

Analytical solutions for conformable fractional Bratu-type equations

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Abstract

Solving fractional differential equations have a prominent function in different science such as physics and engineering. Therefore, are different definitions of the fractional derivative presented in recent years. The aim of the current paper is to solve the fractional differential equation by a semi-analytical method based on conformable fractional derivative. Fractional Bratu-type equations have been solved by the method and to show its capabilities. The obtained results have been compared with the exact solution.

Keywords: Adomian Decomposition Method; Fractional Differential Equations; Conformable Fractional Derivative; Fractional Bratu-Type Equations.

1. Introduction

Solving fractional differential equations is very important but there are many fractional differential equations, which can't be solved analytically. Due to this fact, finding an approximate solution of fractional differential equations is clearly significant. In recent years, many effective methods have been proposed for finding approximate solution to fractional differential equations [3-13], such as Adomian decomposition method [14,15], homotopy perturbation method [16-19], homotopy analysis method [20], variational iteration method [21], generalized, differential transform method [22], finite difference method [23], semi-discrete scheme and Chebyshev collocation method [24], Wavelet Operational [25], Perturbation-iteration algorithm [26], and other methods [27-29]. Definitions of the fractional derivative presented in different forms such as Caputo, Liouville, and so on. Most recently a novel definition called comfortable fractional has been suggested to obtain fractional Derivative [1, 2].]. The purpose of the paper is solving fractional equation by Adomian Decomposition method based on above-mentioned definition.

The organization of the paper is as follows: In Section 2, the basic definitions' amenable fractional derivative and conformable fractional integral are described. In section 3, extended Adomian decomposition method [30-33], based on conformable fractional derivative is described. In Section 4, the method is used to solve fractional Bratu-type equations with different initial value. Finally, discussions are given, in section 4.

2. Basic definitions

In this section some basic definitions and properties of the comfortable fractional derivative and integral have been explained.

2.1. Conformable fractional derivative

Given a function $f: [0, \infty) \rightarrow \mathbb{R}$. Then conformable fractional derivative of f of order α is defined as follows

$$T_{\alpha}(f)(x) = \lim_{\varepsilon \rightarrow 0} \frac{f(x+\varepsilon x^{1-\alpha})-f(x)}{\varepsilon}, \quad (1)$$

For all $x > 0, \alpha \in (0,1)$ [1]. If f is α -differentiable in some $(0, a), a > 0$, and provided that $\lim_{x \rightarrow 0^+} T_{\alpha}(f)(x)$ exists, then define $T_{\alpha}(f)(0) = \lim_{x \rightarrow 0^+} T_{\alpha}(f)(x)$.

If the conformable derivative of f of order α exists, then we simply say that f is α -differentiable [1, 2].

One can easily show that T_{α} satisfies all the properties in the following properties [1]:

Let $\alpha \in (0, 1)$ and f , and g be α -differentiable at a point $x > 0$, Then.

A. For $a, b \in \mathbb{R} \quad T_{\alpha}(af + bg) = a T_{\alpha}(f) + b T_{\alpha}(g),$

B. For all $p \in \mathbb{R} \quad T_{\alpha}(x^p) = px^{p-\alpha},$

C. For all constant functions $f(x) = \lambda, \quad T_{\alpha}(\lambda) = 0,$

D. $T_{\alpha}(f \cdot g) = g \cdot T_{\alpha}(f) + f \cdot T_{\alpha}(g),$

E. $T_{\alpha}\left(\frac{f}{g}\right) = \frac{g \cdot T_{\alpha}(f) - f \cdot T_{\alpha}(g)}{g^2},$

F. $T_{\alpha}(f) = x^{1-\alpha} \frac{df}{dx}.$

2.2. Conformable fractional integral

Given a function $f: [a, \infty) \rightarrow \mathbb{R}$, $a \geq 0$. Then conformable fractional integral of f is defined as the following

$$I_\alpha^a(f)(x) = \int_a^x \frac{f(t)}{t^{1-\alpha}} dt, \tag{2}$$

Where the integral is the usual Riemann improper integral, and $\alpha \in (0,1)[1, 2]$.

