# Time-changed generalized mixed fractional Brownian motion and application to arithmetic average Asian option pricing

### Ömer Önalan *

*Faculty of Business Administration, Marmara University, Bahçelievler Campus, 34180, Bahçelievler - Istanbul, Turkey*
*\*Corresponding author E-mail: omeronalan@marmara.edu.tr*

## Abstract

In this paper we present a novel model to analyze the behavior of random asset price process under the assumption that the stock price process is governed by time-changed generalized mixed fractional Brownian motion with an inverse gamma subordinator. This model is constructed by introducing random time changes into generalized mixed fractional Brownian motion process. In practice it has been observed that many different time series have long-range dependence property and constant time periods. Fractional Brownian motion provides a very general model for long-term dependent and anomalous diffusion regimes. Motivated by this facts in this paper we investigated the long-range dependence structure and trapping events (periods of prices stay motionless) of CSCO stock price return series. The constant time periods phenomena are modeled using an inverse gamma process as a subordinator. Proposed model include the jump behavior of price process because the gamma process is a pure jump Levy process and hence the subordinated process also has jumps so our model can be capture the random variations in volatility. To show the effectiveness of proposed model, we applied the model to calculate the price of an average arithmetic Asian call option that is written on Cisco stock. In this empirical study first the statistical properties of real financial time series is investigated and then the estimated model parameters from an observed data. The results of empirical study which is performed based on the real data indicated that the results of our model are more accuracy than the results based on traditional models.

*Keywords*: *Arithmetic Average Asian Option Pricing; Generalized Mixed Fractional Brownian motion; Inverse Gamma Subordination; Time-Chanced Processes.*

## 1. Introduction

Stochastic models have ability to represent the complex phenomena in a simple fashion diffusion process is one of the most elemental stochastic processes in a variety of applications in natural and social sciences. Brownian motion is the only stochastic process which is Gaussian, stationary and Markovian. In a broad range of contexts, Brownian motion used to model as a driving force for the random behavior of many natural random phenomena(for log returns) whose value is the result of a large number of small shots occurring in time. In the Black-Scholes model, the asset prices are assumed to behavior according to the geometric Brownian motion. This model assumes that the logarithmic returns are independent and satisfies the efficient market hypothesis. But empirical studies represent that financial asset price processes have heavy tails, leptokurtosis, stochastic volatility and jump behaviors [24], [25], [27].The tail distributions of logarithmic returns in financial time series behavior as $P\left(|r_t|>x\right)\sim x^{-\alpha}$ , $\alpha\cong3$ where

$r_t$ log returns on time interval $\Delta t$ . and $\alpha$ is Pareto exponent[29].

The models which based on classic Brownian motion are unable to match the stylized facts as self-similarity, long range correlations, heavy-tailed and skewed marginal distributions, volatility clustering[1], [12].
We don't know completely what the causes of these statistical properties are. The discovering the long-range dependence in financial markets is stimulates the application of fractional Brownian motion. Long range dependence properties of a time series is founded in a variety of fields for example physiology, finance, solar physic. Long range dependence denotes the property of time series to exhibit persistence behavior. A process has the long range dependence properties if the autocorrelations decay to zero so slowly so $\rho(k)\sim c|k|^{\gamma}$ , $\gamma\in(0,1)$ . The empirical studied suggest to use Fractional Brownian motion with $H\in(1/2,1)$ as a model for logarithmic returns [23]. Fractional Brownian motion is a prototypical model for displaying the long term depending stochastic processes. It is an important model for fractional dynamical systems. Fractional Brownian motion is a generalization of Brownian motion. Such that dropping the Markovian property in Brownian motion is obtained the fractional Brownian motion. So Ito calculus does not apply to the FBM. It is not semi martingale for $H\neq1/2$ . To overcome this shortcoming, [5] was proposed a mixed fractional Brownian (MFBM) model with $H\in(3/4,1)$ . MFBM model has been employment for pricing currency option [16], pricing European options pricing Asian power options [2].
Especially in the emerging markets, the financial data represent the motionless periods, i.e. the price path shows constant periods [14]. Normal diffusion is characterized by Gaussian probability density function whose variance increases linearly in time. The anomalous diffusion processes refers to the processes which the mean square displacement (MSD) of a price process is no longer

linear in time. They are characterized by variance growing slower (sub diffusion) or faster (super diffusion) than normal diffusion. To model the anomalous behavior one of ways is change the real time in Brownian diffusion by inverse subordinators. The time changed process exhibits properties of anomalous diffusion [19]. Subordination of fractional Brownian motion consists of time-changing the paths of FBM by an independent subordinator.

We investigated the stochastic origin of financial asset price process and the arithmetic average Asian option pricing problem. In this study, the time-changed generalized mixed fractional Brownian motion model was based on a fractional stochastic differential equation which is proposed by [31] .We introduce the subordinator process into the generalized mixed fractional Brownian motion, that is, we change the original clock as calendar time of main stochastic process with a new random clock as a business time which a non-negative, non-decreasing, nonlinear process $M_{\alpha,\lambda,1}(t)$. The randomness in business time generates randomness in volatility. We obtain the time-changed generalized mixed fractional Brownian motion, i.e., $Z_{H,\alpha}\left(M_{\alpha,\lambda,1}(t)\right)$ where parent process $Z_{H,\alpha}(.)$ is a generalized mixed fractional Brownian motion(GMFBM) and $M_{\alpha,\lambda,1}(t)$ is the inverse Gamma subordinator. We can deduce properties of $Z_{H,\alpha}\left(M_{\alpha,\lambda,1}(t)\right)$ from properties of the subordinator $M_{\alpha,\lambda,1}(t)$.

The remainder of this paper is organized as follows. In section 2, we review definitions and the basic properties that we will need for the rest of the paper of fractional Brownian motion and we describe the different methods to estimate Hurst exponent $H$ . In section 3, we give a time-changed fractional Brownian motion and time-changed mixed generalized fractional Brownian motion. In section 4, we review the Asia option pricing problem, in section 5, the performance of the proposed model is illustrated by some numerical experiments. Finally section 6 presents the conclusions.

