $P_\psi^i = g(\varphi, \theta)dv(\varphi) \otimes h(\mu, \beta)du(\mu) \otimes Q_\varphi^{x^i,T_i}$ denote the joint distribution of $(\emptyset_i, \mu_i, X_i(t))$ and let Q_ψ^i denote the marginal distribution of $(X_i(t), t \in [0, T_i])$. Let us consider the following assumption:

H2 For $i = 1, \dots, n$ and for all $\varphi, \mu, \varphi', \mu'$,

$$Q_{\varphi,\mu}^{x^i,T_i}\left(\int_0^{T_i} \frac{b^2\left(X_i^{\varphi,\mu}(t)\right)}{\sigma^2\left(X_i^{\varphi,\mu}(t), \varphi', \mu'\right)} dt < +\infty\right) = 1$$

Under **H1-H2**, the derivative of the distribution $Q_{\varphi,\mu}^{x^i,T_i}$ with respect to derivative of $Q^i = Q_{\varphi_0,\mu_0}^{x^i,T_i}$ has the density

$$\frac{dQ_{\varphi,\mu}^{x^i,T_i}}{dQ^i}(X_i) = L_{T_i}(X_i, \varphi, \mu)$$

$$= \exp\left(\int_0^{T_i} \frac{b(X_i(s))}{\sigma^2(X_i(s), \varphi, \mu)} dX_i(s) - \frac{1}{2}\int_0^{T_i} \frac{b^2(X_i(s))}{\sigma^2(X_i(s), \varphi, \mu)} ds\right) \quad (3)$$

(See Liptser and Shiryaev [9]).

H3 for $i = 1, \dots, n$, we assume that $U_i - \frac{1}{2}V_i < \infty$, where:

$$U_i = \int_0^{T_i} \frac{b(X_i(s))}{\sigma^2(X_i(s))} dX_i(s), \quad V_i = \int_0^{T_i} \frac{b^2(X_i(s))}{\sigma^2(X_i(s))} ds .$$

By independent of individuals, $P_\psi = \otimes_{i=1}^n P_\psi^i$ is the distribution of $(\emptyset_i, \mu_i, X_i(.))$, $i = 1, \dots, n$ and $Q_\psi = \otimes_{i=1}^n Q_\psi^i$ is the distribution of the sample$(X_i(t), t \in [0, T_i], i = 1, \dots, n)$. We can compute the density of Q_ψ w.r.t. $Q = \otimes_{i=1}^n Q^i$ as follow:

$$\gamma_i(X_i, \psi) = \frac{dQ_\psi}{dQ^i}(X_i) = \int_{\mathbb{R}^+}\int_{\mathbb{R}} L_{T_i}(X_i, \varphi, \mu) g(\varphi, \theta) h(\mu, \beta) dv(\varphi) du(\mu)$$

And the exact likelihood of whole sample $(X_i(t), t \in [0, T_i], i = 1, \dots, n)$ is

$$\xi_n(\psi) = \prod_{i=1}^n \gamma_i(X_i, \psi).$$

3.2 The distributions of the random effects

Consider model (1) with nonlinear random effects in the diffusion coefficient $\sigma(x, \emptyset_i, \mu_i) = (\emptyset_i + \mu_i)^{-1}\sigma(x)$ where $\varphi \in \mathbb{R}$, $\mu \in \mathbb{R}^+$ and $b(.), \sigma(.)$ are known functions. We assume that:

$$\int_0^{T_i} \frac{b^2(X_i(s))}{\sigma^2(X_i(s))} ds < \infty, \ Q_{\varphi,\mu}^{x^i,T_i} - a.s,$$

for all φ, μ and for $i = 1, \dots, n$; $T_i = T, x^i = x$, so that $(X_i(t), t \in [0, T], i = 1, \dots, n)$ are $i.i.d$. We will use the define statistics as follow:

$$U_i = \int_0^T \frac{b(X_i(s))}{\sigma^2(X_i(s))} dX_i(s), \quad V_i = \int_0^T \frac{b^2(X_i(s))}{\sigma^2(X_i(s))} ds \quad (4)$$

So that the density $\gamma_i(X_i, \psi)$ is given by:

$$\gamma_i(X_i, \psi) = \int_{\mathbb{R}^+}\int_{\mathbb{R}} \exp\left((\varphi + \mu)^2\left(U_i - \frac{1}{2}V_i\right)\right) g(\varphi, \theta) h(\mu, \beta) dv(\varphi) du(\mu) \quad (5)$$

For a general distributions, $g(\varphi, \theta)dv(\varphi)$ for the random effect $\emptyset_i$ and $h(\mu, \beta)du(\mu)$ for the random effect μ, it is not possible find an explicit expression for $\gamma_i(X_i, \psi)$ above, therefor we propose a specific distributions, Gaussian (λ, ω^2) for the random effect φ and an exponential (β) for the random effect μ , which will give an explicit likelihood and then find the maximum likelihood estimators of the unknown parameters. In the next proposition an evident expression for $\gamma_i(X_i, \psi)$is obtained when the above distributions of the random effects is with unknown parameter $\psi = (\lambda, \omega^2, \beta) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$ .The true value is denoted by $\psi_0 = (\lambda_0, {\omega^2}_0, \beta_0)$.

