EXISTENCE AND UNIQUENESS OF SOLUTION FOR CAHN-HILLIARD HYPERBOLIC PHASE FIELD SYSTEM WITH DIRICHLET BOUNDARY CONDITIONS AND POLYNOMIAL POTENTIAL.

A. J. Bissouesse, D. Moukoko, F. D. R.Langa, M. Batchi
Faculté des Sciences et Techniques, Université Marien NGouabi P.P.69 Brazzaville Congo.
E-mail: primphabissouesse@gmail.com

October 24, 2017

Abstract

Our aim in this article is to study the existence and the uniqueness of solution for Cahn-Hilliard hyperbolic phase-field system, with initial conditions, Dirichlet boundary homogeneous conditions, polynomial potential in a bounded and smooth domain.

Keywords: *Cahn-Hilliard hyperbolic phase-field system, polynomial potential, Dirichlet boundary conditions.*

1 Introduction

G. Caginalp introduced in [3] the following phase-field system

$$\frac{\partial u}{\partial t} - \Delta^2 u - \Delta f(u) = -\Delta\theta \tag{1.1}$$

$$\frac{\partial \theta}{\partial t} - \Delta\theta = -\frac{\partial u}{\partial t} \tag{1.2}$$

where u is the order parameter and θ is the (relative) temperature. These systems model phase transition processus such as melting solidification processes and have studied (see [1] and [9]) for a similar phase-field model with a nonlinear term.
These Cahn-Hilliard phase-field systems are known as conserved phase-field system (see [7] and [16]) based on type III heat conduction and with two temperatures (see [15]), the authors have proven the existence and the uniqueness of the solutions, the existence of global attractor and exponential attractors.

In [19], Ntsokongo and Batangouna have studied the following Cahn-Hillard hyperbolic phase-field system

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = -\Delta\Big(\frac{\partial \alpha}{\partial t} - \beta\Delta\frac{\partial \alpha}{\partial t}\Big) \tag{1.3}$$

$$\frac{\partial^2 \alpha}{\partial^2 t} - \Delta\frac{\partial^2 \alpha}{\partial^2 t} - \Delta\frac{\partial \alpha}{\partial t} - \Delta\alpha = -\frac{\partial u}{\partial t} \tag{1.4}$$

where $\beta = 1$, u is the order parameter and α is the temperature. They have proven the existence and the uniqueness of solution with Dirichlet boundary condition and the regular potential $f(s) = s^3 - s$.

In [13], Jean De Dieu Mangoubi and al. have studied the following Cahn-Hillard hyperbolic phase field system

$$\begin{aligned}
\varepsilon(-\Delta)\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) &= -\Delta\frac{\partial \alpha}{\partial t} \quad in \quad \mathbb{R}_+ \times \Omega \qquad (1.5)\\
\frac{\partial^2 \alpha}{\partial t^2} + \frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t} - \Delta\alpha &= -\frac{\partial u}{\partial t} \quad in \quad \mathbb{R}_+ \times \Omega \qquad (1.6)\\
u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = \alpha|_{\partial\Omega} &= 0\\
u(0,x) = u_0(x),\ \frac{\partial u}{\partial t}(0,x) &= u_1(x) \ \ \forall\, x \in \Omega\\
\alpha(0,x) = \alpha_0(x),\ \frac{\partial \alpha}{\partial t}(0,x) &= \alpha_1(x)\ \forall\, x \in \Omega.
\end{aligned}$$

They have proven the existence and the uniqueness of the solution with Dirichlet boundary condition and the potential $f(s) = s^3 - s$.

In this paper, we consider the following Cahn-Hilliard hyperbolic phase-field system

$$\begin{aligned}
\epsilon(-\Delta)\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) &= -\Delta\left(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right) \quad in \quad \mathbb{R}_+^* \times \Omega \qquad (1.7)\\
\frac{\partial^2 \alpha}{\partial t^2} + \frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t} - \Delta\alpha &= -u - \frac{\partial u}{\partial t} \quad in \quad \mathbb{R}_+^* \times \Omega \qquad (1.8)\\
u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = \alpha|_{\partial\Omega} &= 0 \qquad (1.9)\\
u(0,x) = u_0(x),\ \frac{\partial u}{\partial t}(0,x) &= u_1(x),\ \ \forall\, x \in \Omega \qquad (1.10)\\
\alpha(0,x) = \alpha_0(x),\ \frac{\partial \alpha}{\partial t}(0,x) &= \alpha_1(x),\ \forall\, x \in \Omega \qquad (1.11)
\end{aligned}$$

where ϵ is a relaxation parameter and Ω is a bounded and regular domain of $\mathbb{R}^n$ with $n = 1, 2$ or 3 and f is a polynomial potential of order $2p - 1$.

$$f(s) = \sum_{i=1}^{2p-1} a_i s^i,\ a_{2p-1} > 0,\ p \geq 2.$$

In this paper we prove the existence and the uniqueness of solution of the hyperbolic system (1.7)– (1.11).