For simplicity, we have, $I_\alpha^0(f)(x) = I_\alpha(f)(x)$.

One of the well results is the following [1]:

For all $x \geq a$, and any continuous function in the domain of I_α^a , we have $T_\alpha(I_\alpha^a f(x)) = f(x)$.

3. Modify adomian decomposition method based on conformable fractional derivative

Consider a fractional nonlinear differential equation in the form

$$T_\alpha^n(u)(x) - \mathcal{N}(u(x)) = f(x), \tag{3}$$

$$T_\alpha^k(u)(0) = c_k, \quad k = 0,1,2, \dots, n - 1,$$

Where $T_\alpha^n = T_\alpha T_\alpha T_\alpha \dots T_\alpha$, n time, \mathcal{N} is a nonlinear operator, and $f(x)$ is a known analytic function.

It is assumed that the unknown function $u(x)$ can be expressed by an infinite series in the following form

$$u(x) = \sum_{m=0}^\infty u_m(x), \tag{4}$$

And the nonlinear term $\mathcal{N}(u)$ can be presented as an infinite series of polynomials, say

$$\mathcal{N}(u) = \sum_{m=0}^\infty \mathcal{A}_m(u_0, u_1, \dots, u_m), \tag{5}$$

Where $\mathcal{A}_n(u_0, u_1, \dots, u_n)$ are called the Adomian polynomials and are defined by

$$A_m = \frac{1}{m!} [T_\alpha^m [N(\sum_{i=0}^\infty u_i (\frac{1}{\alpha} \lambda^\alpha)^i)]]_{\lambda=0}, \quad m = 0,1,2, \dots \tag{6}$$

By applying the inverse operator,

$$(T_\alpha^n)^{-1} = I_\alpha^n = I_\alpha I_\alpha \dots I_\alpha = \int_0^x \int_0^{t_1} \dots \int_0^{t_{n-1}} (\cdot) dt_n dt_{n-1} \dots dt_1$$

On both sides of Eq. (3), the following equation has been derived,

$$u(x) = \sum_{k=0}^{n-1} \frac{(T_\alpha^k u)(0)}{k!} (\frac{1}{\alpha} x^\alpha)^k + I_\alpha^n(f) + I_\alpha^n(\mathcal{N}(u(x))). \tag{7}$$

Considering (4) and (5), we obtain

$$\sum_{m=0}^\infty u_m = \sum_{k=0}^{n-1} \frac{(T_\alpha^k u)(0)}{k!} (\frac{1}{\alpha} x^\alpha)^k + I_\alpha^n(f) + \sum_{m=0}^\infty I_\alpha^n(\mathcal{A}_m(u_0, u_1, \dots, u_m)). \tag{8}$$

Now, let us to define the following iterative equations, as Adomian scheme

$$u_0 = \sum_{k=0}^{n-1} \frac{(T_\alpha^k u)(0)}{k!} (\frac{1}{\alpha} x^\alpha)^k + I_\alpha^n(f),$$

$$u_{m+1} = I_\alpha^n(\mathcal{A}_m(u_0, u_1, \dots, u_m)), \quad m = 1,2, \dots \tag{9}$$

If the equation (3) has an analytic solution, the series (4) would be Taylor expansion of fractional order, and one may recognize the exact solution, otherwise the partial sum

$$u_0(x) + \sum_{m=1}^{M-1} u_m(x), \tag{10}$$

Demonstrates the M th-order approximation of the problem.

4. Examples

We solve fractional Bratu-type equation with different initial condition by the presented method in this section.

Example 1 Consider fractional Bratu-type equation with the following initial condition [15]

$$T_\alpha^2(u)(x) - 2e^{u(x)} = 0, \quad 0 < \alpha \leq 1, 0 \leq x < 1, \quad u(0) = T_\alpha(u)(0) = 0. \tag{11}$$

The exact solution of Eq. (11) is $u(x) = -2\ln(\cos(\frac{1}{\alpha} x^\alpha))$.

