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Research paper

On the polynomial solution of the first Painlevé equation

D. Sierra-Porta[1,2*] and L. A. Núñez[2,3]

[1] *Escuela de Física, Universidad Industrial de Santander, Carrera 27 y Calle 9, 640002 Bucaramanga, Colombia.*
[2] *Centro de Modelado Científico (CMC) and Laboratorio de Astronomía y Física Teórica (LAFT). Facultad Experimental de Ciencias. Universidad del Zulia. Maracaibo 4001, Venezuela.*
[3] *Centro de Física Fundamental, Departamento de Física, Universidad de Los Andes, 5101 Mérida, Venezuela.*
[*] *Corresponding author E-mail: dsporta@cmc.org.ve*

Abstract

The Painlevé equations and their solutions arises in pure, applied mathematics and theoretical physics. In this manuscript we apply the Optimal Homotopy Asymptotic Method (OHAM) for solving the first Painlevé equation. Our approximation technique is based on the use of polynomial solutions, which are shown to be accurate when compared to the computed numerical solutions, thus providing a very close description of the evolution of the system.

Keywords: *Painlevé transcendent, first Painlevé equation, optimization methods.*

1. Introduction

The special functions play an important role in the study of linear differential equations, that are also of great importance in mathematical physics. Examples of this functions are the Airy, Bessel, Parabolic cylindrical, Whittaker, Confluent Hypergeometric and the Hypergeometric functions. Some of them are solutions of linear ordinary differential equations with rational coefficients which receive the same name as those functions. For example, the Bessel functions are solutions of the Bessel equation, the simplest second-order linear differential equation with one irregular singularity, and are used to describe the motion of planets and artificial satellites via the Kepler equation.

Painlevé equations play an analogous role for the non-linear differential equations. These equations were discovered by Painlevé and Gambier more than 100 years ago [1], and their solutions have the so-called Painlevé property; i.e., any movable singularity must be a pole. In fact some specialists consider that during the 21st century, Painlevé functions will be a new member of the special functions. The corresponding equations are non-linear second-order ordinary differential equations, used by physicists and mathematicians since their discovery to describe a growing variety of systems. Some examples involve the description of the asymptotic behavior of non-linear equations [2], statistical mechanics [3], correlation functions of the XY model [4], bidimensional ising model [5], superconductivity [6], Bose-Einstein condensation [6], stimulated Raman dispersion [7], quantum gravity and quantum field theory [8], aleatory matrix models [9], topologic field theory (e.g., the so-called Witten-Dijkgraaf-Verlinde-Verlinde equations) [10], general relativity [11], solutions of Einstein axialsymmetric equations [11], negative curvature surfaces [12], plasma physics [6], Hele-Shaw problems [13] and non-linear optics [14].

During the last years, more and more researchers are interested in these equations and they have found interesting analytic, geometric, and algebraic properties. The ideas of Paul Painlevé allowed to distinguish six families of non-linear second-order differential equations, the first equation being

$$P_I : u'' = 6u^2 + t, \tag{1}$$

where $u = u(t)$, and u satisfies the initial conditions $(u(0), u(0)') = (1,0)$.
To illustrate a concrete example involving the first Painlevé equation, consider the case of quantum curves, which attract both mathematicians and physicists since they are expected to encode the information of many quantum topological invariants, such as Gromov-Witten invariants and quantum knot invariants [15, 16]. A quantum curve is an ordinary differential (or difference) equation containing a formal parameter $\hbar$ playing the role of the Planck constant; a typical example is the Schrödinger equation. The quantum invariants appear in the coefficients of the WKB (Wentzel-Kramers-Brillouin) solution of the quantum curve. In this case, the first Painlevé equation is a quantum curve equation that can be written as

$$P_{Iq} : \hbar\frac{d^2q}{d\tau^2} = 6q^2 + \tau. \tag{2}$$

This equation is obtained from (1) via the rescaling $\tau = \hbar^{-4/5}t$, $q = \hbar^{-2/5}u$. We will assume $\hbar$ a small parameter (as in the Planck's constant). In this context, the last equation has the following formal power series solution:

$$q(\tau,\hbar) = \sum_{n=0}^{\infty} \hbar^{2n} q_{2n}(\tau), \tag{3}$$

The solution contains only even-order terms of $\hbar$ since P_{Iq} is invariant under $\hbar \to -\hbar$. The leading term $q_0 = q_0(\tau)$ satisfies