The potential f satisfies the following properties

$$\frac{1}{2}a_{2p-1}s^{2p} - c_1 \leq f(s)s \leq \frac{3}{2}a_{2p-1}s^{2p} + c_1,\ c_1 > 0\ ,\ \ \forall\, s \in \mathbb{R}, \qquad (1.12)$$

$$-\kappa \leq \frac{2p-1}{2p}a_{2p-1}s^{2p-2} - c_2 \leq f'(s) \leq 3pa_{2p-1}s^{2p-2} + c_2; \kappa,\ c_2 > 0\ \forall\, s \in \mathbb{R}\ , \qquad (1.13)$$

$$\frac{1}{4p}a_{2p-1}s^{2p} - c_3 \leq F(s) \leq \frac{3}{4p}a_{2p-1}s^{2p} + c_3,\ c_3 > 0, \forall\, s \in \mathbb{R}, \qquad (1.14)$$

where $$F(s) = \int_0^s f(\tau)d\tau\ .$$

2 Notations

We denote by $(.,.)$ the scalar product in $L^2(\Omega)$, $\|.\|$ the associated norm usual and $\|.\|_{-1} = \| (-\Delta)^{\frac{-1}{2}} .\|$, where $-\Delta$ denotes the minus Laplace operator Dirichlet boundary conditions. More generally, $\|.\|_X$ denote the norm of Banach space X.
Throughout this paper, the letter $C_i > 0$, denote (generally positive) constants wich may change from line to line, or ever or same line.

3 A priori estimates

Multiplying (1.7) by $(-\Delta)^{-1} \dfrac{\partial u}{\partial t}$ and (1.8) by $\left(\dfrac{\partial \alpha}{\partial t} - \Delta \dfrac{\partial \alpha}{\partial t}\right)$, integrating over Ω and adding the two resulting differential equalities, we find

$$\begin{aligned}
\frac{d}{dt}E_1 + 2\|\frac{\partial u}{\partial t}\|_{-1}^2 + 2\|\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\|^2 &= -2\left(u, \frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right) \\
&\leq 2\|u\|\|\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\| \\
&\leq \|u\|^2 + \|\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\|^2 \\
\frac{d}{dt}E_1 + 2\|\frac{\partial u}{\partial t}\|_{-1}^2 + \|\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\|^2 &\leq C\|\nabla u\|^2
\end{aligned}$$

where

$$E_1 = \varepsilon\|\frac{\partial u}{\partial t}\|^2 + \|\nabla u\|^2 + 2\left(F(u) + c_3, 1\right) + \|\frac{\partial \alpha}{\partial t}\|^2 + \|\nabla\frac{\partial \alpha}{\partial t}\|^2 + \|\nabla \alpha\|^2 + \|\Delta \alpha\|^2.$$

Thanks to the propertie (1.14), we have

$$(F(u) + c_3, 1) \geqslant 0,$$

which implies

$$\frac{d}{dt}E_1 + 2\|\frac{\partial u}{\partial t}\|_{-1}^2 + \|\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\|^2 \leqslant K_1 E_1.$$

Applying the Gronwall's lemma, we obtain

$$E_1(t) + 2\int_0^t \|\frac{\partial u}{\partial t}\|_{-1}^2 d\tau + \int_0^t \|\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\|^2 d\tau \leqslant E_1(0)e^{K_1 T}$$

for all $t \in [0, T]$.
Using again the propertie (1.14), E_1 satisfies

$$E_1 \geq C\left(\varepsilon\|\frac{\partial u}{\partial t}\|^2 + \|u\|_{H^1}^2 + \|u\|_{L^{2p}}^{2p} + \|\nabla\frac{\partial \alpha}{\partial t}\|^2 + \|\Delta \alpha\|^2\right) + C', \quad C > 0.$$