To solve this equation by ADM according to (7), we have

$$u(x) = u(0) + (T_\alpha u)(0) (\frac{1}{\alpha} x^\alpha) + I_\alpha^2(2e^{u(x)}). \tag{12}$$

By substituting (4) and (5) into (12), we derive:

$$\sum_{m=0}^\infty u_m = 0 + \sum_{m=0}^\infty I_\alpha^2(\mathcal{A}_m(u_0, u_1, \dots, u_m)).$$

So, the solution by ADM consists of following scheme

$$u_0(x) = 0,$$

$$u_{m+1}(x) = I_\alpha^2(\mathcal{A}_m(u_0, u_1, \dots, u_m)),$$

Where

$$\mathcal{A}_m(u_0, u_1, \dots, u_m) = \frac{1}{m!} \left(T_\alpha^m \left(2e^{\sum_{i=0}^\infty u_i (\frac{1}{\alpha} \lambda^\alpha)^i} \right) \right)_{\lambda=0}.$$

So

$$\mathcal{A}_0(u_0) = 2e^{u_0},$$

$$\mathcal{A}_1(u_0, u_1) = 2u_1 e^{u_0},$$

$$\mathcal{A}_2(u_0, u_1, u_2) = \frac{1}{2!} (2(2u_2 e^{u_0} + u_1^2 e^{u_0})),$$

⋮

First few terms will be obtained as follows

$$u_1 = (\frac{1}{\alpha} x^\alpha)^2,$$

$$u_2 = \frac{1}{6} (\frac{1}{\alpha} x^\alpha)^4,$$

$$u_3 = \frac{2}{45} (\frac{1}{\alpha} x^\alpha)^6,$$

⋮

Therefore, the following solution will be obtained

$$u(x) = (\frac{1}{\alpha} x^\alpha)^2 + \frac{1}{6} (\frac{1}{\alpha} x^\alpha)^4 + \frac{2}{45} (\frac{1}{\alpha} x^\alpha)^6 + \dots = -2 \ln \left(\cos \left(\frac{1}{\alpha} x^\alpha \right) \right)$$

Which is the exact solution.

Example 2 Consider fractional Bratu-type equation with as follows initial condition [15]

$$T_\alpha^2(u)(x) - \pi^2 e^{u(x)} = 0, \quad 0 < \alpha \leq 1,$$

$$0 \leq x < 1, \quad u(0) = 0, \quad T_\alpha(u)(0) = \pi. \tag{13}$$

The exact solution of Eq. (13) is $u(x) = -\ln(1 - \sin(\frac{\pi}{\alpha} x^\alpha))$.

By using $I_\alpha^2 = \int_0^x \int_0^{t_1} (\cdot) dt_2 dt_1$, we obtain:

$$u(x) = u(0) + (T_\alpha u)(0) \left(\frac{1}{\alpha} x^\alpha\right) + I_\alpha^2(\pi^2 e^{u(x)}).$$

By substituting $u(x) = \sum_{m=0}^\infty u_m$, into this equation, we get

$$\sum_{m=0}^\infty u_m = \pi \frac{1}{\alpha} x^\alpha + \sum_{m=0}^\infty I_\alpha^2(\mathcal{A}_m(u_0, u_1, \dots, u_m)),$$

So, the Adomian scheme would be as follows

$$u_0(x) = \pi \frac{1}{\alpha} x^\alpha, u_{m+1}(x) = I_\alpha^2(\mathcal{A}_m(u_0, u_1, \dots, u_m)), \quad m = 0, 1, 2, \dots$$

Where

$$\mathcal{A}_0(u_0) = \pi^2 e^{u_0},$$

$$\mathcal{A}_1(u_0, u_1) = \pi^2 u_1 e^{u_0},$$

$$\mathcal{A}_2(u_0, u_1, u_2) = \frac{1}{2!}(\pi^2(2u_2 e^{u_0} + u_1^2 e^{u_0})),$$