$$6q_0^2 + \tau = 0, \quad \text{hence}: \quad q_0(\tau) = \sqrt{\frac{-\tau}{6}} \tag{4}$$

and the subleading terms are recursively determined by

$$q_{2(k+1)}(\tau)=\frac{1}{12q_0(\tau)}\left(\frac{d^2q_{2k}}{d\tau^2}-6\sum_{k_1+k_2=k+1}q_{2k_1}(\tau)q_{2k_2}(\tau)\right), \quad (5)$$

for $1\leq k$ and $k_i>0$.
Although several numerical methods [17, 18, 19] have been implemented to solve the first Painlevé equation (1), in this paper we explore analytical solutions using Optimal Homotopy Asymptotic Method (OHAM). Our approximation technique leads in a natural way to a polynomial approach. Polynomial approximation formulas are applicable for approximation over finite intervals, or finite contours. OHAM is effective and accurate, showing great potential for solving strongly non-linear problems. The basic idea of the Optimal Homotopy Asymptotic Method was initially introduced by Marinca and Herisanu [20]. OHAM reduces the size of the computational domain and it has been successfully applied to a number of nonlinear differential equations in science and engineering, e.g., to study steady flow of a fourth-grade fluid through a porous medium [20], oscillators with discontinuities and fractional-power restoring force [21], periodic solutions for the motion of a particle on a rotating parabola [22], thin film flow of a fourth-grade fluid [23], nonlinear heat transfer equations [24], and nonlinear problems in elasticity [25]. In particular, using OHAM Islam et al. [26] investigated Couette and Poiseuille flows of a third-grade fluid with heat transfer analysis, Idrees et al. [27] analyzed the Korteweg–de Vries (KDV) equation, Mohsen et al. [28] studied viscous flow in a semi-porous channel with uniform magnetic field, and Ghoreishi et al. [29] provided a comparative study for nth-order integral-differential equations.
In the next section we present the mathematical formalities that will be used in the rest of the paper.

2. Basic ideas of OHAM

Consider the following general differential equation

$$L[u(t)]+g(t)+N[u(t)]=0, \quad (6)$$

that satisfies de initial/boundary conditions

$$B\left[u(t),\frac{du(t)}{dt}\right]=B\left[u(t),u'(t)\right]=0, \quad (7)$$

where t denotes the independent variable, $u(t)$ is a function to solve, $g(t)$ is a given function, L, N, and B are linear, nonlinear and boundary operators, respectively.
Applying OHAM to the given problem, a general deformation (Homotopy) equation is presented as:

$$(1-\varepsilon)\left(L[H(t,\varepsilon)]+g(t)\right)=h(\varepsilon)\left[L[H(t,\varepsilon)]+g(t)+N[H(t,\varepsilon)]\right], \quad (8)$$

and

$$B\left[H(t,\varepsilon),\frac{\partial H(t,\varepsilon)}{\partial t}\right]=0, \quad (9)$$

where $\varepsilon\in[0,1]$ is an embedding parameter, $h(\varepsilon)$ is a nonzero auxiliary function for $\varepsilon\neq 0$ and $h(0)=0$, $H(t,\varepsilon)$ is a unknown function. Clearly, when $\varepsilon=0$ and $\varepsilon=1$ it holds $H(t,0)=u_0(t)$ and $H(t,1)=u(t)$, respectively.
Thus, as ε changes from 0 to 1, the solution $H(t,\varepsilon)$ changes from $u_0(t)$ to the solution $u(t)$, where $u_0(t)$ is obtained from equation (8) for $\varepsilon=0$:

$$L[u_0(t)]+g(t)=0, \quad B\left[u_0(t),u_0'(t)\right]=0. \quad (10)$$

Now, we propose the auxiliary function $h(\varepsilon)$ to be of the form:

$$h(\varepsilon)=\varepsilon K_1+\varepsilon^2K_2+\varepsilon^3K_3+\cdots+\varepsilon^mK_m=\sum_{i=1}^{m}\varepsilon^iK_i, \quad (11)$$

where K_i are constants. For actual applications K_i, are finite, say, $i=1,2,3,\ldots,m$.
Expanding $H(t,\varepsilon)$ in Taylor's series about ε, we obtain:

$$H(t,\varepsilon)=u_0(t)+\sum_{i=1}^{\infty}u_n(t,K_i)\varepsilon^n. \quad (12)$$