Finally, we deduce that

$$u \in L^\infty\left(0, T; L^{2p}(\Omega) \cap H_0^1(\Omega)\right),\ \alpha \in L^\infty\left(0, T; H^2(\Omega) \cap H_0^1(\Omega)\right)$$

$$\frac{\partial u}{\partial t} \in L^\infty\left(0, T; L^2(\Omega)\right) \cap L^2\left(0, T; H^{-1}(\Omega)\right)$$

and

$$\tfrac{\partial \alpha}{\partial t} \in L^\infty\left(0, T; H_0^1(\Omega)\right) \cap L^2\left(0, T; H^2(\Omega) \cap H_0^1(\Omega)\right), \ \forall \quad T > 0.$$

4 Existence and the uniqueness of the solution

Theorem 4.1. *(Existence) We assume that* $(u_0, u_1, \alpha_0, \alpha_1) \in \left(L^{2p}(\Omega) \cap H_0^1(\Omega)\right) \times L^2(\Omega) \times \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H_0^1(\Omega)$. *Then the system (1.7)-(1.11) possesses at least one solution* (u, α) *such that* $u \in L^\infty\left(0, T; L^{2p}(\Omega) \cap H_0^1(\Omega)\right)$,
$\alpha \in L^\infty\left(0, T; H^2(\Omega) \cap H_0^1(\Omega)\right)$, $\dfrac{\partial u}{\partial t} \in L^\infty\left(0, T; L^2(\Omega)\right) \cap L^2\left(0, T; H^{-1}(\Omega)\right)$
and
$\frac{\partial \alpha}{\partial t} \in L^\infty\left(0, T; H_0^1(\Omega)\right) \cap L^2\left(0, T; H^2(\Omega) \cap H_0^1(\Omega)\right)$, $\forall \quad T > 0.$

The proof is based on a priori estimates obtained in the previous section and on the standard Galerkin scheme.

Theorem 4.2. *(Uniqueness) Let the assumptions of theorem 4.1 hold. The system (1.7)-(1.11) possesses a unique solution* (u, α) *such that* $u \in L^\infty\left(0, T; L^{2p}(\Omega) \cap H_0^1(\Omega)\right)$,
$\alpha \in L^\infty\left(0, T; H^2(\Omega) \cap H_0^1(\Omega)\right)$, $\dfrac{\partial u}{\partial t} \in L^\infty\left(0, T; L^2(\Omega)\right) \cap L^2\left(0, T; H^{-1}(\Omega)\right)$
and
$\frac{\partial \alpha}{\partial t} \in L^\infty\left(0, T; H_0^1(\Omega)\right) \cap L^2\left(0, T; H^2(\Omega) \cap H_0^1(\Omega)\right) \forall \quad T > 0.$

Proof. Let (v, α^1) and (w, α^2) be two solutions of the system $(1.7) - (1.11)$, with initial data $(v_0, v_1, \alpha_0^1, \alpha_1^1)$ and $(w_0, w_1, \alpha_0^2, \alpha_1^2) \in \left(L^{2p}(\Omega) \cap H_0^1(\Omega)\right) \times L^2(\Omega) \times \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H_0^1(\Omega)$, respectively.
We set $u = v - w$ and $\alpha = \alpha^1 - \alpha^2$, then (u, α) is a solution of the following system

$$\begin{aligned}
\epsilon(-\Delta)\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(v) - \Delta f(w) &= -\Delta\left(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right) && (4.1)\\
\frac{\partial^2 \alpha}{\partial t^2} + \frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t} - \Delta\alpha &= -u - \frac{\partial u}{\partial t} && (4.2)\\
u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = \alpha|_{\partial\Omega} &= 0\\
u|_{t=0} = v_0 - w_0 \ , \frac{\partial u}{\partial t}|_{t=0} &= v_1 - w_1\\
\alpha|_{t=0} = \alpha_0^1 - \alpha_0^2 \ \ , \frac{\partial \alpha}{\partial t}|_{t=0} &= \alpha_1^1 - \alpha_1^2.
\end{aligned}$$

Multiply (4.1) by $(-\Delta)^{-1}\dfrac{\partial u}{\partial t}$ and (4.2) by $\dfrac{\partial \alpha}{\partial t} - \Delta\dfrac{\partial \alpha}{\partial t}$, integrate over Ω and add the two resuling differential equalities. We find

$$\frac{d}{dt}E_2 + 2\|\frac{\partial u}{\partial t}\|_{-1}^2 + 2\int_\Omega (f(v) - f(w))\frac{\partial u}{\partial t}dx + 2\|\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\|^2 = -2\left(u, \frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right)$$

where

$$E_2 = \epsilon\|\frac{\partial u}{\partial t}\|^2 + \|\nabla u\|^2 + \|\frac{\partial \alpha}{\partial t}\|^2 + \|\nabla\frac{\partial \alpha}{\partial t}\|^2 + \|\nabla\alpha\|^2 + \|\Delta\alpha\|^2.$$