Proposition 3.1 suppose that $g(\varphi, \theta)dv(\varphi) = \mathcal{N}(\lambda, \omega^2)$, and $h(\mu, \beta)du(\mu) = exp(\beta)$ then:

$$\gamma_i(X_i, \psi) = \frac{\sqrt{\pi}\beta}{\sqrt{M_i}} \exp\left(-\frac{1}{4}\frac{(\beta(1 - 2M_i\omega^2) - 2\lambda M_i)^2}{M_i(1 - 2\omega^2 M_i)} + \frac{\lambda^2 - \lambda(1 - 2M_i\omega^2)}{2\omega^2(1 - 2M_i\omega^2)}\right),$$

Where $M_i = U_i - \frac{1}{2}V_i$.

Proof: from (5) we compute the joint density of$(\emptyset_i, \mu_i, X_i)$:

$$exp\left((\varphi + \mu)^2\left(U_i - \frac{1}{2}V_i\right)\right)\frac{1}{\sqrt{2\pi\omega^2}} \times exp\left(-\frac{1}{2\omega^2}(\varphi - \lambda)^2\right) \times \beta exp(-\beta\mu).$$

Let $M_i = U_i - \frac{1}{2}V_i$, then the exponent become:

$$D_i = \varphi^2 M_i - \frac{1}{2\omega^2}(\varphi-\lambda)^2 + 2\varphi\mu M_i + \mu^2 M_i - \beta\mu. \quad (6)$$

We will compute the first part $(\varphi^2 M_i - \frac{1}{2\omega^2}(\varphi-\lambda)^2 + 2\varphi\mu M_i)$ of the exponent as follow:

$$\varphi^2 M_i - \frac{1}{2\omega^2}(\varphi-\lambda)^2 + 2\varphi\mu M_i$$

$$= \left(M_i - \frac{1}{2\omega^2}\right)\varphi^2 + \left(\frac{\lambda}{\omega^2} + 2\mu M_i\right)\varphi - \frac{\lambda^2}{2\omega^2}$$

$$= \frac{-1}{2}\left(\frac{1}{\omega^2} - 2M_i\right)\left(\varphi^2 - 2\frac{\lambda+2\omega^2 M_i\mu}{1-2\omega^2 M_i}\varphi\right) - \frac{\lambda^2}{2\omega^2}$$

$$= \frac{-1}{2}\left(\frac{1}{\omega^2} - 2M_i\right)\left[\left(\varphi^2 - \frac{\lambda+2\omega^2 M_i\mu}{1-2\omega^2 M_i}\right)^2 - \left(\frac{\lambda+2\omega^2 M_i\mu}{1-2\omega^2 M_i}\right)^2\right] - \frac{\lambda^2}{2\omega^2}$$

$$= \frac{-1}{2}\left(\frac{1-2\omega^2 M_i}{\omega^2}\right)\left(\varphi - \frac{\lambda+2\omega^2 M_i\mu}{1-2\omega^2 M_i}\right)^2$$

$$+ \frac{(\lambda+2\omega^2 M_i\mu)^2}{2\omega^2(1-2\omega^2 M_i)} - \frac{\lambda^2}{2\omega^2}.$$

Now, by split the result into two parts that are independent and dependent on the random effect φ respectively, we find that the integral of the dependent part is the integral of a Gaussian density. Then the first integral in (5) with respect to φ yields the following result:

$$\frac{1}{\sqrt{1-2\omega^2 M_i}} exp\left(\frac{(\lambda+2\omega^2 M_i\mu)^2}{2\omega^2(1-2\omega^2 M_i)} - \frac{\lambda^2}{2\omega^2}\right)$$