Substituting (12) into (8), and equating the coefficient of like powers of ε, we obtain that zeroth-order problem is given by (10), while the first- and second-order problems are given by

$$\begin{aligned} L[u_1(t)]&=K_1N_0[u_0(t)], \quad B\left[u_1(t),u_1'(t)\right]=0,\\ L[u_2(t)]-(1+K_1)L[u_1(t)]&=K_2N_0[u_0(t)]+K_1N_1[u_0(t)],\\ &B\left[u_2(t),u_2'(t)\right]=0. \end{aligned} \quad (13)$$

It is then possible to write

$$L[u_n(t)]-L[u_{n-1}(t)]=K_nN_0[u_0(t)]$$
$$+\sum_{i=1}^{n-1}K_i\left[L[u_{n-i}(t)]+N_{n-i}[u_0(t),u_1(t),\ldots,u_{n-1}(t)]\right], \quad (14)$$

$$B\left[u_n(t),u_n'(t)\right]=0. \quad (15)$$

In the last equation $N_m[u_0(t),u_1(t),\ldots,u_{n-1}(t)]$ is the coefficient of ε^m in the expansion of $N[H(t,\varepsilon)]$:

$$N[H(t,\varepsilon,K_i)]=N_0[u_0(t)]+\sum_{m=1}^{\infty}N_m[u_0(t),u_1(t),\ldots,u_m(t)]\varepsilon^m. \quad (16)$$

Here, convergence of the series (12) depends upon the constants K_i, $i=1,2,3,\ldots$.
When $\varepsilon=1$, the equation (12) can be written as

$$\bar{u}(t,K_m)=u_0(t)+\sum_{i=1}^{n}\bar{u}_i(t,K_m), \quad (17)$$

and the sum converges, because that in practical application n es finite for effect to approximate a solution. Substituting (17) into (8), we obtain the residual:

$$R(t,K_m)=L[\bar{u}(t,K_m)]+g(t)+N[\bar{u}(t,K_m)]. \quad (18)$$

If $R=0$, then $\bar{u}$ yields the exact solution. However, this does not happen in general, especially when dealing with non-linear problems. In order to determine K_i, there are various methods like Ritz Method, Galerin's Method, and Collocation Method, or the Method of Least Squares,

$$J(t,K_m)=\int_a^b R^2(t,K_m)dt, \quad (19)$$

where the residual $R=L[\bar{u}]+g(t)+N[\bar{u}]$, and

$$\frac{\partial J(t,K_m)}{\partial K_i}=0, \quad (20)$$

with a and b properly chosen numbers to locate the desired K_i. With these constants known, the approximate solution (of order m) is well-defined.

3. Approximate solution of the first Painlevé equation by OHAM

Here we develop a solution for equation (1) using OHAM. First we note that in this case we can make the identification

$$L[A]:=\frac{d^2}{dt^2}A, \quad g(t):=0, \quad N[A]:=-6A^2-t, \quad (21)$$

The zeroth-order of approximation is given by

$$u_0''(t)=0, \quad u_0(0)=1, \quad u_0'(0)=1, \quad (22)$$

with solution $u_0(t) = 1$. For the first-order problem, we obtain

$$u_1''(t) + K_1(6+x) = 0, \quad u_1(0) = 0, \quad u_1'(0) = 0, \tag{23}$$

with solution $u_1(t) = a_1x^2 + a_2x^3$, and $a_1 := -3K_1$, $a_2 := -K_1/6$.
For the second-order of approximation, we can see that $u_2(t)$ is a higher-order polynomial function. Actually, $u_2(t)$ is of the form $u_2(t) = Q_2(t) = \sum_{n=2}^{8} b_n t^n$, with the coefficients b_n expressed in terms of a_1, a_2, K_1 and K_2, provided by (13) and (13).
Higher orders of approximation are given by functions with a higher order of powers in t. For example, polynomials of third- ($Q_3(t)$) and fourth-order ($Q_4(t)$) have powers of 18 and 36 in the variable t, respectively.
There are many ways to choose the polynomials $Q_n(t)$, $n = 1,2,3$. We illustrate some simple cases that give very accurate results when compared to the numerical solutions:

$$\text{Case 1: } u^{(1)}(t) = 1 + \sum_{n=1}^{15} a_n x^{2n}, \tag{24}$$

$$\text{Case 2: } u^{(2)}(t) = 1 + \sum_{n=1}^{11} a_n x^{3n-1}, \tag{25}$$

$$\text{Case 3: } u^{(3)}(t) = 1 + \sum_{n=1}^{8} a_n x^{4n-2}, \tag{26}$$

$$\text{Case 4: } u^{(4)}(t) = 1 + \sum_{n=1}^{6} a_n x^{5n-3}, \tag{27}$$

where a_n are unknown parameters ($n = 1,2,\ldots 15$).

