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Common fixed point theorems for generalized contractive mappings on cone metric spaces

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Abstract

In this paper, we present the generalization of contractive type mappings in the setting of cone metric spaces.

Keywords: *Cone metric spaces,fixed point, Generalized contractive mappings.*

1 Introduction

Recently, many authors have established and extended different types of contractive mappings in cone metric spaces, see for instance [1],[4],[5],[6],[8],[9] and [10]. The author [4] proved fixed point theorems for generalized contractive mappings on cone metric spaces.
We obtained the generalization of results in [1,3,4].

2 Preliminaries

Definition 2.1 : Let (E,τ) be a topological vector space and $P \subset E$. Then P is called a cone whenever
i) P is closed, non-empty and $P \neq \{0\}$;
ii) $ax + by \in \mathrm{P}$ for all $x, y \in P$ and non-negative real numbers a, b;
iii) $x \in P$ and $-x \in P \Rightarrow x = 0 \Leftrightarrow \mathrm{P} \cap (-\mathrm{P}) = \{0\}$.

Given a cone $P \subset E$, a partial ordering is defined as $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. It is denoted as $x \ll y$ will stand for $y - x \in int\ P$, where $int\ P$ denotes the interior of P.

Definition 2.2 [1]: Let X be a non-empty set, a mapping $d: X \times X \to E$ is called cone metric on X if the following conditions are satisfied:
(i) $0 \leq d(x,y)$ for all $x, y \in X$, and $d(x,y) = 0$ if and only if $x = y$;
(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(iii) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$;

From now on, we assume that E is a normed space, P is a cone in E with $int\ (P) \neq \emptyset$ and $\leq$ is a partial ordering with respect to P, and (X,d) is called cone metric space.

Definition 2.3 [1]: Let (X,d) be a cone metric space and $\{x_n\}$ be a sequence of points of X. Then
(i) $\{x_n\}$ converges to $x \in X$ and denoted by $\lim_{n\to\infty} x_n = x$ *or* $x_n \to x$, if for any $c \in int\ (P)$, there exists N such that for all $n > N$, $d(x_n, x) \ll c$.
(ii) $\{x_n\}$ is called Cauchy if for every $c \in int\ (P)$, there exists N such that for all $n, m > N$, $d(x_n, x_m) \ll c$.
(iii) (X,d) is complete if every Cauchy sequence in X is convergent.

Definition 2.4 [4]: A function $F: P \to P$ is called $\ll$ - increasing if, for each $x, y \in P$; $x \ll y$ if and only if $f(x) \ll f(y)$.

Let $F: P \to P$ be a function such that
(F1) $F(t) = 0$ if and only if $t = 0$;
(F2) F is $\ll -$ increasing;
(F3) F is surjective.
We denote by $\Upsilon(P,P)$ the family of functions satisfying $(F1), (F2)$ and $(F3)$.

Lemma 2.5 [8]: Let E be a topological vector space. If $c_n \epsilon E$ and $c_n \to 0$, then for each $c \epsilon\ int\ (P)$ there exists N such that $c_n \ll c$ for all n>N.

3 Main results

Theorem 3.1: Let (X, d) be a complete cone metric space. Suppose that mappings $T_1, T_2: X \to X$ satisfy

$$F(d(T_1x, T_2y)) \leq k\{F\big(d(x, T_1x) + d(y, T_2y)\big)\} \tag{3.1}$$

For all $x, y \in X$; where $k \in \left[0, \frac{1}{2}\right)$ and $F \in \Upsilon(\mathrm{P}, \mathrm{P})$ such that
(1) F is sub-additive;
(2) if, for $\{c_n\} \subset P, \lim_{n\to\infty} F(c_n) = 0$ $then$ $\lim_{n\to\infty} c_n = 0$.

Then T_1 and T_2 have a unique common fixed in X. For each $x \in X$, the iterate sequences $\{T_1^{2n+1}x\}$ and $\{T_2^{2n+2}x\}$ are converge to the common fixed point.
Proof: Let x_o be an arbitrary point in X. Define the sequences,
$x_{2n+1} = T_1x_{2n} = T_1^{2n+1}x_o$ And
$x_{2n+2} = T_2x_{2n+1} = T_2^{2n+2}x_o$
For all $n \in \mathbb{N}$.
From (3.1), we have

$$\begin{aligned} F(d(x_{2n+1,}x_{2n})) &= F(d(T_1x_{2n}, T_2x_{2n-1})) \\ &\leq k\{F\left(d\left(x_{2n,}T_1x_{2n}\right) + d\left(x_{2n-1,}T_2x_{2n-1}\right)\right)\} \\ &= k\{F\left(d\left(x_{2n,}x_{2n+1}\right) + d\left(x_{2n-1,}x_{2n}\right)\right)\} \leq kF\left(d\left(x_{2n+1,}x_{2n}\right)\right) + kF(d\left(x_{2n,}x_{2n-1}\right)\big) \end{aligned}$$