## 2. Fractional Brownian motion

Fractional Brownian motion (FBM) first was introduced by Kolmogorov (1940) in Hilbert spaces and a stochastic integral representation provided by [15]. The FBM is the usual candidate to model the phenomena which have self-similarity and long range dependence properties can be observed from the empirical data. Fractional Brownian motion is only stochastic process which is Gaussian, non- stationary, non-Markovian and statistically self-similar with continuous sample path.

**Definition 2. 1:** *A fractional Brownian motion (FBM),* $\left(B_H(t)\right)_{t\in R}$ *with Hurst exponent* $H \in (0,1)$ *on the probability spaces* $(\Omega, \Im, P)$ *is a stationary, mean-zero Gaussian process with the features,*

i).    $B_H(0) = 0$

ii).    $E\left(B_H(t)\right) = 0 \quad, t \geq 0$

iii).    $Cov\left[B_H(t), B_H(s)\right] = \dfrac{\sigma_H^2}{2}\left[t^{2H} + s^{2H} - |t-s|^{2H}\right], s, t \geq 0$

Where $\sigma_H^2 = \left(\Gamma(1-2H)\cos(\pi H)/\pi H\right)$ . If $\sigma^2 = 1$ case is called a normalized fractional Brownian motion. If $H = 1/2$ corresponding FBM is standard Brownian motion $B(t)$ , for $H \neq 1/2$ , $B_H(t)$ is neither a semi-martingale nor a Markov process. The main difference between them is that the increments of $B(t)$ are independent while the increments of $B_H(t)$ are dependent.

Fractional Brownian motion is characterized by the following properties [17]:

- $B_H(t)$ is a self-similar process, i.e for every constant $c > 0$ , $\left(B_H(ct); t \geq 0\right) \overset{d}{=} \left(c^H B_H(t); t \geq 0\right)$ so

$$B_H(t) \overset{d}{=} t^H B_H(1) \quad E\left[B_H(t)^2\right] = E\left[\left(t^H B_H(1)\right)^2\right] = t^{2H} E\left[B_H(1)^2\right] = t^{2H}$$

Let $b_H(k) = B_H(k) - B_H(k-1) \quad, k = 1,2,...$ then $b_H(.)$ are called fractional Gaussian noise (fGn) and are stationary and they have following covariance function

- $b_H(t_k) = B_H(t_k) - B_H(t_{k-1}) \quad, k = 1,2,...,n \quad, b_H(.)$ are called fractional Gaussian noise (fGn) and are stationary. $b_H(t_k) = B_H(t_k) - B_H(t_{k-1}) \sim N\left(0, |t_k - t_{k-1}|^{2H}\right)$ [20].Fractional Gaussian noise process has following covariance function,

$$\rho(n) = Cov\left(b_H(k), b_H(k+n)\right) = \frac{1}{2}\left[(n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\right]$$
$$\approx H(2H-1)n^{2H-2} \quad, n \to \infty$$

- The useful measure of the degree of dependence between the values of a time series at different times is given by the following expected value,

$$E\left[\left(B_H(t) - B_H(s)\right)^{2k}\right] = \frac{\sigma^2(2k)!}{k!2^k}|t-s|^{2Hk}$$

- For $H > 1/2$ the process $\left(B_H(t): t \geq 0\right)$ exhibit a long range dependence, that is, and for $n$ large enough, $\sum_{n=1}^{\infty}\rho(n) = \infty$ . The parameter $H$ determines the memory of the process and also controls the roughness of its path. If $H \in (0, 1/2)$ the process has short memory and its trajectory is rough and for $H \in (1/2, 1)$ far away observations have significant influence on the present values of the process. The autocorrelation function $\rho(n)$ behaves like $\rho(n) \sim |n|^{2-2H}$ . A fBm can be represented as an integral with respect to Brownian motion as [15] , [17] , [14],

$$B_H(t) = \frac{1}{\Gamma(H+1/2)}\left\{\begin{array}{l}\int_{-\infty}^{0}\left[(t-u)^{H-1/2} - (-u)^{H-1/2}\right]dB(u) \\ +\int_{0}^{t}(t-u)^{H-1/2}dB(u)\end{array}\right\} \quad (1)$$

Where, $B(t)$ is standard Brownian motion, $\Gamma(H+1/2) = \int_{0}^{\infty}x^{H-1/2}e^{-x}dx$ is Euler's gamma function and $d = H - 1/2$ is called the memory parameter. For each $t$ , the random variable $B_H(t)$ has a Gaussian distribution with $B_H(t) \sim N\left(0, \sigma^2 t^{2H}\right)$ [7]. The probability density function (pdf) of FBM given by

$$f_{B_H}(x, t) = \frac{1}{\sqrt{2\pi}\,t^H}\exp\left(-\frac{1}{2}x^2 t^{-2H}\right) \quad, x \in R$$

The Laplace transform of FBM is described as

$$E\left(\exp(-uB_H(t))\right) = \exp\left(-\frac{1}{2}u^2 t^{2H}\right), t \geq 0,$$

We can approximate to the Mandelbrot and van Ness representation with Riemann type sums as follows,

$$\hat{B}_H(t) = c_H\left\{\sum_{k=-d}^{0}\left[(t-k)^{H-1/2} - (-k)^{H-1/2}\right]B_1(k) + \sum_{k=0}^{t}(t-k)^{H-1/2}B_2(k)\right\} \quad (2)$$

Where,

$$c_H = \sqrt{\Gamma(2H+1)\sin(\pi H)/\Gamma(H+1/2)}$$

Remark: We denotes the increments of fractional Brownian motion as $dB_H(t) = \xi_t(dt)^H$ , $\xi_t \sim N(0,1)$ .