⋮

Therefore, the following results will be derived

$$u_1 = e^{\pi(\frac{1}{\alpha} x^\alpha)} - \pi \left(\frac{1}{\alpha} x^\alpha\right) - 1,$$

$$u_2 = -\frac{5}{4} - \frac{\pi}{2} \left(\frac{1}{\alpha} x^\alpha\right) + \frac{1}{4} e^{2\pi(\frac{1}{\alpha} x^\alpha)} - \pi \left(\frac{1}{\alpha} x^\alpha\right) e^{\pi(\frac{1}{\alpha} x^\alpha)} + e^{\pi(\frac{1}{\alpha} x^\alpha)},$$

$$u_3 = -\frac{11}{6} + \frac{\pi^2}{2} \left(\frac{1}{\alpha} x^\alpha\right)^2 e^{\pi(\frac{1}{\alpha} x^\alpha)} - \frac{\pi}{2} \left(\frac{1}{\alpha} x^\alpha\right) e^{2\pi(\frac{1}{\alpha} x^\alpha)} - \frac{3\pi}{2} e^{\pi(\frac{1}{\alpha} x^\alpha)}$$

$$- \frac{\pi}{2} \left(\frac{1}{\alpha} x^\alpha\right) + \frac{1}{2} e^{2\pi(\frac{1}{\alpha} x^\alpha)} + \frac{5}{4} e^{\pi(\frac{1}{\alpha} x^\alpha)} + \frac{1}{12} e^{3\pi(\frac{1}{\alpha} x^\alpha)},$$

⋮

Four-terms approximation to the solution will be obtained as the following form

$$u(x) \approx -\frac{49}{12} - \pi \left(\frac{1}{\alpha} x^\alpha\right) + \frac{3}{4} e^{2\pi(\frac{1}{\alpha} x^\alpha)} - \frac{5\pi}{2} \left(\frac{1}{\alpha} x^\alpha\right) e^{\pi(\frac{1}{\alpha} x^\alpha)} + \frac{13}{4} e^{\pi(\frac{1}{\alpha} x^\alpha)}$$

$$+ \frac{\pi^2}{2} \left(\frac{1}{\alpha} x^\alpha\right)^2 e^{\pi(\frac{1}{\alpha} x^\alpha)} - \frac{\pi}{2} \left(\frac{1}{\alpha} x^\alpha\right) e^{2\pi(\frac{1}{\alpha} x^\alpha)} + \frac{1}{12} e^{3\pi(\frac{1}{\alpha} x^\alpha)}$$

In Figure 1, the exact and approximate solutions of fractional Bratu-type equation for $\alpha = 0.5$, up to 1.0, is plotted, It can be concluded from the result that whatever α is closer to 1, the convergence as well as the accuracy of approximate solution will be better.