4. Numerical Results

In order to show the validity and accuracy of the OHAM, we compare previously obtained approximate solutions (24)-(27) with numerical integrations obtained by means of a forth-order Runge-Kutta method, using Maple 18 software. Using the least-square method for determination of the parameters a_n (more precisely: convergence-control parameters) we obtain

$$\begin{aligned}
u^{(1)}(t) &= 1 + 2.9150x^2 + 2.4133x^4 + 13.9042x^6 \\
&\quad -37.2086x^8 + 37.4900x^{10} + 85.8166x^{12} \\
&\quad -174.9123x^{14} + 41.6388x^{16} + 56.7217x^{18} \\
&\quad +106.7576x^{20} - 139.5246x^{22} - 54.5454x^{24} \\
&\quad +115.0371x^{26} - 36.9073x^{28} + 1.7745x^{30}, \quad (28) \\
u^{(2)}(t) &= 1 + 3.2003x^2 + 6.9920x^5 + 0.7213x^8 \\
&\quad -2.5434x^{11} + 38.4064x^{14} - 54.1081x^{17} \\
&\quad +2.7455x^{20} + 92.5541x^{23} - 110.2190x^{26} \\
&\quad +52.1268x^{29} - 7.5301x^{32}, \quad (29) \\
u^{(3)}(t) &= 1 + 3.4747x^2 + 11.8083x^6 - 14.7294x^{10} \\
&\quad +53.9107x^{14} - 84.1765x^{18} + 90.8881x^{22} \\
&\quad -52.5900x^{26} + 14.4486x^{30}, \quad (30) \\
u^{(4)}(t) &= 1 + 3.8303x^2 + 13.3965x^7 - 9.1998x^{12} \\
&\quad +23.6756x^{17} - 16.5473x^{22} + 7.35461x^{27}, \quad (31)
\end{aligned}$$

In Figure 1 we present a comparison between the approximate solutions given by equations (28)-(31) and the corresponding numerical result.
Some values of the approximate solutions obtained, i.e., equations (28)-(31), and the numerical results for different values of x, are given in Table 1.
In Table 2, we compare the relative error (%) $E_{OHAM} = \frac{|u_{approx}(t) - u_{Num}|}{u_{Num}} \times 100$, for 22 points in the range $0 \le x \le 1$.

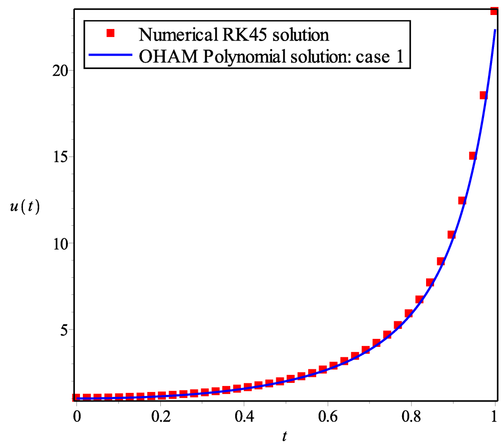

Figure 1: Comparison between the results obtained by means OHAM, Eq. (28) and numerical results: numerical results in solidbox mark and approximate solution in continuun line.

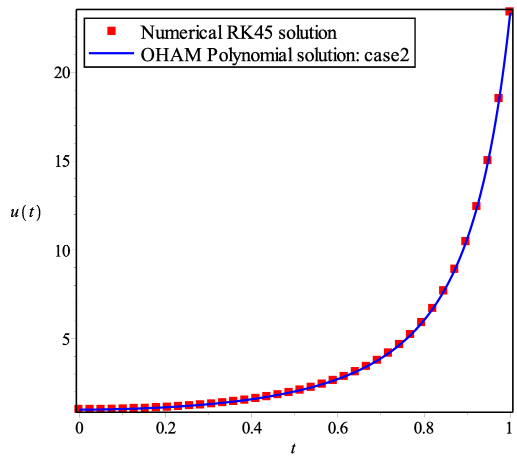

Figure 2: Comparison between the results obtained by means OHAM, Eq. (29) and numerical results: numerical results in solidbox mark and approximate solution in continuun line.