## 2.1. Hurst exponent $H$ estimation method

There are many methods to estimate the Hurst parameter $H$ . Let $S_k$ denotes the price of a financial asset which follows a FBM of stock (index) at time $k$ $(k = 0,1,2,...,n)$ , $Y_k = \ln(S_k)$ and $\Delta t$ is the observation interval $\Delta t = T/n$ , $T$ is observation period. Let $Z_i = Y_{i\Delta t} - Y_{(i-1)\Delta t}$ , $i = 1,2,...,n$ and $x_i = Z_i - \hat{\mu}\Delta t$ [22], [26]

$$\hat{H} = \frac{1}{2\ln 2}\ln \frac{\frac{1}{2[n/2]}\left(\sum_{i=1}^{[n/2]-1}(x_{2i+1}+x_i)^2 + \sum_{i=1}^{[n/2]-1}(x_{2i+2}+x_{2i+1})^2\right)}{\frac{1}{n}\sum_{i=1}^{n}x_i^2} \tag{3}$$

$$\hat{H}_{Robust} = \frac{\sum_{i=2}^{n}|x_{i-1}||x_i|}{\sum_{i=3}^{n}|x_{i-2}||x_i|} \tag{4}$$

R/S method: Given a return series $X = (x_t : t = 1,2,...)$ with average $\bar{x}(n) = \frac{1}{n}\sum_{i=1}^{n}x_t$ and variance $S^2(n) = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x}(n))^2$ then,

$$\frac{R(n)}{S(n)} = \frac{1}{S(n)}\{\max(0,w_1,w_2,...,w_n) - \min(0,w_1,w_2,...,w_n)\}$$

Where $w_k = (x_1 + x_2 + ... + x_k) - k\bar{x}(n)$ , $k = 1,2,...,n$

If the sample series satisfy long range dependence, then

$$E(R(n)/S(n)) \sim cn^H , n \to \infty$$

If we take the logarithm both of side the equation, we obtain that,

$$\log(R(n)/S(n)) = \log(c) + H.\log(n)$$

If we assume that the time series displays a power law as $p(x) \sim x^{-\alpha}$ , $1 < \alpha < 3$ .
Then Hurst exponent $H$ is related with $\alpha$ as $H = (3-\alpha)/2$ . The probability density of power-law distribution is given by

$$p(x) = ((\alpha-1)/x_{min})(x/x_{min})^{-\alpha} \tag{5}$$

The MLE of the power-law exponent $\alpha$ is obtained as

$$\hat{\alpha}_{MLE} = 1 + n\left(\sum_{i=1}^{n}\log(x_i/x_{min})\right)^{-1} \tag{6}$$

Where $x_i$ , $i = 1,2,...,n$ are independent observed values as $x_i \geq x_{min}$ .
Remarks: Fractional Brownian motion exhibits sub diffusive behavior for $H < 1/2$ and super diffusive behavior for $H > 1/2$ and normal diffusion for $H = 1/2$ .

## 3. Time-changed generalized mixed fractional Brownian motion

In this section, we will consider a superposition of two independent mechanisms, namely a generalized mixed FBM and a time process. Here time process can be considered an operational time of main process. The generalized mixed fractional Brownian motion (GMFBM) is combination of standard Brownian motion and fractional Brownian motions. In [13] is introduced the sub-diffusive geometric Brownian motion to model the asset prices. [9] Proposed a time-changed mixed Brownian fractional Black-Scholes model for the price of the underlying stock. The details of subordinated generalized mixed fractional Brownian motion model, you can look [21].

### 3.1. The subordinator process

The changing time $t$ of main stochastic process with another increasing process (subordinator) $M_{\alpha,\lambda,1}(t)$ is a way to construct new stochastic process from a main process. A subordinator is a real-valued Levy process which only takes nonnegative values. From the Levy-Ito decomposition, for a subordinator the $\sigma$ must be zero, the drift $\mu$ must be nonnegative. We denote the Laplace exponent of a subordinator process $X$ as

$$\phi(\lambda) = \mu\lambda + \int_0^{\infty}(1-e^{-\lambda x})v(dx) , \lambda > 0 \tag{7}$$

The Laplace exponent of a subordinator is closely related to the concept of Bernstein functions. The Laplace exponent $\phi$ of any subordinator is a Bernstein function with $\lim_{t\downarrow0}\phi(t) = 0$ . It can be shown that the converse is also valid. That is, a function $\phi:(0,\infty)\to[0,\infty)$ is the Laplace exponent of a subordinator if and only if $\phi$ is a Bernstein function with $\lim_{t\downarrow0}\phi(t) = 0$ .We consider the time change process $M_{\alpha,\lambda,1}(t)$ is given by inverse first hitting time process of $(G_{\lambda,1}(t))_{t\geq0}$ which is an increasing right continuous process with left limits. The relationship between the two processes is expressed as $M_{\alpha,\lambda,1}(t) = \inf\{\tau > 0 : G_{\lambda,1}(\tau) > t\}$ . $M = (M_{\alpha,\lambda,1}(t))_{t\geq0}$ Is an inverse subordinator associated with the Bernstein function $\varphi$ . $P(G_{\lambda,1}(\tau) < t) = P(M_{\alpha,\lambda,1}(t) < \tau)$

Choosing the different stochastic process as $G_{\lambda,1}(t)$ , we can generate wide range of stochastic time processes [23]. In this study we consider the physical time $t$ as $t = M_{\alpha,\lambda,1}(t)$ with the drift less strictly increasing and pure jump Levy process $G_{\lambda,1}(t)$ .The Laplace transform of $G_{\lambda,1}(t)$ is given by $\varphi(u) = \frac{1}{\lambda}\log(1+u)$ [16] , [19]. We change the time process of main process by an inverse gamma subordinator. It is continuously increasing process and denoted by $M_{\alpha,\lambda,1}(t)$ [31].

### 3.2. Mixed fractional Brownian motion

We consider the a linear combination of a Brownian motion $B(t)$ and independent fractional Brownian motion $B_H = (B_H(t))_{t\geq0}$ with Hurst exponent $H \in (0,1)$ , MFBM is defined as $Z = (Z(t))_{t\geq0}$

$$Z(t) = aB(t) + bB_H(t) , a,b \in R/\{0\} \tag{9}$$

The process $Z = (Z(t))_{t\geq0}$ is centered Gaussian process with $Z(0) = 0$ and with covariance function [2],

$$Cov(Z(t),Z(s)) = a^2 + \min(t,s) + \frac{b^2}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) \tag{10}$$

### 3.3. The time changed generalized mixed fractional Brownian motion model

The generalized mixed fractional Brownian motion process introduced by [34] as a linear combination of a countable number of Brownian motions and fractional Brownian motions. This process is defined as follows.