By substituting in (5), the second part of the exponent is become:

$$E_i = \frac{(\lambda+2\omega^2 M_i\mu)^2}{2\omega^2(1-2\omega^2 M_i)} + \mu^2 M_i - \beta\mu - \frac{\lambda^2}{2\omega^2}$$

$$= \left(\frac{2\omega^2 M_i{}^2}{1-2\omega^2 M_i} + M_i\right)\mu^2 - \left(\beta - \frac{2\lambda M_i}{1-2\omega^2 M_i}\right)\mu$$

$$+ \frac{\lambda^2-\lambda(1-2\omega^2 M_i)}{2\omega^2(1-2\omega^2 M_i)}$$

$$= \frac{M_i}{1-2\omega^2 M_i}\left(\mu^2 - \left(\frac{\beta(1-2\omega^2 M_i)-2\lambda M_i}{M_i}\right)\mu\right)$$

$$+ \frac{\lambda^2-\lambda(1-2\omega^2 M_i)}{2\omega^2(1-2\omega^2 M_i)}$$

$$= \frac{M_i}{1-2\omega^2 M_i}\left[\left(\mu - \frac{1}{2}\frac{\beta(1-2\omega^2 M_i)-2\lambda M_i}{M_i}\right)^2 - \left(\frac{1}{2}\frac{\beta(1-2\omega^2 M_i)-2\lambda M_i}{M_i}\right)^2\right]$$

$$+ \frac{\lambda^2-\lambda(1-2\omega^2 M_i)}{2\omega^2(1-2\omega^2 M_i)}$$

$$= -\frac{1}{2}\frac{M_i}{\omega^2 M_i - \frac{1}{2}}\left(\mu - \frac{1}{2}\frac{\beta(1-2\omega^2 M_i)-2\lambda M_i}{M_i}\right)^2$$

$$-\frac{1}{4}\frac{(\beta(1-2\omega^2 M_i)-2\lambda M_i)^2}{M_i(1-2\omega^2 M_i)} + \frac{\lambda^2-\lambda(1-2\omega^2 M_i)}{2\omega^2(1-2\omega^2 M_i)}$$

Now, by rearrange the second integral we see that the first part is normal depend on the random effect μ with mean is

$$m_i = \frac{1}{2}\frac{\beta(1-2\omega^2 M_i)-2\lambda M_i}{M_i},$$

And variance,

$$\sigma_i^2 = \frac{\omega^2 M_i - \frac{1}{2}}{M_i},$$

Then, the conditional distribution of $(\emptyset_i, \mu_i)$ given X_i is $\mathcal{N}(m_i, \sigma_i^2)$.
And hence,

$$\gamma_i(X_i,\psi) = \frac{\sqrt{\pi}\beta}{\sqrt{M_i}} exp\left(-\frac{1}{4}\frac{(\beta(1-2M_i\omega^2)-2\lambda M_i)^2}{M_i(1-2\omega^2 M_i)} + \frac{\lambda^2-\lambda(1-2M_i\omega^2)}{2\omega^2(1-2M_i\omega^2)}\right).$$

3.2 The estimators of the parameters of the random effects

A natural approach to estimate $\psi = (\lambda, \omega^2, \beta)$ is the maximum likelihood estimation, so, the likelihood function is written as:

$$\xi_n(\psi) = \prod_{i=1}^n \frac{\sqrt{\pi}\beta}{\sqrt{M_i}} exp\left(-\frac{1}{4}\frac{(\beta(1-2M_i\omega^2)-2\lambda M_i)^2}{M_i(1-2\omega^2 M_i)} + \frac{(\lambda^2-\lambda(1-2M_i\omega^2))}{2\omega^2(1-2M_i\omega^2)}\right).$$

And hence, the logarithm of likelihood function is,

$$\mathcal{L}_n(\psi) = log\xi_n(\psi)$$

$$= log\pi^{\frac{n}{2}}\beta^n - \frac{1}{2}\sum_{i=1}^n log(M_i)$$

$$-\sum_{i=1}^n\left[\frac{1}{4}\frac{(\beta(1-2M_i\omega^2)-2\lambda M_i)^2}{M_i(1-2\omega^2 M_i)} - \frac{(\lambda^2-\lambda(1-2M_i\omega^2))}{2\omega^2(1-2M_i\omega^2)}\right]. \quad (7)$$

We will study the following score function

$$G_n(\psi) = \left(\frac{\partial}{\partial\lambda}\mathcal{L}_n(\psi)\ \frac{\partial}{\partial\beta}\mathcal{L}_N(\psi)\ \frac{\partial}{\partial\omega^2}\mathcal{L}_n(\psi)\right)'$$