Applying Hölder and Young inequalities, we get

$$\frac{d}{dt}E_2 + 2\|\frac{\partial u}{\partial t}\|_{-1}^2 + \|\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\|^2 \leqslant 2\int_\Omega |f(v) - f(w)||\frac{\partial u}{\partial t}|dx + \|u\|^2. \tag{4.3}$$

We know that

$$\begin{aligned} f(v)-f(w) &= \sum_{k=1}^{2p-1} a_k(v^k-w^k) \\ &= (v-w)\Big(a_1+a_2(v+w)+\sum_{k=3}^{2p-1} a_k \sum_{i=0}^{k-1} v^{k-i-1}w^i\Big) \end{aligned}$$

which implies

$$|f(v)-f(w)| \leq |u|\Big(|a_1|+|a_2|(|v|+|w|)+\sum_{k=3}^{2p-1}|a_k|\sum_{i=0}^{k-1}|v|^{k-i-1}|w|^i\Big).$$

Applying Young's inequality, we obtain

$$\begin{aligned} |v|^{k-1-i}|w|^i &\leq \frac{k-1-i}{k-1}(|v|^{k-1-i})^{\frac{k-1}{k-1-i}}+\frac{i}{k-1}(|w|^i)^{\frac{k-1}{i}} \\ &\leq \frac{k-1-i}{k-1}|v|^{k-1}+\frac{i}{k-1}|w|^{k-1} \end{aligned}$$

which implies

$$\begin{aligned} \sum_{i=0}^{k-1}(|v|^{k-i-1}|w|^i) &\leqslant \sum_{i=0}^{k-1}\frac{k-i-1}{k-1}|v|^{k-1}+\sum_{i=0}^{k-1}\frac{i}{k-1}|w|^{k-1} \\ &\leqslant \frac{k}{2}\left(|v|^{k-1}+|w|^{k-1}\right). \end{aligned}$$

Then we obtain

$$\begin{aligned} |f(v)-f(w)| &\leq |u|\Big(|a_1|+|a_2|(\frac{1}{2p-2}|v|^{2p-2}+\frac{1}{2p-2}|w|^{2p-2}+C) \\ &+ \sum_{k=3}^{2p-1}\frac{k}{2}|a_k|(|v|^{k-1}+|w|^{k-1})\Big) \\ &\leq |u|\Big(|a_1|+|a_2|(\frac{1}{2p-2}|v|^{2p-2}+\frac{1}{2p-2}|w|^{2p-2}+C) \\ &+ \sum_{k=3}^{2p-1}|a_k|(\frac{(k-1)k}{4(p-1)}|v|^{2p-2}+\frac{(k-1)k}{4(p-1)}|w|^{2p-2}+C)\Big) \\ &\leq C|u|\Big(\frac{1}{2p-2}|v|^{2p-2}+\frac{1}{2p-2}|w|^{2p-2}+\frac{|v|^{2p-2}+|w|^{2p-2}}{4(p-1)}\sum_{k=3}^{2p-1}k(k-1)+1\Big) \end{aligned}$$

which implies

$$|f(v)-f(w)| \leqslant C|u|(|v|^{2p-2}+|w|^{2p-2}+1).$$

Hence

$$\int_\Omega |f(v)-f(w)||\frac{\partial u}{\partial t}|dx \leq C\int_\Omega |u|(|v|^{2p-2}+|w|^{2p-2}+1)|\frac{\partial u}{\partial t}|dx.$$

In order to obtain the estimate of $\int_\Omega |f(v)-f(w)||\frac{\partial u}{\partial t}|dx$, we consider the two following cases.
If $n=1$.

We know that $H^1(\Omega) \subset L^\infty(\Omega)$. Since $v, w \in L^\infty\left(0, T; H_0^1(\Omega)\right)$, then $v, w \in L^\infty\left((0,T) \times \Omega\right)$, there exists $C_1, C_2 > 0$ such that $\sup_{(t,x)\in(0,T)\times\Omega} |v(t,x)| \leqslant C_1$ and $\sup_{(t,x)\in(0,T)\times\Omega} |w(t,x)| \leqslant C_2$, then

$$\begin{aligned}
\int_\Omega |f(v) - f(w)||\frac{\partial u}{\partial t}|dx &\leq C_3(\|v\|_{L^\infty}^{2p-2} + \|w\|_{L^\infty}^{2p-2} + 1)\int_\Omega |u||\frac{\partial u}{\partial t}|dx \\
&\leqslant C\|u\|\|\frac{\partial u}{\partial t}\| \\
&\leq C\|u\|_{H^1}\|\frac{\partial u}{\partial t}\|.
\end{aligned}$$