Which implies
$F(d(x_{2n+1}, x_{2n})) \leq hF(d(x_{2n}, x_{2n-1}))$ for all $n \in \mathbb{N}$
Where $h = \frac{k}{1-k}$.
Hence
$F\big(d(x_{2n+1}, x_{2n})\big) \leq hF\big(d(x_{2n}, x_{2n-1})\big) \leq h^2F\big(d(x_{2n-1}, x_{2n-2})\big) \ldots\ldots\ldots\ldots\ldots. \leq h^{2n}F\big(d\big(x_{1,}x_o\big)\big)$.

We now show that $\{x_{2n}\}$ is a Cauchy sequence in X. For $m > n$ we have

From $F(d(x_{2n}, x_{2m})) \leq F(d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + ---- \ d(x_{2m-1}, x_{2m}))$

$$\leq F\left(d\left(x_{2n,}x_{2n+1}\right)\right) + F\big(d(x_{2n+1}, x_{2n+2})\big) + ----+F(d(x_{2m-1}, x_{2m}))$$

$$\leq k^{2n}F\big(d(x_1, x_o)\big) + k^{2n+1}F\big(d(x_1, x_0)\big) + ----- + k^{2m-1}F(d(x_1, x_0))$$

$$\leq \frac{k^{2m}}{1-k}F(d(x_1, x_0)) \to 0.$$

Hence $\lim_{n,m\to\infty} d(x_{2n}, x_{2m}) = 0$ by (2). By Lemma 2.5, $\{x_{2n}\}$ is a Cauchy sequence in X. Since X is complete, there exists $z \in X$ such that $\lim_{n\to\infty} x_{2n} = z$.
Let $c \in int\ (P)$ be given. Choose $N \in \mathbb{N}$ such that
$d(x_{2n+1}, x_{2n}) \ll F^{-1}\left(\frac{c(1-k)}{2k}\right)$ and
$d\big(x_{2n,}, z\big) \ll F^{-1}\left(\frac{c(1-k)}{2}\right)$ for all $n > N$.

By (F2) and (F3),
$F(d(x_{2n+1},x_{2n})) \ll \frac{c(1-k)}{2k}$ and
$F(d(x_{2n},z)) \ll \frac{c(1-k)}{2}$ for all $n > N$.
Then we have

$$F(d(T_1z,z)) \leq F(d(T_1z,T_1x_{2n}) + d(T_1x_{2n},z)) \leq k\{F(d(z,T_1z) + d(x_{2n},T_1x_{2n}))\} + F(d(T_1x_{2n,}z))$$
$$= k\{F(d(z,T_1z) + d(x_{2n},x_{2n+1}))\} + F(d(x_{2n+1,}z))$$

Hence we have
$F(d(T_1z,z)) \leq \frac{k}{1-k}F(d(x_{2n+1},x_{2n})) + \frac{1}{1-k}F(d(x_{2n+1,}z)) \ll \frac{c}{2} + \frac{c}{2} = c.$
Thus, $F(d(T_1z,z)) \ll \frac{c}{n}$ for all $n \in \mathbb{N}$, and so $\frac{c}{n} - F(d(T_1z,z)) \in P$. Since $\frac{c}{n} \to 0$ and P is closed, $-F(d(T_1z,z)) \in P$.
Hence $F(d(T_1z,z)) = 0$.
By $(F1)$, $d(T_1z,z) = 0$ and so $z = T_1z$.
Assume that u is another fixed point of T_1.
Then from (3.1) we have

$$F(d(z,u)) = F(d(T_1z,T_1u)) \leq k\{F(d(z,T_1z) + d(u,T_1u))\}$$
$$= k\{F(d(z,z) + d(u,u))\} = 0.$$

Hence $F(d(z,u)) \in -P$, and hence $F(d(z,u)) = 0$.
By $(F1)$, $d(z,u) = 0$, and so $z = u$.
Therefore, T_1 has a unique fixed point in X.
Similarly, it can be established that $z = T_2z$, that is $z = T_1z = T_2z$.
Thus z is the unique common fixed point of T_1 and T_2.