Where x' denotes the transpose of x, such that:

$$\frac{\partial}{\partial\lambda}\mathcal{L}_n(\psi) = \sum_{i=1}^n\left[\frac{\beta(1-2M_i\omega^2)-2\lambda M_i}{1-2\omega^2 M_i} + \frac{2\lambda-(1-2M_i\omega^2)}{2\omega^2(1-2M_i\omega^2)}\right],$$

$$= \sum_{i=1}^n\left[\beta + \frac{2\lambda(1-2M_i\omega^2)}{2\omega^2(1-2M_i\omega^2)} - \frac{1}{2\omega^2}\right].$$

$$= \sum_{i=1}^n\left[\beta + \frac{2\lambda-1}{2\omega^2}\right].\frac{\partial}{\partial\omega^2}\mathcal{L}_n(\psi) = \sum_{i=1}^n\left[\frac{(\beta(1-2M_i\omega^2)-2\lambda M_i)\beta}{1-2M_i\omega^2} - \frac{1}{2}\frac{(\beta(1-2M_i\omega^2)-2\lambda M_i)^2}{(1-2\omega^2 M_i)^2} - \frac{\lambda^2(2-8M_i\omega^2)}{\left(2\omega^2(1-2M_i\omega^2)\right)^2} + \frac{\lambda}{2(\omega^2)^2}\right].$$

$$= \sum_{i=1}^n\left[\frac{1}{2}\beta^2 - \frac{2\lambda^2 M_i{}^2}{(1-2M_i\omega^2)^2} - \frac{2\lambda^2-8\lambda M_i\omega^2}{\left(2\omega^2(1-2M_i\omega^2)\right)^2} + \frac{\lambda}{2(\omega^2)^2}\right]$$

$$= \sum_{i=1}^n\left[\frac{1}{2}\beta^2 - \frac{2\lambda^2 M_i{}^2}{(1-2M_i\omega^2)^2} - \frac{2\lambda^2(1-2M_i\omega^2-2M_i\omega^2)}{(2\omega^2)^2(1-2M_i\omega^2)^2} + \frac{\lambda}{2(\omega^2)^2}\right]$$

$$= \sum_{i=1}^n\left[\frac{1}{2}\beta^2 - \frac{2\lambda^2 M_i{}^2}{(1-2M_i\omega^2)^2} - \frac{2\lambda^2(1-2M_i\omega^2)}{(2\omega^2)^2(1-2M_i\omega^2)^2} + \frac{4M_i\omega^2\lambda^2}{(2\omega^2)^2(1-2M_i\omega^2)^2} + \frac{\lambda}{2(\omega^2)^2}\right]$$

$$= \sum_{i=1}^n\left[\frac{1}{2}\beta^2 - \frac{2\lambda^2 M_i{}^2}{(1-2M_i\omega^2)^2} - \frac{\lambda^2 M_i}{\omega^2(1-2M_i\omega^2)^2} - \frac{2\lambda^2}{(2\omega^2)^2(1-2M_i\omega^2)^2} + \frac{\lambda}{2(\omega^2)^2}\right]$$

$$= \sum_{i=1}^n\left[\frac{1}{2}\beta^2 + \frac{\lambda^2 M_i(1-2M_i\omega^2)}{\omega^2(1-2M_i\omega^2)^2} - \frac{\lambda^2}{2(\omega^2)^2(1-2M_i\omega^2)} + \frac{\lambda}{2(\omega^2)^2}\right]$$

$$= \sum_{i=1}^n\left[\frac{1}{2}\beta^2 - \frac{\lambda^2(1-2M_i\omega^2)}{2(\omega^2)^2(1-2M_i\omega^2)} + \frac{\lambda}{2(\omega^2)^2}\right]$$

$$= \sum_{i=1}^n\left[\frac{1}{2}\beta^2 - \frac{\lambda^2-\lambda}{2(\omega^2)^2}\right],$$

$$\frac{\partial}{\partial\beta}\mathcal{L}_n(\psi) = \sum_{i=1}^n\left[\frac{1}{\beta} - \frac{\beta(1-2M_i\omega^2)}{2M_i} + \lambda\right].$$

$$= \sum_{i=1}^n\left[\frac{1}{\beta} - \frac{\beta}{2M_i} + \beta\omega^2 + \lambda\right].$$

When ω_0^2, β_0 are known, the explicit estimator for λ_0:

$$\hat{\lambda}_n = \frac{2\beta_0\omega_0^2 - 1}{2},$$

And when ω_0^2, λ_0 are known, the explicit estimator for β_0 :

$$\hat{\beta}_n = \begin{cases} \frac{2n\lambda_0 + \sqrt{4n^2\lambda_0^2 + 8nK_i}}{2K_i} \\ \frac{2n\lambda_0 - \sqrt{4n^2\lambda_0^2 + 8nK_i}}{2K_i} \end{cases}$$

Where $K_i = \sum_{i=1}^{n} \frac{1-2M_i\omega_0^2}{M_i}$.
When λ_0, β_0 are known, the explicit estimator for ω_0^2:

$$\widehat{\omega_n^2} = \sqrt{\frac{\lambda_0^2 - \lambda_0}{2\beta_0^2}},$$

Such that, $\frac{\lambda_0^2 - \lambda_0}{2\beta_0^2} > 0$.