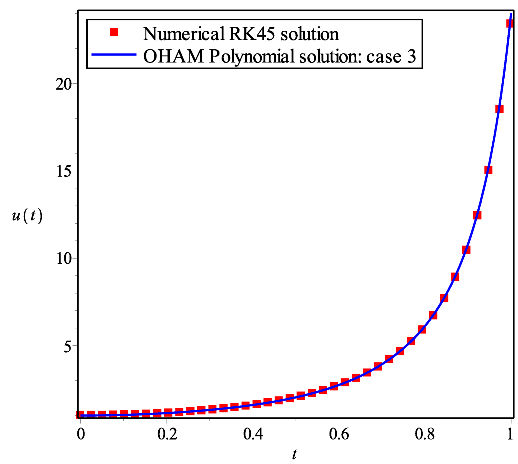

Figure 3: Comparison between the results obtained by means OHAM, Eq. (30) and numerical results: numerical results in solidbox mark and approximate solution in continuun line.

5. Conclusions

In this paper we introduced the OHAM approach to propose analytic approximate solutions to the First Painlevé equation. The procedure is valid even if the non-linear equation does not contain

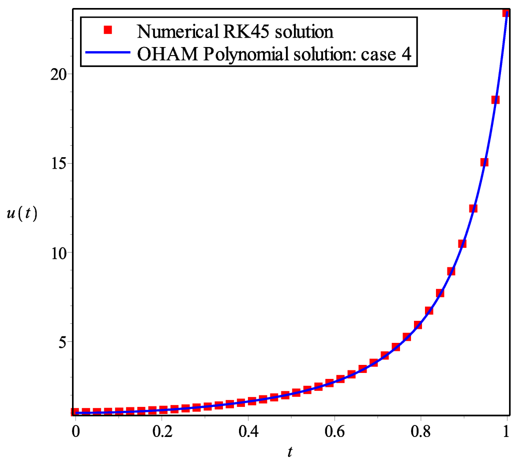

Figure 4: Comparison between the results obtained by means OHAM, Eq. (31) and numerical results: numerical results in solidbox mark and approximate solution in continuun line.

Table 1: Approximate solutions obtained OHAM - Eq. (28)-(31) and numerical solutions for different values of t

t	u_{Num}	Eq. (28)	Eq. (29)	Eq. (30)	Eq. (31)
0	1	1	1	1	1
0.01	1.0003	1.0002	1.0003	1.0003	1.0003
0.05	1.0075	1.0073	1.0080	1.0086	1.0095
0.1	1.0304	1.0294	1.0320	1.0347	1.0383
0.15	1.0696	1.0669	1.0725	1.0783	1.0862
0.2	1.1263	1.1212	1.1302	1.1397	1.1533
0.25	1.2027	1.1944	1.2068	1.2200	1.2402
0.3	1.3014	1.2898	1.3050	1.3212	1.3476
0.35	1.4264	1.4117	1.4289	1.4469	1.4778
0.4	1.5830	1.5656	1.5841	1.6029	1.6346
0.45	1.7784	1.7585	1.7783	1.7973	1.8251
0.5	2.0227	1.9993	2.0221	2.0418	2.0601
0.55	2.3301	2.3006	2.3294	2.3515	2.3564
0.6	2.7212	2.6818	2.7203	2.7475	2.7377
0.65	3.2261	3.1728	3.2245	3.2600	3.2371
0.7	3.8908	3.8192	3.8879	3.9353	3.9019
0.75	4.7882	4.6914	4.7838	4.8477	4.8029
0.8	6.0383	5.9025	6.0322	6.1220	6.0575
0.85	7.8514	7.6498	7.8424	7.9755	7.8757
0.9	10.6226	10.3056	10.6079	10.8178	10.6584
0.95	15.1665	14.6291	15.1416	15.5000	15.2278
1	23.3936	22.3707	23.3459	24.0347	23.5100