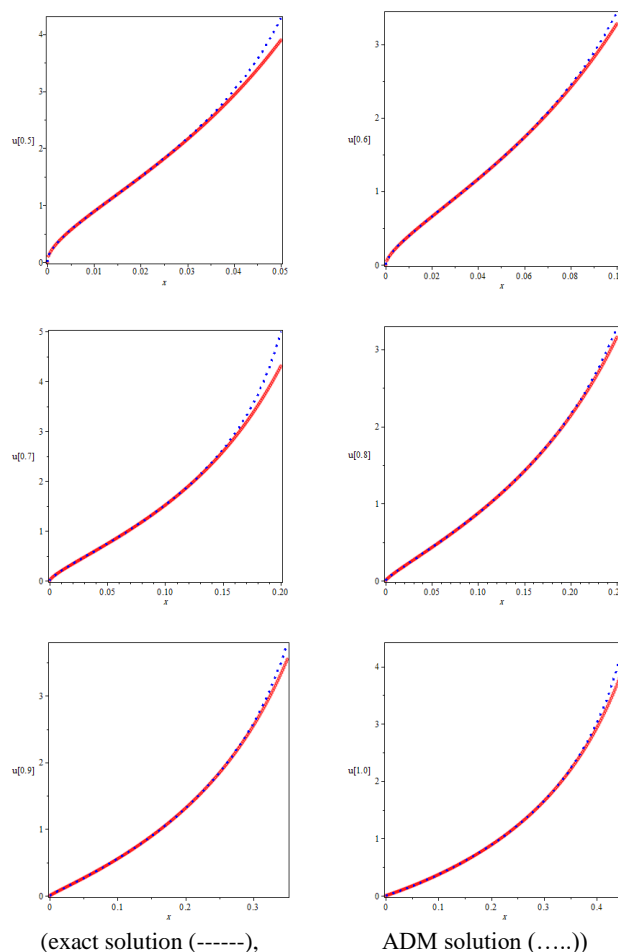


Fig. 1: The Comparison 4th-Order Approximation of ADM and Exact Solution for Example 2.

Example 3 Consider fractional Bratu-type equation [15]

$$T_\alpha^2(u)(x) + \pi^2 e^{-u(x)} = 0, \quad 0 < \alpha \leq 1, \quad 0 \leq x < 1, \quad (14)$$

With initial condition $u(0) = 0, T_\alpha(u)(0) = \pi$.

The exact solution of Eq. (14) is $u(x) = \ln(1 + \sin(\frac{\pi}{\alpha} x^\alpha))$.

By using Eq. (7), Eq. (13) can be written as follows:

$$u(x) = u(0) + (T_\alpha u)(0) \left(\frac{1}{\alpha} x^\alpha\right) - I_\alpha^2(\pi^2 e^{-u(x)}).$$

By substituting $u(x) = \sum_{m=0}^\infty u_m$, we get

$$u_0(x) = \pi \frac{1}{\alpha} x^\alpha,$$

$$u_{m+1}(x) = -I_\alpha^2(\mathcal{A}_m(u_0, u_1, \dots, u_m)), \quad m = 0, 1, 2, \dots$$

Where Adomian polynomial will be obtained as the following form

$$\mathcal{A}_0(u_0) = \pi^2 e^{-u_0},$$

$$\mathcal{A}_1(u_0, u_1) = -\pi^2 u_1 e^{-u_0},$$

$$\mathcal{A}_2(u_0, u_1, u_2) = \frac{1}{2!}(\pi^2(-2u_2 e^{-u_0} + u_1^2 e^{-u_0})),$$

⋮

So, we have

$$u_1 = -e^{-\pi(\frac{1}{\alpha} x^\alpha)} - \pi \left(\frac{1}{\alpha} x^\alpha\right) + 1,$$

$$u_2 = \frac{5}{4} - \frac{\pi}{2} \left(\frac{1}{\alpha} x^\alpha\right) - \frac{1}{4} e^{-2\pi \left(\frac{1}{\alpha} x^\alpha\right)} - \pi \left(\frac{1}{\alpha} x^\alpha\right) e^{-\pi \left(\frac{1}{\alpha} x^\alpha\right)} - e^{-\pi \left(\frac{1}{\alpha} x^\alpha\right)},$$

$$u_3 = \frac{11}{6} - \frac{\pi^2}{2} \left(\frac{1}{\alpha} x^\alpha\right)^2 e^{-\pi \left(\frac{1}{\alpha} x^\alpha\right)} - \frac{\pi}{2} \left(\frac{1}{\alpha} x^\alpha\right) - \frac{1}{2} e^{-2\pi \left(\frac{1}{\alpha} x^\alpha\right)} - \frac{3\pi}{2} \left(\frac{1}{\alpha} x^\alpha\right) e^{2\pi \left(\frac{1}{\alpha} x^\alpha\right)}$$

$$- \frac{\pi}{2} \left(\frac{1}{\alpha} x^\alpha\right) e^{-2\pi \left(\frac{1}{\alpha} x^\alpha\right)} - \frac{5}{4} e^{-\pi \left(\frac{1}{\alpha} x^\alpha\right)} - \frac{1}{12} e^{-2\pi \left(\frac{1}{\alpha} x^\alpha\right)},$$