**Definition ([21] definition2.1:** *Let be* $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ *coefficients and* $H = (H_1, H_2, ..., H_n)$ *Hurst parameters. A gmfBm is a stochastic process* $Z_H = (Z_{H,\alpha}(t))_{t \geq 0}$ *where,* $Z_{H,\alpha} = \sum_{k=1}^{N} \alpha_k B_{H_k}(t)$.

*The mean and variance of gmfBm are* $E(Z_H) = 0$ *and* $Var(Z_H) = \sum_{k=1}^{N} \alpha_k^2 t^{2H_k}$ , *respectively[21].*

$$Cov\left(Z_{H,\alpha}(t), Z_{H,\alpha}(s)\right) = \frac{1}{2}\sum_{k=1}^{N} \alpha_k^2\left(t^{2H_k} + s^{2H_k} - |t-s|^{2H_k}\right) \tag{11}$$

### 3.4. Pricing model

In this study we describe the time- changed generalized mixed fractional Brownian motion model as follows,

$$Z_{H,G}\left(M_{\alpha,\lambda,1}(t)\right) = \left[B\left(M_{\alpha,\lambda,1}(t)\right) + B^H\left(M_{\alpha,\lambda,1}(t)\right)\right] \tag{12}$$

We assume that the dynamics of underlying stock price $S_t$ follow following time-changed generalized mixed fractional Brownian motion,

$$S_t = S_{t-1}\exp\left\{\begin{matrix} \mu\xi(t) - \frac{1}{2}\sigma^2\left(\xi(t)\right)^2 + \sigma Z_{H,G}\left(M_{\alpha,\lambda,1}(t)\right) \\ + \eta\left(B^H\left(M_{\alpha,\lambda,1}(t)\right)\right)^2 \end{matrix}\right\} \tag{13}$$

Where

$$\xi(t) = M_{\alpha,\lambda,1}(t) - M_{\alpha,\lambda,1}(t-1)$$

### 3.5. Parameter estimation of model

Let $t = (0, \Delta t, 2\Delta t, ..., n\Delta t)$ is the observation times and $X = (x_0, x_{\Delta t}, ..., x_{n\Delta t})$ are returns. The maximum likelihood estimation of parameters $\mu$ and $\sigma$ are determined as follows [31], [10], [26] [32], [33].

$$\hat{\mu} = \frac{tR_H^{-1}X^T}{tR_H^{-1}t^T}$$

$$\hat{\sigma}^2 = \frac{1}{N}\frac{(XR_HX^T)(tR_Ht^T) - (tR_HX^T)^2}{tR_H^{-1}t^T}, \quad \hat{\eta} = \sqrt[4]{E\left[X - \bar{X}\right]^4} \tag{14}$$

$$\begin{aligned}R_H &= \left[Cov\left(B_H(i\Delta t), B_H(j\Delta t)\right)\right]_{i,j=0,1,...,n} \\ &= \frac{1}{2}(\Delta t)^{2H}\left[i^{2H} + j^{2H} - |i-j|^{2H}\right]_{i,j=0,1,...,n}\end{aligned} \tag{15}$$

### 3.6. Simulation of the model

We use an inverse gamma process as new time process in proposed model. The probability density function of inverse gamma distribution is given by

$$IG\left(x \mid \alpha, \beta\right) = \frac{\beta^\alpha x^{-\alpha-1}}{\Gamma(\alpha)}\exp\left(-\frac{\beta}{x}\right) \tag{16}$$

Parameter estimation with methods of moments: The observed data is given by $\{x_1, x_2, ..., x_n\}, x_i > 0$

$$\hat{\alpha} = \frac{\mu^2}{\nu} + 2 \text{ and } \hat{\beta} = \mu\left[\left(\frac{\mu^2}{\nu}\right) + 1\right] \tag{17}$$

Where

$$\mu = \frac{1}{n}\sum_{i=1}^{n}x_i \text{ and } \nu = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \mu)^2$$

Maximum Likelihood method:

$$\hat{\alpha} = \mu^2/\nu + 2 \text{ and } \hat{\beta} = (n\hat{\alpha})\bigg/\sum_{i=1}^{n}\left(\frac{1}{x_i}\right) \tag{18}$$

We can simulate the time- changed generalized mixed fractional Brownian motion process as follows [30].

$$M_{\alpha,\lambda,1}(t) = S_\alpha\left(G_{\lambda,1}(t)\right) = \left(G_{\lambda,1}(t)\right)^{1/\alpha}S_\alpha(1) \tag{19}$$

Where

$$S_\alpha(1) = \frac{\sin(\alpha\pi U_1)\left[\sin(1-\pi)U_2\right]^{1/\alpha-1}}{\left[\sin(\pi U_1)\right]^{1/\alpha}|\ln U_2|^{1/\alpha-1}}\frac{|\ln U_3|^{1/\alpha}}{\nu^{1/\alpha}} ,$$

Where $U_1, U_2$ and $U_3$ are uniform distributed random variables over $[0,1]$ , $\nu$ is mean of exponential distribution and $\alpha$ is the stable exponent. The Simulation algorithm is can be summarize as follows:

a) Determine the time interval $[0,t]$ and $h = t/n$ , $t_1 = h, t_2 = 2h, ..., t_{n-1} = (n-1)h, t_n = t$ ,

b) Generate the gamma variates $G = \{G_1, G_2, ..., G_n\}$ with $G_i \sim gamma(\lambda, t_i - t_{i-1})$ ,

c) Generate $\alpha$ -stable random variates $S = \{S_1, S_2, ..., S_n\}$ ,

d) Set $Y_k = G_k^{1/\alpha}S_k$ , $k \geq 1$ and $M_i = \sum_{k=1}^{i}Y_k$ , $i = 1, 2, ..., n$ .