If all the parameters are unknown, the MLEs of $\psi_0 = (\lambda_0, \omega_0^2, \beta_0)$ are given by the system:

$$\hat{\lambda}_n = \frac{2\hat{\beta}_n\omega_n^2 - 1}{2},$$

$$\hat{\beta}_n = \begin{cases} \frac{2n\hat{\lambda}_n + \sqrt{4n^2\hat{\lambda}_n^2 + 8nK_i}}{2K_i} \\ \frac{2n\hat{\lambda}_n - \sqrt{4n^2\hat{\lambda}_n^2 + 8nK_i}}{2K_i} \end{cases},$$

$$\widehat{\omega_n^2} = \sqrt{\frac{\hat{\lambda}_n^2 - \hat{\lambda}_n}{2\widehat{\beta_n}^2}}.$$

Such that $\frac{\hat{\lambda}_n^2 - \hat{\lambda}_n}{2\widehat{\beta_n}^2} > 0$

And $\hat{\beta}_n$ is a maximum likelihood estimator defined as any solution of $\mathcal{L}_n(\hat{\beta}_n) = \sup_{\psi\in\Theta} \mathcal{L}_N(\psi)$.
The second derivatives of $\mathcal{L}_N(\psi)$ with respect to the paramters is as follow:

$$\frac{\partial^2}{\partial\lambda^2}\mathcal{L}_n(\psi) = \sum_{i=1}^{n}\left[\frac{-2M_i}{1-2\omega^2 M_i} + \frac{1}{\omega^2(1-2M_i\omega^2)}\right]$$

$$= \sum_{i=1}^{n}\left[\frac{1-2M_i\omega^2}{\omega^2(1-2M_i\omega^2)}\right]$$

$$= \frac{n}{\omega^2}, \tag{8}$$

$$\frac{\partial^2}{\partial\omega^2\partial\omega^2}\mathcal{L}_n(\psi) = \sum_{i=1}^{n}\frac{\lambda^2-\lambda}{(\omega^2)^3}, \tag{9}$$

$$\frac{\partial^2}{\partial\beta^2}\mathcal{L}_n(\psi) = \sum_{i=1}^{n}\left[2\omega^2 - \frac{1}{M_i}\right] \tag{10}$$

$$\frac{\partial^2}{\partial\lambda\partial\omega^2}\mathcal{L}_n(\psi) = \frac{n(1-2\lambda)}{2(\omega^2)^2} = \frac{\partial^2}{\partial\omega^2\partial\lambda}\mathcal{L}_N(\psi) \tag{11}$$

$$\frac{\partial^2}{\partial\lambda\partial\beta}\mathcal{L}_N(\psi) = \frac{\partial^2}{\partial\beta\partial\lambda}\mathcal{L}_N(\psi) = n,$$

$$\frac{\partial^2}{\partial\omega^2\partial\beta}\mathcal{L}_n(\psi) = \frac{\partial^2}{\partial\beta\partial\omega^2}\mathcal{L}_n(\psi) = n\beta. \tag{12}$$

And the information matrix

$$I(\psi) = -\begin{pmatrix} E_\psi\left(\frac{\partial^2}{\partial\lambda^2}\mathcal{L}_n(\psi)\right) & E_\psi\left(\frac{\partial^2}{\partial\lambda\partial\omega^2}\mathcal{L}_n(\psi)\right) & E_\psi\left(\frac{\partial^2}{\partial\lambda\partial\beta}\mathcal{L}_n(\psi)\right) \\ E_\psi\left(\frac{\partial^2}{\partial\omega^2\partial\lambda}\mathcal{L}_n(\psi)\right) & E_\psi\left(\frac{\partial^2}{\partial\omega^2\partial\omega^2}\mathcal{L}_n(\psi)\right) & E_\psi\left(\frac{\partial^2}{\partial\omega^2\partial\beta}\mathcal{L}_n(\psi)\right) \\ E_\psi\left(\frac{\partial^2}{\partial\beta\partial\lambda}\mathcal{L}_n(\psi)\right) & E_\psi\left(\frac{\partial^2}{\partial\beta\partial\omega^2}\mathcal{L}_n(\psi)\right) & E_\psi\left(\frac{\partial^2}{\partial\beta^2}\mathcal{L}_n(\psi)\right) \end{pmatrix}, \tag{13}$$

Is the covariance matrix of the vector

$$\begin{pmatrix} \frac{\partial}{\partial\lambda}\mathcal{L}_n(\psi) \\ \frac{\partial}{\partial\omega^2}\mathcal{L}_n(\psi) \\ \frac{\partial}{\partial\beta}\mathcal{L}_n(\psi) \end{pmatrix}.$$

We need the following additional assumptions to prove the asymptotic properties:
H4 The parameter set Θ is a compact subset of $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$.
H5 The true value ψ_0 belongs to Θ^o.
H6 The matrix $I(\psi_0)$ is invertible.