Theorem 3.2: Let (X,d) be a complete cone metric space. Suppose that mappings $T_1, T_2: X \to X$ satisfy

$$F(d(T_1x,T_2y)) \leq k\{F(d(y,T_1x) + d(x,T_2y))\} \tag{3.2}$$

For all $x,y \in X$; where $k \in \left[0,\frac{1}{2}\right)$ and $F \in \Upsilon(\mathrm{P},\mathrm{P})$ such that
(1) F is sub-additive;
(2) if, for $\{c_n\} \subset P$, $\lim_{n\to\infty} F(c_n) = 0$ then $\lim_{n\to\infty} c_n = 0$.

Then T_1 and T_2 have a unique common fixed in X. For each $x \in X$, the iterative sequences $\{T_1^{2n+1}x\}$ and $\{T_2^{2n+2}x\}$ are converge to the common fixed point.
Proof: Let x_o be an arbitrary point in X. Define the sequences,

$x_{2n+1} = T_1x_{2n} = T_1^{2n+1}x_o$ and
$x_{2n+2} = T_2x_{2n+1} = T_2^{2n+2}x_o$
For all $n \in \mathbb{N}$.
From (3.2), we have

$$F(d(x_{2n+1,}x_{2n})) = F(d(T_1x_{2n},T_2x_{2n-1})) \leq k\{F\left(d(x_{2n-1,}T_1x_{2n}) + d(x_{2n,}T_2x_{2n-1})\right)\}$$
$$= k\{F\left(d(x_{2n-1,}x_{2n+1}) + d(x_{2n,}x_{2n})\right)\} \leq kF\left(d(x_{2n-1,}x_{2n})\right) + kF(d(x_{2n,}x_{2n+1}))$$

Which implies
$F(d(x_{2n+1},x_{2n})) \leq hF(d(x_{2n},x_{2n-1}))$ for all $n \in \mathbb{N}$,
Where $h = \frac{k}{1-k}$
Hence
$F(d(x_{2n+1},x_{2n})) \leq hF(d(x_{2n},x_{2n-1})) \leq h^2F(d(x_{2n-1},x_{2n-2})) \ldots \leq h^{2n}F(d(x_{1,}x_o)).$

As in proof of Theorem 3.1, $\{x_{2n}\}$ is a Cauchy sequence in X. Since X is complete, there exists $z \in X$ such that $\lim_{n\to\infty} x_{2n} = z$.
Let $c \in int\ (P)$ be given. Choose $N \in \mathbb{N}$ such that
$d(x_{2n,},z) \ll F^{-1}\left(\frac{c(1-k)}{3}\right)$ for all $n > N$.
By $(F2)$ and $(F3)$,
$F(d(x_{2n},z)) \ll \frac{c(1-k)}{3}$ for all $n > N$.
Then we have

$F(d(T_1z,z)) \leq F(d(T_1z,T_2x_{2n-1}) + d(T_2x_{2n-1},z)) \leq k\{F(d(x_{2n-1},T_1z) + d(z,T_2x_{2n-1}))\} + F(d(T_2x_{2n-1,}z))$
$= k\{F(d(x_{2n-1},T_1z) + d(z,x_{2n}))\} + F(d(x_{2n,}z)) \leq k\{F(d(x_{2n-1},z) + d(z,T_1z) + d(z,x_{2n}))\} + F(d(x_{2n,}z))$
$F(d(T_1z,z)) \leq \frac{k}{1-k}\{F(d(x_{2n-1},z) + d(z,x_{2n}))\} + \frac{1}{1-k}F(d(x_{2n,}z)) \ll \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c.$

As in proof of Theorem 3.1, we have $-F(d(T_1z,z)) \in P$, and so $F(d(T_1z,z)) = 0$. By $(F1)$, $d(T_1z,z) = 0$. Hence $T_1z = z$.
Suppose that u is another fixed point of T_1 such that $u \neq z$.
Then from (3.2) we have

$$F(d(z,u)) = F(d(T_1z,T_1u)) \leq k\{F(d(u,T_1z) + d(z,T_1u))\}$$
$$= k\{F(d(u,z) + d(z,u))\} \leq 2kF(d(u,z)) < F(d(u,z))$$

which is contradiction. Therefore, T_1 has a unique fixed point in X.

Similarly, it can be established that z = T_2z. Hence $T_1z = z = T_2z$. Thus z is the unique common fixed point of T_1 *and* T_2.