The increments of FBM are discritized by Maruyama symbols as [10], [31]

$$\Delta B_H(t_i) = B_H(t_i + \Delta t) - B_H(t_i) = \xi_i(\Delta t)^H \tag{20}$$

Where $t_i$ represent the *ith* sub-interval. We get the FBM simulation curves according to,

$$B_H(t_{i+1}) = B_H(t_i) + \xi_i(\Delta t)^H \tag{21}$$

## 4. Asian option pricing

A standard option is a financial contract which gives the owner of the contract the right but not the obligations, to buy or sell a specified asset to a pre specified price (strike price) at a per specified time (maturity). Estimating option pricing is an important topic in mathematical finance. The payoff of an option is described by the difference between the underlying asset price and strike price. Path dependent exotic options are options whose payoff is affected by how the price of the underlying asset at maturity was reached, the price path of the underlying asset. Asian options are popular hedging instruments for financial risk managers. Its payoff is determined by the average value over some predetermined time interval. From the stabilization effect which is short price changes in the market of average Asian options (average options) are widely used in commodity and stock markets as cheaper alternatives to European and American options for hedging and risk management. It reduce the volatility in option. The payoff of an Asian option is determined by the average value of the stock price over a prefixed time interval. The terminal payoff Asia call options with fixed

strike and floating strike prices are $\max(A_r - X, 0)$ and $\max(S(T) - A(T), 0)$ respectively. Where $S(T)$ is the asset price time $T$ and $X$ stands for the strike price and $A(T)$ denotes for the time average of the underlying asset's prices from initial time to maturity date $T$. The average is defined as either the arithmetic or the geometric mean form of the underlying asset prices. Let $t_1, t_2, ..., t_n$ be predetermined times. Then the discrete geometric average is defined by $A_n = \left(\prod_{i=1}^{n} S(t_i)\right)^{1/n}$ and continuous geometric average is $A(T) = \exp\left(\frac{1}{T}\int_0^T \log S(t)dt\right)$ in continuous case, arithmetic average is obtained by $A(T) = \frac{1}{T}\int_0^T \log S(t)dt$ where $t_0 \in [0, T]$, $t_n = T$ are fixed and $\Delta t = \frac{T - t_0}{n}$. Then $t_i = t_0 + i\Delta t$, $i = 1, 2, ..n$. The probability distribution of average is generally unknown in Black-Scholes framework, geometric average Asian options can be formula easily ([4] Theorem 2.2, pp.17).

There is not analytically valuation formula to pricing arithmetic average Asian options. Different approaches proposed to pricing Asian options are based on numerical inversion method and control variates Monte Carlo methods, in this study we use the Monte Carlo approach to pricing the arithmetic average Asian options is given by

Set Sum = 0

For $i = 1$ to $n$

Generate $S(T/m), S(2T/m), ..., S(T)$

$$Sum_k = Sum_{k-1} + \max\left(\left(1/(m+1)\right)\sum_{i=1}^{m} S(iT/m) - X, 0\right) \qquad (22)$$

$$\hat{C}_A = \exp(-rT)\left[Sum_n/m\right].$$

## 5. Application to real data

In this section, we empirically analyze performance of time changed generalized mixed fractional Brownian process to evaluate the prices of arithmetic average Asian option which is written on the Cisco System Inc (CSCO) stock. Cisco System Inc (CSCO) stock adjusted prices data across a range of 06.24.2010 to 06.23.2016. Total number of the observed data is $n = 1511$. Analyzed data are obtained from Yahoo finance services. We assume that the price process $S(t_i)$ is the stock adjusted closing price on a time $t_i$. It is observed at times $t_i = i\Delta, i = 1, 2, ..., n$ and $T = n\Delta t$ denotes the length of the observations window and we choice as $\Delta t = 1/252$ (data collected once a day). We have taken the logarithmic adjusted closing prices of the CSCO stock, over a time scale of one day is defined as $r_t = \log(S_t/S_{t-1})$, $1 \le t \le n$. To give an intuitive illustration, we present the plots of prices and returns as follows.
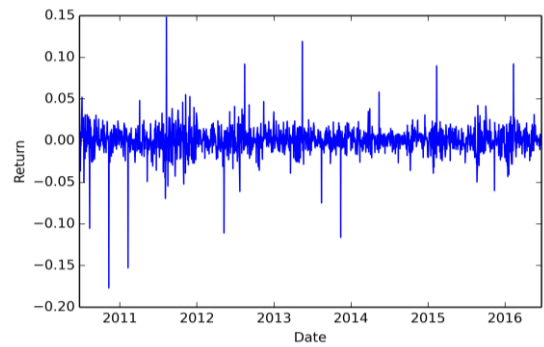


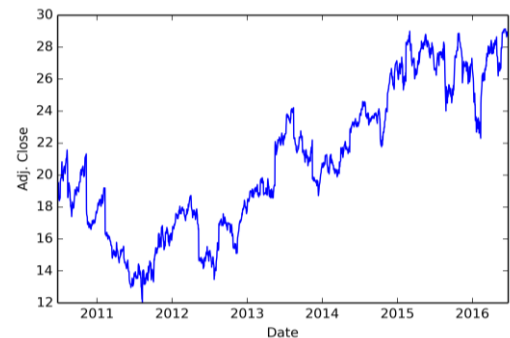**Fig. 1:** (A) Daily Adjusted Closing Log Returns of CSCO.



**Fig. 1:** (B) Daily Adjusted Closing Prices of CSCO.

In figure1, we observe that the analyzed data have "volatility clustering" i.e., the large moves follow large moves and small moves follow the small moves, but it has not trends. Furthermore the trapping events are observed. Before empirical research, we examine the empirical characteristics of CSCO logarithmic return data. We tabulated the basic descriptive statistics for return data during the sampled period.