4. Asymptotic properties of the Bayesian posterior distribution

4.1. Consistency of the Bayesian posterior

We consider the theorem 7.80 of schervish (1995) [14] and verify the regularity conditions in this theorem for our purpose with suppose that Ω is compact.

Theorem 1: *[14]: Let $\{x_n\}_{n=1}^{\infty}$ be conditionally $i.i.d$ given θ with density $f_1(x|\theta)$ with respect to a measure v on a space$(\chi^1, \mathcal{B}^1)$. Fix $\theta_o \in \Omega$, and define, for each $M \subseteq \Omega$ and $x \in \chi^1$,*

$$Z(M, x) = \inf_{\alpha\in M} log\frac{f_1(x|\theta_o)}{f_1(x|\alpha)}.$$

Assume that for each $\theta \neq \theta_o$, there is an open set N_θ such that $\theta \in N_\theta$ and that $E_{\theta_o}Z(N_\theta, X_i) > -\infty$.
Also assume that $f_1(x|.)$ is continuous at θ for every θ, a.s. $[P_{\theta_o}]$.For$\in > 0$, defin$C_\in = \{\theta: K_1(\theta_o, \theta) < \in\}$, where

$$K_1(\theta_o, \theta) = E_{\theta_o}\left(log\frac{f_1(X_1|\theta_o)}{f_1(X_1|\theta)}\right),$$

Is the kullback-leibler divergence measure associated with observation X_1.let π be a prior distribution such that $\pi(C_\in) > 0$, for every $\in > 0$.Then for every $\in > 0$ and open set N_0 containing $C_\in$, the posterior satisfies

$$\lim_{n\to\infty}\pi(N_0|X_1, \dots, X_n) = 1 \text{ , a.s. } [P_{\theta_o}]. \tag{14}$$

In our purpose we investigate the conditions in the theorem above as follow:
We note that for any, $f_1(x|\psi) = \gamma_1(x, \psi) = \gamma(x, \psi)$, Which is clearly continuous in ψ, so for every $\psi \neq \psi_o$ we get:

$$log\frac{f_1(x|\psi_o)}{f_1(x|\psi)} = log\frac{\beta_0}{\beta} - \frac{1}{4}\frac{\left(\beta_0(1-2\omega_0^2M_i) - 2\lambda_0M_i\right)^2}{M_i(1-2\omega_0^2M_i)}$$

$$+\frac{\lambda_0^2 - \lambda_0(1-2\omega_0^2M_i)}{2\omega_0^2(1-2\omega_0^2M_i)} + \frac{1}{4}\frac{(\beta(1-2\omega^2M_i) - 2\lambda M_i)^2}{M_i(1-2\omega^2M_i)}$$

$$+\frac{\lambda^2 - \lambda(1-2\omega^2M_i)}{2\omega^2(1-2\omega^2M_i)}$$

$$= log\frac{\beta_0}{\beta}$$

$$-\frac{1}{4}\left(\frac{\beta_0^2\left(1-2\omega_0^2M_i\right)^2-4\beta_0\lambda_0M_i\left(1-2\omega_0^2M_i\right)+4\lambda_0^2M_i^2}{M_i(1-2\omega_0^2M_i)}\right)$$

$$+\frac{\lambda_0^2}{2\omega_0^2(1-2\omega_0^2M_i)}-\frac{\lambda_0}{2\omega_0^2}$$

$$+\frac{1}{4}\left(\frac{\left(\beta^2(1-2\omega^2M_i)^2-4\beta\lambda M_i(1-2\omega^2M_i)+4\lambda^2M_i^2\right)^2}{M_i(1-2\omega^2M_i)}\right)$$

$$+\frac{\lambda^2}{2\omega^2(1-2\omega^2M_i)}-\frac{\lambda}{2\omega^2}.$$

$$=log\frac{\beta_0}{\beta}-\frac{1}{4}\frac{\beta_0^2\left(1-2\omega_0^2M_i\right)}{M_i}+\beta_0\lambda_0-\frac{\lambda_0^2M_i}{1-2\omega_0^2M_i}$$