If $n = 2$ or 3.
We have

$$\int_\Omega |u||v|^{2p-2}|\frac{\partial u}{\partial t}|dx \leqslant \|u\|_{L^6}\||v|^{2p-2}\|_{L^3}\|\frac{\partial u}{\partial t}\|$$

and

$$\||v|^{2p-2}\|_{L^3} = \|v\|_{L^{3(2p-2)}}^{2p-2}.$$

Since $3(2p-2) < 6p$, $\|v\|_{L^{6p}} = \||v|^p\|_{L^6}^{\frac{1}{p}}$ and $H^1(\Omega) \subset L^6(\Omega)$ (the continuous embending), we have $\|v\|_{L^{6p}} \leqslant C\||v|^p\|^{\frac{1}{p}} \leqslant C'\|v\|_{H^1}$.
Since $v \in L^\infty(0, T; H_0^1(\Omega))$, we have

$$\begin{aligned}
\int_\Omega |u||v|^{2p-2}|\frac{\partial u}{\partial t}|dx &\leqslant C\|v\|_{H^1}^{2p-2}\|u\|_{L^6}\|\frac{\partial u}{\partial t}\| \\
&\leqslant C\|u\|_{L^6}\|\frac{\partial u}{\partial t}\| \\
&\leq C\|u\|_{H^1}\|\frac{\partial u}{\partial t}\|.
\end{aligned}$$

We have the same estimate for w.
Finally, we have for $n = 1,\ 2$ or 3

$$\begin{aligned}
\int_\Omega |f(v) - f(w)||\frac{\partial u}{\partial t}|dx &\leq C\|u\|_{H^1}\|\frac{\partial u}{\partial t}\| \\
&\leq C\left(\|u\|_{H^1}^2 + \|\frac{\partial u}{\partial t}\|^2\right).
\end{aligned}$$

Iserting the above estimate into (4.3) we have

$$\frac{d}{dt}E_2 + 2\|\frac{\partial u}{\partial t}\|_{-1}^2 + \|\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\|^2 \leqslant KE_2, K > 0.$$

Applying Gronwall's lemma, we obtain for all $t \in [0, T]$

$$E_2(t) \leqslant E_2(0)e^{KT}.$$

Then we deduce the continuous dependence of solution wich respect to the initial conditions, and the uniqueness of the solution is proven. □

The existence and the uniqueness of the solution of problem (1.7) – (1.11) being proven in a larger space, we now establish the solution with more regularity.

Theorem 4.3. *Assume* $(u_0, u_1, \alpha_0, \alpha_1) \in \left(L^{2p}(\Omega) \cap H_0^1(\Omega) \cap H^2(\Omega)\right) \times H_0^1(\Omega) \times \left(H^3(\Omega) \cap H_0^1(\Omega)\right) \times \left(H^2(\Omega) \cap H_0^1(\Omega)\right)$.
Then the system $(1.7) - (1.11)$ *possesses a unique solution* (u, α) *such that*

$u \in L^\infty\left(0, T; (L^{2p}(\Omega) \cap H_0^1(\Omega) \cap H^2(\Omega)\right),\ \alpha \in L^\infty\left(0, T; H^3(\Omega) \cap H_0^1(\Omega)\right),$

$\dfrac{\partial u}{\partial t} \in L^\infty\left(0, T; H_0^1(\Omega)\right) \cap L^2\left(0, T; L^2(\Omega)\right),$

$\frac{\partial \alpha}{\partial t} \in L^\infty\left(0, T; H^2(\Omega) \cap H_0^1(\Omega)\right) \cap L^2\left(0, T; H^3(\Omega) \cap H_0^1(\Omega)\right),\ \dfrac{\partial^2 \alpha}{\partial t^2} \in L^2\left(0, T; L^2(\Omega)\right)$ *and* $\frac{\partial^2 u}{\partial t^2} \in L^2(0, T; L^2(\Omega))\ \forall \quad T > 0.$