**Table1:** Summarize Descriptive Statistics of Daily Return of (CSCO) Stock

| Min | Max | Mean | Std.dev. | Skew.(S) | Kurt.(K) | JB |
|---|---|---|---|---|---|---|
| -0.177 | 0.148 | 0.0003 | 0.017 | -0.943 | 25.145 | 31078.9 |

As seen Table1, we reject null hypothesis that the sample data of the Cisco System Inc (CSCO) returns is from a normal distribution using Jarque-Bera (JB) statistic calculated the return series is defined as $JB = (n/6)\left(S^2 + (K-3)^2/4\right)$. Its null hypothesis is that the observations are iid normal distributed. JB statistic is asymptotically distributed as chi-square with two degree of freedom i.e $JB \sim \chi_2^2(0.05) = 5.9915$. Hence the distribution of CSCO return series is not a normal distribution. It has been exhibits leptokurtic and fat tails properties with skew value -0.9432 and kurtosis value 25.1452. These findings represent that the return distribution of CSCO return has a long left tail and the series have a kurtosis that is more than three, which is the kurtosis of the normal distribution. This means that the distribution is more peak than the normal distribution. We can say that the observed data has not normally distributed. Stationary and non-stationary processes have different properties. If a time series have a trend or a varying volatility then it is not stationary. To represent a time series stationary or not, generally we use its increments. To test for the stationary the ADF test is used usually. We applied our time series to following model, $\Delta(S_t) = \alpha + \beta t + \gamma S_{t-1}$,

Where $\alpha$ is a constant, $\beta$ is the coefficient of the time trend. ADF test statistics is the t- statistics of the OLS estimate of $\gamma$. The null hypothesis of the ADF test is $\gamma = 0$, as can be seen from following table, Augmented Dickey-Fuller (ADF) statistics have values are below the critical values at 1%, 5% and 10% level of significance. Therefore we accept the stationary of the daily returns of CSCO stock prices during the period 2010-2016. This

mean that return, squared return and absolute returns series may be consider stationary so our time series can be modeled by the long memory process.

**Table 2:** Augmented Dickey-Fuller (ADF) Statistics for CSCO

| Null hypothesis $H_0: \ \gamma = 0$ | | | | |
|---|---|---|---|---|
| Series | ADF Test | t-stat. | Sign. level | Critical value |
| Returns | -26.77 | -38.76 | 1%<br>5%<br>10% | -3,4593<br>-2.8738<br>-2.5732 |
| Squared returns | -27.81 | -39.01 | 1%<br>5%<br>10% | -3,4593<br>-2.8738<br>-2.5732 |
| Absolute returns | -26.25 | -37.81 | 1%<br>5%<br>10% | -3,4593<br>-2.8738<br>-2.5732 |

We can use a visual stationary test based on the behavior of the empirical second moment of the return series as $C_j = \sum_{i=1}^{j} r_i^2$. If the analyzed time series obtained from a sample with elements which has the same distribution, then the $C_j$ statistics is a linear function with respect to $j$.
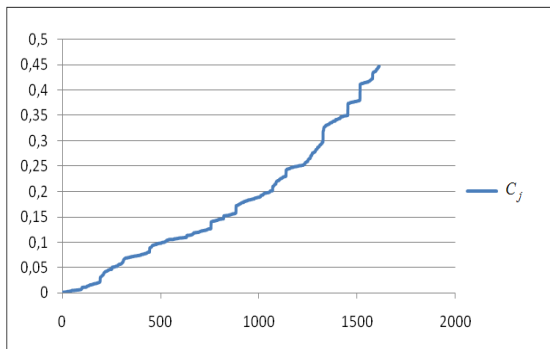


**Fig. 2:** $C_j$ Statistics for CISCO Stock Returns.

From above figure, we see that the returns obtained from dependent random variables. Fractional Brownian motion and Gamma process both have stationary increments hence; the subordinated process also has stationary increments. The periods of time which stock prices stay motionless are captured by an anomalous diffusion process. In analyzed data observed the motionless periods. We calculated an annualized historical volatility of total returns with a window function of 60 point and 255 trading days (estimated number of trade days in a year).

$$\hat{\sigma}_k = \frac{1}{59} \sum_{j=0}^{59} \left( x_{k-j} - \frac{1}{60} \sum_{j=0}^{59} x_{k-j} \right)^2 \times 255$$

The characterization of short and long-range dependence in the time series required the knowledge of the auto covariance functions. The slowly decaying auto covariance function is representing a long-range dependent process (LRD). The fractional Brownian motion can be a suitable model for sub diffusive or super diffusive behavior exhibit time series. For a random return series sample $\{x_i : i = 1, 2, ..., n\}$, autocorrelation function (ACF) is described by

$$\rho(h) = \sum_{i=1}^{n-h} (x_i - \bar{x})(x_{i+h} - \bar{x}) \Big/ \sum_{i=1}^{n} (x_i - \bar{x})^2 \quad , h = 0, 1, 2, ...$$
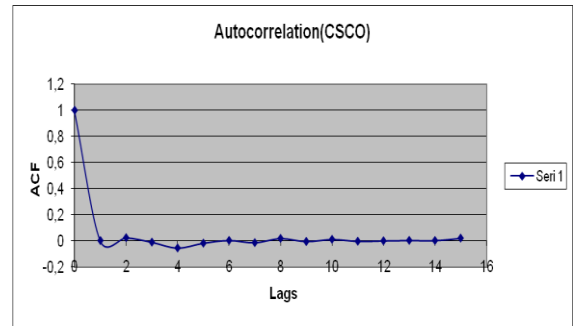


**Fig. 2:** Autocorrelation Function for CSCO.

The time series is called LRD if its autocorrelation function is non-sum able. The decay rates of the sample autocorrelations of return (squared and absolute returns) appear to be slow and this is the evidence of long-term dependence behaviour. The result of degree of autocorrelation proposed that the assumption of independence for log returns of CSCO stock is not provided. So, we can use a model which is based on fractional Brownian motion for CSCO log return series. The Hurst parameter can be used to measure the self-similarity property of time series. Using the historical data we estimated the Hurst parameter as $\hat{H} = 0.3985$. In order to describe the stock price data exhibiting periods of constant values. We are used the inverse Gamma subordinator process as the time for price process.
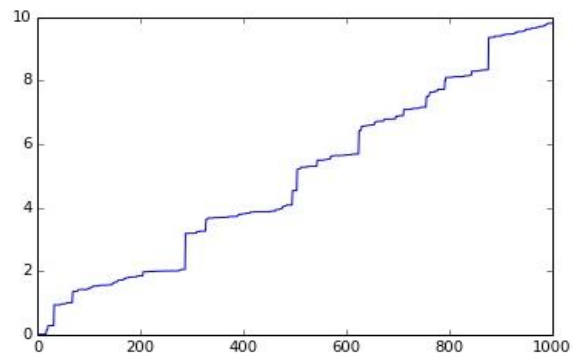