$$+\frac{\lambda_0^2}{2\omega_0^2(1-2\omega_0^2M_i)}-\frac{\lambda_0}{2\omega_0^2}+\frac{1}{4}\frac{\beta^2(1-2\omega^2M_i)}{M_i}-\beta\lambda$$

$$+\frac{\lambda^2M_i}{1-2\omega^2M_i}+\frac{\lambda^2}{2\omega^2(1-2\omega^2M_i)}-\frac{\lambda}{2\omega^2}$$

$$=log\frac{\beta_0}{\beta}-\frac{1}{4}\left(\frac{\beta_0^2\left(1-2\omega_0^2M_i\right)}{M_i}-\frac{\beta^2(1-2\omega^2M_i)}{M_i}\right)$$

$$-\left(\frac{\lambda_0^2M_i}{1-2\omega_0^2M_i}-\frac{\lambda^2M_i}{1-2\omega^2M_i}\right)$$

$$+\left(\frac{\lambda_0^2}{2\omega_0^2(1-2\omega_0^2M_i)}+\frac{\lambda^2}{2\omega^2(1-2\omega^2M_i)}\right)$$

$$-(\beta\lambda-\beta_0\lambda_0)-\left(\frac{\lambda_0}{2\omega_0^2}-\frac{\lambda}{2\omega^2}\right).$$

$$=log\frac{\beta_0}{\beta}-\frac{1}{4}\frac{\beta_0^2}{M_i}+\frac{1}{2}\ \omega_0^2\beta_0^2-\frac{\beta^2}{M_i}+2\beta^2\omega^2$$

$$+\frac{\lambda_0^2\left(1-2\omega_0^2M_i\right)}{2\omega_0^2(1-2\omega_0^2M_i)}+\frac{\lambda^2(1-2\omega^2M_i)}{2\omega^2(1-2\omega^2M_i)}-(\beta\lambda-\beta_0\lambda_0)$$

$$-\left(\frac{\lambda_0}{2\omega_0^2}-\frac{\lambda}{2\omega^2}\right)$$

$$=log\frac{\beta_0}{\beta}-\frac{1}{4}\frac{\beta_0^2}{M_i}+\frac{1}{2}\ \omega_0^2\beta_0^2-\frac{\beta^2}{M_i}+2\beta^2\omega^2$$

$$-(\beta\lambda-\beta_0\lambda_0).$$

We note that $E_{\psi_o}log\frac{\beta_0}{\beta}$, $E_{\psi_o}\left(\frac{1}{2}\ \omega_0^2\beta_0^2\right)$, $E_{\psi_o}(2\beta^2\omega^2)$ and $E_{\psi_o}(\beta\lambda-\beta_0\lambda_0)$ are finite. Under H3, $E_{\psi_o}\left(\frac{1}{4}\frac{\beta_0^2}{M_i}\right)$ and $E_{\psi_o}(\frac{\beta^2}{M_i})$ are also finite, and by assume that $N_\psi=(\underline{\lambda},\overline{\lambda})\times(\underline{\omega^2},\overline{\omega^2})\times(\underline{\beta},\overline{\beta})$, follows that $E_{\psi_o}Z\left(N_\psi,X_i\right)>-\infty$.

Now we must prove there exists a prior π such that it's gives positive probability to C_ϵ for every $\epsilon>0$.

As we know, the kullback equal zero if and only if $\psi=\psi_o$, then for any $\epsilon>0$, the set C_ϵ is non-empty provided that $\Omega\backslash\{\psi_o\}$ is non-empty. From the above we see that $K_1(\psi_o,\psi)$ is continuous in ψ (see [6]), and since the parameter space Ω is compact , then from the properties of real analyses it is clear that $K_1(\psi_o,\psi)$ is uniformly continuous on Ω, that is mean , for any $\epsilon>0$, there exist δ_ϵ such that

$\|\psi-\psi_o\|\le\delta_\epsilon$ Implies

$$\left|E_{\psi_o}(logf_1(X_1|\psi_o))-E_{\psi_o}(logf_1(X_1|\psi))\right|<\epsilon,$$

Then, $K_1(\psi_o,\psi)<\epsilon$.

Hence,

$$\pi(C_\epsilon)\ge\pi(\{\psi:\|\psi-\psi_o\|\le\delta_\epsilon\})$$
$$\ge\left[inf_{\{\psi:\|\psi-\psi_o\|\le\delta_\epsilon\}}g(\psi)\right]\times v(\{\psi:\|\psi-\psi_o\|\le\delta_\epsilon\})$$
$$>0.$$

So, we obtain that (14) holds with any prior with continuous density with respect to the Lebesgue measure.