Proof. According to the theorems 4.1 and 4.2, the hyperbolic system $(1.7)-(1.11)$ possesses the unique solution (u, α) such that
$u \in L^\infty\left(0, T; L^{2p}(\Omega) \cap H_0^1(\Omega) \cap H^2(\Omega)\right), \alpha \in L^\infty\left(0, T; H^2(\Omega) \cap H_0^1(\Omega)\right),$
$\dfrac{\partial u}{\partial t} \in L^\infty\left(0, T; L^2(\Omega)\right) \cap L^2\left(0, T; H^{-1}(\Omega)\right)$

and $\frac{\partial \alpha}{\partial t} \in L^\infty\left(0, T; H_0^1(\Omega)\right) \cap L^2\left(0, T; H^2(\Omega) \cap H_0^1(\Omega)\right),\ \forall \quad T > 0.$

Multiplying (1.7) by $\frac{\partial u}{\partial t}$ and integrating over Ω, we have

$$\frac{d}{dt}\left(\varepsilon \| \nabla \frac{\partial u}{\partial t}\|^2 + \|\Delta u\|^2\right) + 2\|\frac{\partial u}{\partial t}\|^2 = -2\left(\nabla f(u), \nabla \frac{\partial u}{\partial t},\right) + 2\left(\nabla \frac{\partial u}{\partial t}, \nabla(\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t})\right). \tag{4.4}$$

Multiplying (1.8) by $-\Delta\left(\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}\right)$ and integrating over Ω, we get

$$\begin{aligned} & \frac{d}{dt}\left(\|\nabla \frac{\partial \alpha}{\partial t}\|^2 + \|\Delta \frac{\partial \alpha}{\partial t}\|^2 + \|\Delta \alpha\|^2 + \|\nabla \Delta \alpha\|^2\right) + 2\|\nabla\left(\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}\right)\|^2 \\ = & \ -2\left(\nabla u, \nabla(\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t})\right) - 2\left(\nabla \frac{\partial u}{\partial t}, \nabla(\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t})\right). \end{aligned} \tag{4.5}$$

Now summing (4.4) and (4.5), and applying Hölder and Young inequalities, we get

$$\frac{d}{dt}E_3 + 2\|\frac{\partial u}{\partial t}\|^2 + \|\nabla(\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t})\|^2 \leqslant \|\nabla u\|^2 + 2\int_\Omega |\nabla f(u)||\nabla \frac{\partial u}{\partial t}|dx \tag{4.6}$$

where

$$E_3 = \varepsilon \| \nabla \frac{\partial u}{\partial t}\|^2 + \|\Delta u\|^2 + \|\nabla \frac{\partial \alpha}{\partial t}\|^2 + \|\Delta \frac{\partial \alpha}{\partial t}\|^2 + \|\Delta \alpha\|^2 + \|\nabla \Delta \alpha\|^2.$$

We know that

$$\int_\Omega |\nabla f(u)||\nabla \frac{\partial u}{\partial t}|dx = \int_\Omega |f'(u)\nabla u||\nabla \frac{\partial u}{\partial t}|dx$$

Using (1.13), we have

$$\begin{aligned} \int_\Omega |f'(u)\nabla u||\nabla \frac{\partial u}{\partial t}|dx & \leqslant \int_\Omega \left(3pa_{2p-1}|u|^{2p-2} + c_2\right)|\nabla u||\nabla \frac{\partial u}{\partial t}|dx \\ & \leqslant C_4 \int_\Omega |u|^{2p-2}|\nabla u||\nabla \frac{\partial u}{\partial t}|dx + c_2 \int_\Omega |\nabla u||\nabla \frac{\partial u}{\partial t}|dx \\ & \leqslant C_4 \int_\Omega |u|^{2p-2}|\nabla u||\nabla \frac{\partial u}{\partial t}|dx + c_2\|\nabla u\|^2 + c_2\|\nabla \frac{\partial u}{\partial t}\|^2 \\ & \leqslant C_4 \int_\Omega |u|^{2p-2}|\nabla u||\nabla \frac{\partial u}{\partial t}|dx + c_2\|\Delta u\|^2 + c_2\|\nabla \frac{\partial u}{\partial t}\|^2. \end{aligned}$$

Now, we need the estimate of

$$\int_\Omega |u|^{2p-2}|\nabla u||\nabla \frac{\partial u}{\partial t}|dx$$

Similary to the proof theorem 4.2 we have
if $n=1$,

$$\begin{aligned}\int_\Omega |u|^{2p-2}|\nabla u||\nabla \frac{\partial u}{\partial t}|dx &\leq \|u\|^{2p-2}_{L^\infty}\|\nabla u\|\|\nabla \frac{\partial u}{\partial t}\| \\ &\leq C\|\nabla u\|\|\nabla \frac{\partial u}{\partial t}\| \\ &\leq C\Big(\|\Delta u\|^2 + \|\nabla \frac{\partial u}{\partial t}\|^2\Big)\end{aligned}$$