small (or large) parameters. The proposed construction of homotopy is different from other approaches in the presence of parameters a_n, which ensure a very rapid convergence of the solutions. In the range $0 \leq x \leq 0.9$, each approximate solution is very close to the value of the numerical solution, with errors beginning to grow (smoothly) in the range $0.9 \leq x \leq 1$. The average errors do not exceed 1.4%. Moreover, the derivatives for each approximate solution are quite close to the values of the numerical derivatives, i.e., $u'_{(1)} = 211.6947789$, $u'_{(2)} = 225.6800639$, $u_{(3)} = 235.7368580$, $u'_{(4)} = 228.0583361$, compared with $u'_{Num} = 226.373168734935$, respectively. In this case the error in $x = 1$ does not exceed 4.2%.

Finally we can mentioned that at least in the case of the first of the equations of Painlevé this method is a convenient way to control the convergence of the approximation in series of the solution. This method has been tested in various examples of linear and nonlinear and system of initial value problems of DDEs and was seen to yield satisfactory results. The results which are obtained revealed that the proposed method is explicit, effective, and easy to use.

Table 2: Comparison between the relative errors (%) obtained by OHAM - Eq. (28)-(31) for different values of t.

t	$Err(u^{(1)})$	$Err(u^{(2)})$	$Err(u^{(3)})$	$Err(u^{(4)})$
0	0	0	0	0
0.01	0.0008	0.0019	0.0047	0.0082
0.05	0.0234	0.0459	0.1138	0.2021
0.1	0.1033	0.1555	0.4162	0.7602
0.15	0.2490	0.2725	0.8127	1.5502
0.2	0.4529	0.3450	1.1878	2.3988
0.25	0.6832	0.3455	1.4415	3.1187
0.3	0.8907	0.2776	1.5214	3.5502
0.35	1.0322	0.1714	1.4377	3.5986
0.4	1.0970	0.0669	1.2552	3.2597
0.45	1.1187	0.0039	1.0645	2.6229
0.5	1.1588	0.0323	0.9415	1.8500
0.55	1.2674	0.0336	0.9163	1.1275
0.6	1.4469	0.0341	0.9683	0.6055
0.65	1.6530	0.0497	1.0526	0.3416
0.7	1.8415	0.0744	1.1413	0.2829
0.75	2.0206	0.0923	1.2436	0.3076
0.8	2.2485	0.1015	1.3853	0.3173
0.85	2.5684	0.1158	1.5796	0.3086
0.9	2.9839	0.1382	1.8373	0.3367
0.95	3.5434	0.1640	2.1988	0.4041
1	4.3725	0.2042	2.7403	0.4974
Average	**1.3980**	**0.1239**	**1.1482**	**1.2477**

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References

[1] Painlevé P., Sur les équations différentielles du second ordre et d'ordre superieur dont l'intégrale général est uniforme, *Acta Math.* 25 (1902), 1-85.

[2] H. Segur and M.J. Ablowitz. Asymptotic solutions of nonlinear evolution equations and a Painlevé transcedent. *Physica D*, (1993), 3:165.

[3] M. Adler, T. Shiota, and P. van Moerbeke. Random matrices, vertex operators and the Virasoro algebra. *Phys. Lett. A*, 208 (1995), 67-78.

[4] M. Shiroishi, M. Takahashi, and Y. Nishiyama. Emptiness formation probability for the one-dimensional isotropic XY model. *J. Phys. Soc. Jpn.*, 70 (2001), 3535.

[5] B.M. McCoy, J.H. Perk, and R.E. Shrock. Correlation functions of the transverse Ising chain at the critical field for large temporal and spatial separation. *Nucl. Phys. B*, 220 (2002), 269.

[6] T. Kanna, K. Sakkaravarthi, C. Senthil, M. Lakshmanan, and M. Wadati. Painlevé singularity structure analysis of three component Gross-Pitaevskii type equations. *J. Math. Phys.*, 50 (2009), 113520.

[7] E.A. Moskovchenko and V.P. Kotlyarov. Periodic boundary data for an integrable model of stimulated Raman scattering: long-time asymptotic behavior. *J. Phys. A: Math. Gen.*, 43 (2010), 5.

[8] A.S. Fokas, A.R. Its, and A.V. Kitaev. Discrete Painlevé equations and their appearance in quantum gravity. *Commun. Math. Phys.*, 142 (1991), 313.