**Fig. 3:** Simulated values for inverse subordinated gamma process.

We simulated the 1000 trajectories of CSCO with the price process that the time-changed generalized mixed fractional Brownian motion model using the inverse subordinator gamma distribution as the new time process.
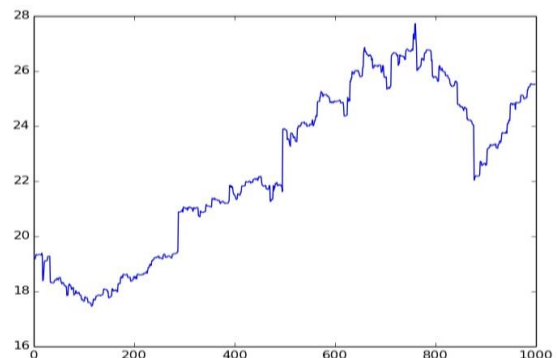


**Fig. 4:** The Plot of Simulated Generalized Mixed Fractional Brownian motion.

Asian options are popular exotic derivatives for risk management since by taking the average price jumps and market manipulations are minimized. In this study we used the Monte Carlo Method to pricing of average Asian option which is writing on CSCO stock. To do this work, we generated random variables from known distribution, applying the law of large numbers; we estimated unknown expected values using the generated outcomes. Then we

obtained the option price discounting the estimated expected value. The price behavior of arithmetic Asian call option prices when number of averages increase. As a result of Jensen's inequality the arithmetic Asian option price is higher than the geometric Asian option price.
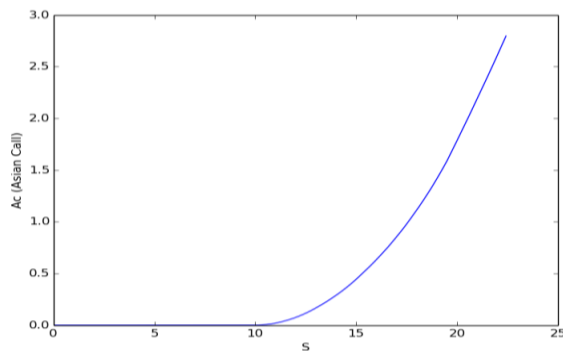


**Fig. 5:** Asian Call Option Price.

The space-time plot of arithmetic average Asian call option values for parameter values $\sigma = 0.02$, $T = 2$, $S_0 = 19$, $X = 10$, $r = 0.05$.
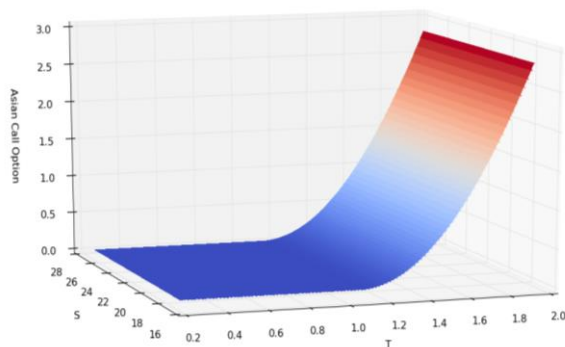


**Fig. 6:** Plot of Arithmetic Average Asian Call Option Values.

## 6. Conclusions

In this paper we studied on a time-changed mixed fractional Brownian motion model to characterize the stock adjusted closing prices behavior. We describe the model as based on Langevin equation that describes the stochastic evolution of base process in micro level that time changed by suitable subordinator. To model the observed long range dependence in the financial price time series, we replaced Brownian motion by fractional Brownian motion in traditional exponential Levy model. FBM has not stationary increment and it is not Markov process. So, we cannot use the traditional parameter estimation methods for FBM. We chose the FBM process to include the long range dependence structure. To eliminate the arbitrage we used a mixed fractional Brownian motion model instead of FBM as a main process. We chanced the time of main process with an inverse gamma subordinator. We applied the inference methods to real data that is observed at discrete time points. Analyzing the real data, we found that the incremental process is not- Gaussian, stationary, non-Markovian and negative correlated. The based on these results we enabled describe the stock price behavior in a fractional Brownian framework. We observed trapping events (motionless periods) in real data. To overcome this problem we modeled the price evaluation of CSCO price process with the time-changed mixed fractional Brownian motion model. Then we calculated the price of average-arithmetic Asian option that is written on CSCO stock, using the values that are simulated proposed model. Empirical studies show that the time changed generalized mixed fractional Brownian motion model provide good for arithmetic average Asian option pricing with different maturities. It has a flexible structure that con-

tains jump components, motionless periods and time changed volatility dynamics.

## References

[1] Ait-Sahalia,Y.(1996), Nonparametric pricing of interest rate derivative securities, *Econometrica*, 64, 3,527-560. https://doi.org/10.2307/2171860.

[2] Prakasa Rao BLS. (2016), Pricing Asian power options under mixed fractional Brownian motion environment. *Physica A: Statistical Mechanics and its Applications*,446 ,92-99. https://doi.org/10.1016/j.physa.2015.11.013.

[3] Bates, D. (1996), Jumps and stochastic volatility: exchange rate processes implicit in deutsche mark options. *Review of Financial Studies,* 9, 69-107 https://doi.org/10.1093/rfs/9.1.69.

[4] Cao G. & Wong Y. (2015), Risk-neutral pricing for geometric average Asian options with floating strike, *Journal of university of Chinese academy of sciences*,32,1,13-17.

[5] Cheridito,P.(2001),Fixed fractional Brownian motion. *Bernoulli*, 7,913-934. https://doi.org/10.2307/3318626.

[6] Cont,R & Tankov, P.(2004), *Financial modeling with jump processes*, Chapman & Hall/CRC Baton; Florida.

[7] Filetova D.V. & Grzywaczewski, M. (2008), Mathematical modeling in selected biological systems with fractional Brownian motion. *HIS*, Krakow, Poland, May 25-27.

[8] Gajda, J. & Magdziarz, M.(2010), Fractional Fokker-Planck equation with tempered alpha-stable waiting times: Langevin picture and computer simulation, *Phys.Rev. E*,82, doi:10.1103/ Phys Rev E. 82.011117 https://doi.org/10.1103/PhysRevE.82.011117.