⋮

Consciously, four-terms approximations to the solution will be as follows:

$$u(x) \approx \frac{49}{12} - \frac{\pi}{2} \left(\frac{1}{\alpha} x^\alpha\right) - 2e^{-\pi \left(\frac{1}{\alpha} x^\alpha\right)} - \pi \left(\frac{1}{\alpha} x^\alpha\right) e^{-\pi \left(\frac{1}{\alpha} x^\alpha\right)} - \frac{1}{4} e^{-2\pi \left(\frac{1}{\alpha} x^\alpha\right)}$$

$$- \frac{\pi^2}{2} \left(\frac{1}{\alpha} x^\alpha\right)^2 e^{-\pi \left(\frac{1}{\alpha} x^\alpha\right)} - \frac{\pi}{2} \left(\frac{1}{\alpha} x^\alpha\right) - \frac{1}{2} e^{-2\pi \left(\frac{1}{\alpha} x^\alpha\right)} - \frac{3\pi}{2} \left(\frac{1}{\alpha} x^\alpha\right) e^{2\pi \left(\frac{1}{\alpha} x^\alpha\right)}$$

$$- \frac{\pi}{2} \left(\frac{1}{\alpha} x^\alpha\right) e^{-2\pi \left(\frac{1}{\alpha} x^\alpha\right)} - \frac{5}{4} e^{-\pi \left(\frac{1}{\alpha} x^\alpha\right)} - \frac{1}{12} e^{-2\pi \left(\frac{1}{\alpha} x^\alpha\right)}.$$

Figures 2 confirmed previous results.

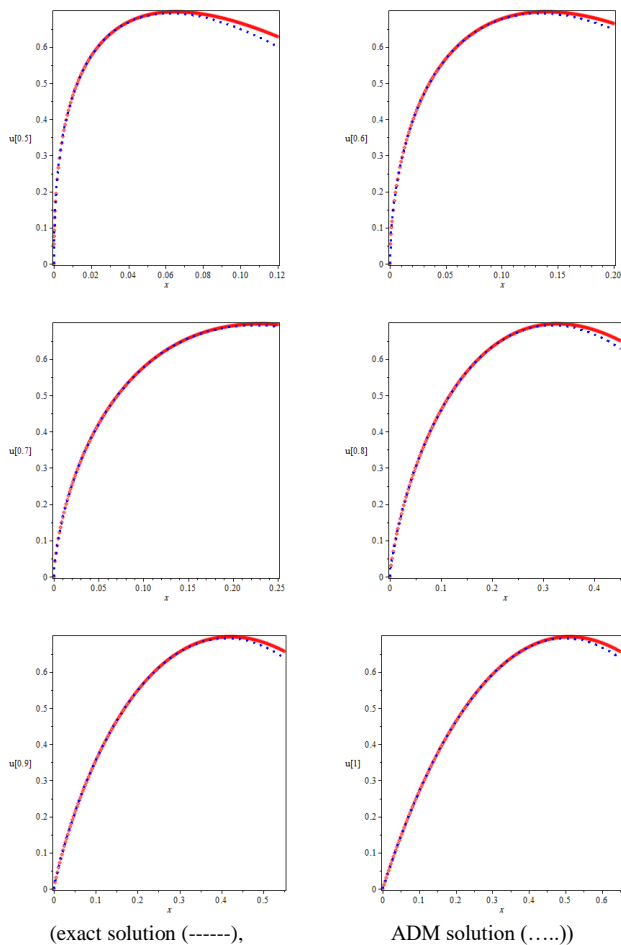


Fig. 2: The Comparison 4th-Order Approximation of ADM and Exact Solution for Example 3.

5. Conclusion

In this paper, Adomian Decomposition method has been applied to obtain the solutions of fractional differential equations. To this aim, a conformable fractional derivative has been used to find the

solution. The results showed that the definition is the simplest tool to obtain the approximation solutions of nonlinear fractional differential equations in comparison to the other definitions. To show the effectiveness and simplicity of the method, fractional Bratu-type equations as an example have been solved with conformable fractional derivative and Adomian Decomposition method.

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