[9] M. Adler, T. Shiota, and P. van Moerbeke. Random matrices, vertex operators and the Virasoro algebra. *Phys. Lett. A*, 208 (1995), 67-78.

[10] B. Dubrovin. Painlevé trascendents and two-dimensional topological field theory. *CRM Math. Phys.*, (1999) pages 287-412.

[11] Gariel, G. Marcilhacy, and N.O. Santos. Stationary axisymmetric solutions involving a third order equation irreducible to Painlevé transcendents. *J. Math. Phys.*, 49 (2008), 022501.

[12] X. Cao and C. Xu. A Backlund transformation for the Burgers hierarchy. Abs. *App. Ana.*, (2010), 241898.
[13] S.Y. Lee, R. Teodorescub, and P. Wiegmannc. Shocks and nite-time singularities in Hele-Shaw ow. *Physica D*, 238 (2009), 1113.
[14] M. Florjanczyk and L. Gagnon. Exact solutions for a higher-order nonlinear Schrodinger equation. *Phys. Rev. A*, 41 (1990), 4478.
[15] Aganagic M., Cheng M.C.N., Dijkgraaf R., Krefl D., Vafa C., Quantum geometry of refined topological strings, *J. High Energy Phys.* (2012), no. 11, 019, 53 pages, arXiv:1105.0630.
[16] Aganagic M., Dijkgraaf R., Klemm A., Mariño M., Vafa C., Topological strings and integrable hierarchies, *Comm. Math. Phys.* 261 (2006), 451-516, hep-th/0312085.
[17] B. Fornberg and J.A.C. Weideman, A numerical methodology for the Painleve equations, *Journal of Computational Physics*, 230 (2011), pp. 5957-5973.
[18] K. El-Kamel, Sinc numerical solution for solitons and solitary waves, *J. Comput. Appl. Math.*, 130 (2001), pp. 283-292.
[19] E. Hesameddini and S. Shekarpaz, Wavelet solutions of the first Painleve equation, 23rd International Conference of Jangjeon Mathematical Society, Shahid Chamran Uinvesity of Ahvaz, Ahvaz, Iran, (2010).
[20] Marinca V, Herisanu N, Bota C, Marinca B. An optimal homotopy asymptotic method applied to the steady flow of fourth-grade fluid past a porous plate. *Appl Math Lett 22* (2009), 245-51.
[21] Herisanu N, Marinca V. Accurate analytical solutions to oscillators with discontinuities and fractional-power restoring force by means of the optimal homotopy asymptotic method. *Comp Math Appl 60* (2010), 1607-15.
[22] Marinca V, Herisanu N. Determination of periodic solutions for the motion of a particle on a rotating parabola by means of the optimal homotopy asymptotic method. *Journal of Sound and Vibration* 329 (2010), 1450-1459.
[23] Marinca V, Herisanu N, Nemes I. Optimal homotopy asymptotic method with application to thin film flow. *Central European Journal of Physics* 6 (2008), 648-53.
[24] Marinca V, Herisanu N. An optimal homotopy asymptotic method for solving nonlinear equations arising in heat transfer. *Int Commun Heat Mass Transfer* 35 (2008), 710-5.
[25] Marinca V, Herisanu N. Nonlinear dynamical systems in engineering. Some Approximate Approaches. (2011), Springer, Verlag.
[26] Islam S, Shah RA, Ali I. Optimal homotopy asymptotic solutions of Couette and Poiseuille flows of a third grade fluid with heat transfer analysis. *Int J Nonlinear Sci Numer Simul* 11(6) (2010), 389-400.
[27] Idrees M, Islam S, Tirmizi SIA, Haq S. Application of the optimal homotopy asymptotic method for the solution of the Korteweg–de Vries equation. *Mathematical and Computer Modelling* 55 (2012), 1324-33.
[28] Mohsen S, Hamid RA, Domairry D, Hashim I. Investigation of the laminar viscous flow in a semi-porous channel in the presence of uniform magnetic field using optimal homotopy asymptotic method. *Sains Malaysiana* 41(10), (2012), 1281-5.
[29] Ghoreishi M, Ismail AIM, Alomari AK. Comparison between homotopy analysis method and optimal homotopy asymptotic method for nth-order integro-differential equation. *Math Meth Appl Sci* 34 (2011), 1833-42.