[9] Guo, Z. & Yuan, H. (2014), Pricing European option under the time-changed mixed Brownian-fractional Brownian model, *Physica A*, 406, 73-79. https://doi.org/10.1016/j.physa.2014.03.032.

[10] Jian-Kai, L.,Wan-Qing, S.& Qing, L.(2015), Research on cutting tool wear based on fractional Brownian motion, *International Journal of Mechanical Engineering and Applications,* 3(1), 1-5. https://doi.org/10.11648/j.ijmea.20150301.11.

[11] Kubiluis, K. & Melichov, D. (2010), Quadratic variations and estimation of the Hurst index of the solution of the SDE driven by a fractional Brownian motion. *Lithuanian Mathematical Journal*, 50(4), 401-417. https://doi.org/10.1007/s10986-010-9095-z.

[12] Lo,A.W.(1991),Long term memory in stock market prices, *Econometrica*, 59,(5),1279-1313. https://doi.org/10.2307/2938368.

[13] Magdziarz,M.(2009), Black-Scholes formula in sub-diffusive regime, *J. Stat. Phys.,* 136, 553–564. https://doi.org/10.1007/s10955-009-9791-4.

[14] Magdziarz,M.& Gajda,J.(2012),Anomalous dynamics of Black-Scholes model time-changed by inverse subordinators, *Acta Physica Polonica B*,43,1093-1109. https://doi.org/10.5506/APhysPolB.43.1093.

[15] Mandelbrot, B.B. & Van Ness, J.W. (1968), Fractional Brownian motion, fractional noise and applications. *SIAM Review,* 10,422-437. https://doi.org/10.1137/1010093.

[16] Shokrollahi,F.& Kılıçman, A.(2014),Pricing currency option in a mixed fractional Brownian motion with jumps environment, *Mathematical Problems in Engineering*.1-13. https://doi.org/10.1155/2014/858210.

[17] Taqqu,M.S.(2013), Benoit Mandelbrot and fractional Brownian motion, *Statistical Science*,28, (1) https://doi.org/10.1214/12-STS389.

[18] Villasenor J.A. & Gonzales-Estrada, E. (2015), A variance ratio test of fit for gamma distributions. *Statistics and Probability Letters*, 96,281-286. https://doi.org/10.1016/j.spl.2014.10.001.

[19] Wyłomańska, A. (2012), Arithmetic Brownian motion subordinated by tempered stable and inverse tempered stable processes. *Physica A*, 391 (22), 5685-5696. https://doi.org/10.1016/j.physa.2012.05.072.

[20] Xu,Y., Guo,R., Liu,D.& Zhang,H.(2014), Stochastic averaging principle for dynamical systems with fractional Brownian motion. *Discrete and Continuous Dynamical Systems Series B.*, 19, (4), 1197-1212. https://doi.org/10.3934/dcdsb.2014.19.1197.

[21] Wenting, C., Bowen, Y., Guanghua, L. & Ying, Z. (2016), Numerical pricing American options under the generalized mixed fractional Brownian motion model. *Physica A*, 451, 180-189. https://doi.org/10.1016/j.physa.2015.12.154.

[22] Xiao, W., Zhang, W. & Zhang, X. (2015), Parameter identification for drift fractional Brownian motion with application to the Chinese stock markets. *Communications in Statistics Simulation and Computation,* 5, 44, 2117-2136 https://doi.org/10.1080/03610918.2013.849738.

[23] Zhang, P.; Sun, Q. & Xiao, W.L. (2014), Parameter identification in mixed Brownian-fractional Brownian motions using Powell's optimization algorithm. *Econ. Model*, 40,314-19. https://doi.org/10.1016/j.econmod.2014.04.026.

[24] Mittinik,S.; Paolella, M.S.& Rachev, S.T.(2000), Diagnosing and treating the fat tails in financial return data. *J. Empirical Finance,* 7, 389-416. https://doi.org/10.1016/S0927-5398(00)00019-0.

[25] Xiao, W., Weiguo, Z., Weijun, X. & Xili, Z. (2012), the valuation of equity warrants in a fractional Brownian environment. *Physica A*, 391,(4),1742-1752, https://doi.org/10.1016/j.physa.2011.10.024.

[26] Xiao, W., Weiguo, Z. & Xili, Z. (2015), Parameter identification for the discretely observed geometric fractional Brownian motion. *Journal of Statistical Computation and Simulation*, 85, (2), 269-283. https://doi.org/10.1080/00949655.2013.814135.

[27] Kleinert, H. & Korbel, J. (2016), Option pricing beyond Black-Scholes based on double-fractional diffusion. *Physica A,* 449, 200-214. https://doi.org/10.1016/j.physa.2015.12.125.

[28] Zhijuan, M. & Zhian, L. (2014), Evaluation of geometric Asian power options under fractional Brownian motion. *Journal of Mathematical Financ*e, 4, 1-9. https://doi.org/10.4236/jmf.2014.41001.

[29] Stanley, H.E., Gabaix,X., Gopikrishnan,P.& Plerou,V.(2007), Quantifying extremely rare and less rare events in finance, *Physica A*, 382,286-301. https://doi.org/10.1016/j.physa.2007.02.023.

[30] Kumar,A.,Wylomanska,A.,Poloczanski,R&Sundar,S.(2017), Fractional Brownian motion time-changed by gamma and inverse gamma process,*Physica A*. 468,648-667. https://doi.org/10.1016/j.physa.2016.10.060.

[31] Wanging,S.,Ming,L.&Jian-Kai Liang(2016), Prediction of bearing fault using fractional Brownian motion and minimum entropy deconvolution,*Entropy,MDPI*,18,418,1-15.hysica A.

[32] Lv,L,Ren,F.Y.&Qiu,W.Y.(2010),The application of fractional derivatives in stochastic models driven by fractional Brownian motion, *Physica A,*21,4809-4818.

[33] Jeon, J. H. & Metzler,R.(2010),;Fractional Brownian motion and motion governed by the fractional Langevin equation in confined geometries. *Phys.Rev.E*,81,21-33. https://doi.org/10.1103/physreve.81.021103.

[34] Thale,C.(2009),Further remarks on mixed fractional Brownian motion,*Appl.Math.Sci.*,3,38,1885-1901.