If $n=2$ **or** 3,

$$\begin{aligned}\int_\Omega |u|^{2p-2}|\nabla u||\nabla \frac{\partial u}{\partial t}|dx &\leqslant \|\nabla u\|_{L^6}\||u|^{2p-2}\|_{L^3}\|\nabla \frac{\partial u}{\partial t}\| \\ &\leqslant C\|\nabla u\|_{L^6}\|\nabla \frac{\partial u}{\partial t}\| \\ &\leqslant C\|\Delta u\|\|\nabla \frac{\partial u}{\partial t}\| \\ &\leqslant C\Big(\|\Delta u\|^2 + \|\nabla \frac{\partial u}{\partial t}\|^2\Big).\end{aligned}$$

Then for $n=1,2,3$,(4.6) can be written as

$$\frac{d}{dt}E_3 + 2\|\frac{\partial u}{\partial t}\|^2 + \|\nabla\left(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right)\|^2 \leqslant K_1E_3, \quad K_1>0.$$

Appling the Gronwall's lemma, we deduce that

$u \in L^\infty\left(0,T; L^{2p}(\Omega)\cap H^2(\Omega)\cap H^1_0(\Omega)\right), \ \alpha \in L^\infty\left(0,T;H^3(\Omega)\cap H^1_0(\Omega)\right),$

$\frac{\partial u}{\partial t} \in L^\infty\left(0,T;H^1_0(\Omega)\right)\cap L^2\left(0,T;L^2(\Omega)\right)$

and
$\frac{\partial \alpha}{\partial t} \in L^\infty\left(0,T;H^2(\Omega)\cap H^1_0(\Omega)\right)\cap L^2\left(0,T;H^3(\Omega)\cap H^1_0(\Omega)\right).$

Multipplying (1.8) by $\frac{\partial^2\alpha}{\partial t^2}$ and integrating over Ω, we get

$$\frac{d}{dt}\left(\|\frac{\partial \alpha}{\partial t}\|^2 + \|\nabla\frac{\partial \alpha}{\partial t}\|^2\right) + 2\|\frac{\partial^2 \alpha}{\partial t^2}\|^2 = 2\left(\Delta\alpha, \frac{\partial^2 \alpha}{\partial t^2}\right) + 2(-u, \frac{\partial^2 \alpha}{\partial t^2}) + 2(-\frac{\partial u}{\partial t}, \frac{\partial^2 \alpha}{\partial t^2})$$

Appling Hölder and Young inequalitiess, we find the following estimate

$$\frac{d}{dt}\left(\|\frac{\partial \alpha}{\partial t}\|^2 + \|\nabla\frac{\partial \alpha}{\partial t}\|^2\right) + \|\frac{\partial^2 \alpha}{\partial t^2}\|^2 \leqslant C\Big(\|\alpha\|^2_{H^2} + \|u\|^2_{H^1} + \|\frac{\partial u}{\partial t}\|^2\Big)$$

which implies

$$\frac{\partial^2 \alpha}{\partial t^2} \in L^2\left(0,T;L^2(\Omega)\right).$$

Multiplying (1.7) by $(-\Delta)^{-1}\frac{\partial^2 u}{\partial t^2}$ and integrating over Ω, we obtain

$$\begin{aligned}
\frac{d}{dt}\|\frac{\partial u}{\partial t}\|_{-1}^2 + 2\epsilon\|\frac{\partial^2 u}{\partial t^2}\|^2 &= 2\left(\Delta u, \frac{\partial^2 u}{\partial t^2}\right) + 2\left(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}, \frac{\partial^2 u}{\partial t^2}\right) - 2\left(f(u), \frac{\partial^2 u}{\partial t^2}\right) \\
\frac{d}{dt}\|\frac{\partial u}{\partial t}\|_{-1}^2 + \epsilon\|\frac{\partial^2 u}{\partial t^2}\|^2 &\leqslant \left(\|\Delta u\|^2 + \|f(u)\|^2 + \|\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\|^2\right),
\end{aligned}$$

which yields, using the fact that $u \in L^\infty\left(0,T;H^2(\Omega)\right)$ and $H^2(\Omega) \subset L^\infty(\Omega)$,

$$\frac{\partial^2 u}{\partial t^2} \in L^2\left(0,T;L^2(\Omega)\right).$$

Then the proof of theorem 4.3 is complete.

References

[1] Brochet, D., Hilhorst, D. and Novick-Cohen, A.(1996). *Maximal Attractor and Inertial Sets. for a conserved Phase-Field Model.* Advance in Differential Equations, 1, 547-578.