4.2. Asymptotic normality of the Bayesian posterior

To verify Asymptotic normality of the posterior distribution, we investigate the conditions in the next theorem provided in ([14], Theorem 7.102).We use the notations:

$$l_n(\theta)=logf_n(X_n|\theta)\ ,l_n''(t)=\left(\left(\frac{\partial^2}{\partial\theta_i\partial\theta_j}l_n(\theta)\Big|_{\theta=t}\right)\right) \qquad (15)$$

And let

$$\Sigma_n=\begin{cases}-l_n''^{-1}(\hat{\theta}_n)\ if\ the\ invers\ and\ \hat{\theta}_n\ exist\\ \qquad\qquad \mathbb{I}_d\ if\ not,\end{cases} \qquad (16)$$

And $\mathbb{I}_d$ is the identity matrix of order d. Notice that Σ_n^{-1} is the observed fisher information matrix. We need to investigate the following regularity conditions:

1) The parameter space is $\Omega\subseteq\mathbb{R}^d$ for some finite d.
2) θ_0 Is a point interior to Ω .
3) The prior distribution of Θ has a density with respect to Lebesgue measure that is positive and continuous at θ_0.
4) There exists a neighborhood $N_0\subseteq\Omega$ of θ_0 on which $l_n(\theta)=logf(X_1,\dots,X_n|\theta)$ is twice continuously differentiable with respect to all coordinates of, a.s. $[P_{\theta_0}]$.

Theorem 2: *[15]: Let $\{X_n\}_{n=1}^\infty$ be conditionally $i.i.d$ given θ. Assume the above regularity conditions; and suppose that there exist $H_r(x,\theta)$ such that, for each $\theta_0\in int(\Omega)$ and each j ,*

$$\sup_{\|\theta-\theta_0\|\le r}\left|\frac{\partial^2}{\partial\theta_k\partial\theta_j}log\,f_{X_1|\Theta}\,(x|\theta_0)-\frac{\partial^2}{\partial\theta_k\partial\theta_j}log\,f_{X_1|\Theta}\,(x|\theta)\right|\le H_r(x,\theta_0) \qquad (17)$$

With,

$$\lim_{r\to0}E_{\theta_0}H_r(X,\theta_0)=0. \qquad (18)$$

And suppose that the conditions of theorem 1 hold, and that the fisher information matrix $I(\theta_0)$ is positive definite. Define l_n'' as in (15), and Σ_n be defined by (16).

Let $\Gamma_n=\Sigma_n^{-\frac{1}{2}}(\theta-\hat{\theta}_n)$. Then for each compact subset E of $\mathbb{R}^d$ and each $\epsilon>0$, the posterior density of Γ_n given X_n converges in probability to the standard normal distribution with density $\phi(.)$, which is mean:

$$\lim_{n\to\infty}P_{\theta_0}\left(\sup_{\Gamma\in E}|\pi(\Gamma|X_1,\dots,X_n)-\phi(\Gamma)\ |>\epsilon\right)=0 \qquad (19)$$

Now we want to investigate the regularity conditions and the conditions of the above theorem in our case:

It easy to see that the first condition in the regularity conditions is trivial. From assumption **H5**, the second condtion holds, and from the conditions of theorem 1 above, the third condition holds. The differentiation can be passed under the integral sign (see [7], proof of proposition 5).from(8),(9),(10),(11) and (12),we deduce that $\frac{\partial^2}{\partial\psi_k\partial\psi_j}log\,f_{X_1|\Theta}\,(x|\psi)$ is differentiable in $\psi=(\lambda,\omega^2,\beta)$, that is means the fourth condition holds. From remark (3.1) , the derivatives has finite expectation, Hence (17) and (18) holds, we obtain that the information matrix $I(\psi)$ is finite, and from **H6**, $I(\psi)$ is invertible, hence (19) holds.

5. Conclusion

We depend on SDE with random effects model framework and consider the nonlinearity assumption in the diffusion function given by $\sigma(x, \emptyset_i, \mu_i) = (\emptyset_i + \mu_i)^{-1}\,\sigma(x)$ where $\emptyset_i$ are supposed to be Gaussian random variables with mean λ and variance ω^2, and μ_i to be exponential random variables with parameter β. A closed form expression of the likelihood of the parameters of the $i.i.d$ random effects and the maximum likelihood estimator are obtained. We proved posterior consistency and asymptotic posterior normality of the estimators by using the classical asymptotic theory of the Bayesian framework.

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