Theorem 3.3: Let (X,d) be a complete cone metric space. Suppose that mappings $T_1, T_2: X \to X$ satisfy

$$F(d(T_1x,T_2y)) \leq kF(d(x,y)) + lF(d(x,T_2y)) \tag{3.3}$$

For all $x,y \in X$; where $k,l \in [0,1)$ *and* $F \in \Upsilon(\mathrm{P},\mathrm{P})$ such that
(1) F is sub-additive;
(2) if, for $\{c_n\} \subset P$, $\lim_{n\to\infty} F(c_n) = 0$ then $\lim_{n\to\infty} c_n = 0$.

Then T_1 *and* T_2 have a common fixed point in X. For each $x \in X$, the iterative sequences $\{T_1^{2n+1}x\}$ and $\{T_2^{2n+2}x\}$ are converge to the common fixed point.
Moreover, if $k + l < 1$ then T_1 and T_2 have a unique common fixed point in X.
Proof: Let x_o be an arbitrary point in X. Define the sequences,
$x_{2n+1} = T_1x_{2n} = T_1^{2n+1}x_o$ and
$x_{2n+2} = T_2x_{2n+1} = T_2^{2n+2}x_o$
For all $n \in \mathbb{N}$.
From (3.3), we have

$$F(d(x_{2n+1,}x_{2n})) = F(d(T_1x_{2n},T_2x_{2n-1})) \leq kF\left(d(x_{2n,}x_{2n-1})\right) + lF(d(x_{2n,}T_2x_{2n-1}))$$
$$= kF\left(d(x_{2n,}x_{2n-1})\right) + lF(d(x_{2n,}x_{2n})) \leq kF\left(d(x_{2n,}x_{2n-1})\right)$$

Thus we obtain

$F(d(x_{2n+1},x_{2n})) \leq kF(d(x_{2n},x_{2n-1}))$ for all $n \in \mathbb{N}$,

Hence
$F(d(x_{2n+1},x_{2n})) \leq kF(d(x_{2n},x_{2n-1})) \leq k^2F(d(x_{2n-1},x_{2n-2}))\ldots\ldots \leq k^{2n}F(d(x_{1,}x_o))$.

As in proof of Theorem 3.1, $\{x_{2n}\}$ is a Cauchy sequence in X. Since X is complete, there exists $z \in X$ such that $\lim_{n\to\infty} x_{2n} = z$.
Let $c \in int\,(P)$ be given. Choose $N \in \mathbb{N}$ such that $d(x_{2n-1},z) \ll F^{-1}\frac{c}{3}$ for all $n > N$. By $(F2)$ *and* $(F3)$, $F(d(x_{2n-1},z)) \ll \frac{c}{3}$ for all $n > N$.
Thus, for all $n > N$, we obtain

$$F(d(z,T_1z)) \leq F(d(z,x_{2n-1}) + d(x_{2n-1},T_1z)) \leq F(d(z,x_{2n-1})) + F(d(T_1x_{2n-2,}T_1z))$$
$$\leq F(d(z,x_{2n-1})) + kF(d(x_{2n-2,}z)) + lF(d(z,T_1x_{2n-2}))$$
$$= F(d(z,x_{2n-1})) + kF(d(x_{2n-2,}z)) + lF(d(z,x_{2n-1})) \ll \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c.$$

As in proof of Theorem 3.1, we have $T_1 z = z$. Suppose that u is another fixed point of T_1 Then from (3.3) we have

$$F\big(d(z,u)\big) = F(d(T_1 z, T_1 u)) \le kF(d(z,u)) + lF(d(z, T_1 u))$$
$$= kF\big(d(z,u)\big) + lF\big(d(z,u)\big)$$
$$= (k+l)F\big(d(u,z)\big).$$

Thus $(k + l - 1)\, F\big(d(u,z)\big) \in$ P. Since 0$\le k + l < 1$,
$(k + l - 1)\, F\big(d(u,z)\big) \in -P$. Hence, $F\big(d(u,z)\big) = 0$. By $(F1), d(z,u) = 0$, and so z = u.
Therefore, T_1 has a unique fixed point in X. Similarly it can be established that $z = T_2 z$. Thus z is the unique common fixed point of T_1 and T_2.

4 Conclusion

Many fixed point theorems have been established in metric spaces or in the setting of topological spaces. In this work attempt has been made to extend such results in cone metric spaces with different type of contractive conditions.

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