[2] Brochet, D.(1993). *Maximal Attractor and Inertial Sets for Somme Second and Fourth Order Phase-Field Models.* In:Pitman Res. Notes Math. Ser, Vol.296, Longman Sci.Tech,Harloww, 77-85.

[3] Boukhatem, Benattou, B., Bita, R.(2009). *Méthode de faedo-Galerkin pour un probleme aux limites non linéaire.* Anale Universitatii Oradea Fasc. Matematica, Tom, XVI (2009), 167-181.

[4] Caginalp, G.(1990).*The Dynamic of Conserved phase System:Stefan-Like, Heleshaw and Cahn-Hilliard Models as Asymptotic Limits.* IMA Journal of Applied Mathematics, 44, 77-94.

[5] Caginalp, G (1988). *Conserved-phase Field System:Implications for Kinetic Under cooling.* Physical Review B, 789-791.htt://doxdoi.org/10.1103/physRev B.38.789.

[6] Y.Choquet-Bruhat, C. Dewitt-Morete ,M. Dillard-Bleick.*Analysis, Manifolds and physics.*North-Holland Publishing Company, Amsterdam, New york, Oxford (1977).

[7] Colli, P., Gilardi, Grasseli, M. and Schimperna, G (2001). *The Conserved Phase-Field System Wih Momory.* Adv.Math.Sci App, 11, 265-291.

[8] Colli, P., Gilardi, G., Laurenot, ph. and Novick-Cohen, A.(1999). *Uniqueness and Long-Time Behavior for the Conserved phase-System Memory.* Discrete and Continuous Dynamical Systems-Series A, 5, 375-390.http://dx.doi.org/10.3934/dcds.1999.5.375

[9] Gatti, S. and Pata, V.(2004). *Exponential Atractor for a Conserved Phase-Field System with Memory.* Physica D.Nonlinear Phenomena, 189, 31-48.
http:// dx.doi.org/10.16/j.physd.2003.10.005

[10] Gilardi, G.(2007). *On a Conserved Phase-Field Model with Irregular Potenials and Dynamic Boundary Condition*. Istit.Lombardo.Accad.Sci.Lett.Rend.A, 141,129-161.

[11] Goyaud, M.E.I., Moukamba, F.,Moukoko, D. and Langa, F. D. R. (2015). *Existence and Uniqueness of Solution for Caginalp Hyperbolic phase-Field System with polynomial Growth potential*. International Mathematical Forum, 10, 477-486.

[12] Lions J.L. *Quelques méthodes de résolution des problÃÍmes aux limites non linéaires*. Paris, Dunod, (1969).

[13] Mangoubi, J. Moukoko, D., Moukamba, F. and Langa, F.D.R.(2016). *Existence and Uniqueness of solution for Cahn-Hilliard Hyperbolic phase-Field System with Dirichlet Boundary Condition and regular potentials*. Applied Mathematics, 2016,7. http:\\ www.Scirp.org\journal \am ISSN oline:2152-7393
ISSN print:2152-7385.

[14] Mavoungou, Moukamba, F, Moukoko D. and F. D. R *Existence and uniqueness of solution for Caginalp Hyperbolic phase Field System with a singular potential*. Nonlinear Analysis and Eq, Vol.4, 2016, n0, 4,151-160 HIKARI Ltd,www.m-hikari.com. http:\\dx.dviorg \10.12988 \.2016.51255.

[15] Miranville, A.(2013). *On the Conserved Phase-Field Model*. Journal of Mathemaical Analysis and Applications, 400, 143-152. http://dx.doi.org/10.1016/j.maa.2012.11.038

[16] Miranville, A, and Quintanilla, R. (2011). *A Type III phase -Field System with a Logarithmic potential*. Applied Mathematics Letters, 24, 1003-1008.
http://dx.doi.org/10.1016/j.aml.2011.01.016

[17] Moukoko, D.(2014). *Well-posedness and Longtine Behaviors of a Hyperbolic Caginalp System*. Journal of Applied Analysis and Computation, 4, 151-196.

[18] Moukoko,D., Moukamba, D, MOukamba F and F.D.R (2015). *Global Attractor for Caginalp Hyperbolic Field-phase System with Singular potential*. Journal of Mathematics Research, 7, 165-177.

[19] Ntsokongo, A.J. and Batangouna, N. (2016). *Existence and Uniqueness of solutions for a Conserved phase -Field Type model*. AIMS Mathematics, 1, 144-145.
http://dx.doi.org/10.3934/Math.2016.2.144.