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Common fixed point results in cone metric spaces

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Abstract

In this paper, we generalize and prove common fixed point theorems of generalized contractive maps in complete cone metric spaces. Our theorems improve and generalize of the results [7].

Keywords: *Complete cone metric space, common fixed point, Generalized contractive Mapping, Non-normal cone*.

1 Introduction

Recently, Huang and Zhang [1], replaced the real numbers by an ordering Banach space, and defined a cone metric spaces(X, d) of contractive mappings and also discussed some properties of convergence of sequences; many authors have established and extend different types of contractive mappings in cone metric spaces see for instance [3-10]., and also generalized the results [1] by [2].The author [7] proved fixed point results in cone metric spaces.
The purpose of this paper is to obtain the generalization of results in [1] and 2.1, 2.2 of [7], by using non-normality of cone.

2 Preliminary notes

First, we recall some standard notations and definitions in cone metric spaces with some of their properties [1].

Definition 2.1 [1]: Let E be a real Banach space and P be a subset of E. P is called a cone if and only if:
(i) P is closed, non – empty and $P \neq \{0\}$,
(ii) ax + by $\in P$ for all $x, y \in P$ and non – negative real number a, b;
(iii) $x \in P$ and - $x \in P$ => $x = 0$ <=> $P \cap (-P) = \{0\}$.
Given a cone $P \subset E$, we define a partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x \ll$ y if $y - x \in$ int$P$, where int P denotes the interior of P.
The cone P is called normal if there is a number K > 0 such that $x, y \in$ E, $0 \leq$ x $\leq$ y implies $\| x \| \leq K \| y \|$.
The least positive number satisfying the above is called the normal constant P. The cone p is called regular if every increasing sequence which is bounded from above is convergent .that is , if $\{ x_n \}$ is sequence such that $x_1 \leq x_2 \leq \ldots x_n \leq \ldots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \longrightarrow 0$ ($n \longrightarrow \infty$). Equivalently the cone p is regular if and only if every decreasing sequence which is bounded from below is convergent.

Lemma 2.2[**2, 8**]
(i) Every regular cone is normal
(ii) For each k > 1, there is a normal cone with normal constant K > k.

Definition 2.3 [1]: Let X be a non – empty set. Suppose the mapping $d: X\,xX \rightarrow E$ satisfies
(i) $0 < d\,(x, y)$ for all $x, y \in$ X and d $(x,) = 0$ if and only if $x = y$;
(ii) $d\,(x, y) = d\,(y, x)$ for all $x, y \in X$;
(iii) $d\,(x, y) \leq d\,(x,$ z$) + d\,($z$, y)$ for all$x, y \in$ X.
Then d is called a cone metric, on X and pair (X, d) is called a cone metric space. It is obvious that cone metric spaces generalize metric space.

Example 2.4: Let $E = R^2$, $P = \{(x, y) \in E: x, y \geq 0\}$, $X = R$ and $d: X \times X \rightarrow$ E defined by $d(x, y) = (| x - y |, \alpha | x, y |)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 2.5 [1]: Let (X, d) be a cone metric space, $x \in$ X and $\{x_n\}_{n \geq 1}$ a sequence in X. Then,

(i) $\{x_n\}_{n \geq 1}$ converges to x whenever for every c ∈ E with o ≪ c, there is a natural number N such that d (x_n, x) ≪c for all n ≥ N. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$, (n → ∞).

(ii) $\{x_n\}_{n \geq 1}$ is said to be a Cauchy sequence if for every c ∈ E with o ≪ c, there is a natural number N such that d $(x_n, x_m) \ll$ c for all n, m ≥ N.

(iii) (X, d) is called a complete cone metric space if every Cauchy sequence in X is convergent

Definition 2.6 [8]: Cone P is called minihedral cone if sup {x, y} exists for all x, y ∈ E and strongly minihedral if every subset of E which is bounded from above has a supremum.

Lemma 2.7 [9]: Every strongly minihedral normal cone is regular.

3 Main results

Theorem3.1: Let (X, d) be a complete cone metric space and suppose the mapping $T_1, T_2: X \longrightarrow X$ **satisfy** the contractive condition,

$d(T_1x, T_2y) \leq K[(T_1 x, x) + d(x, y) + d(T_2 y, y)$ for all $x, y \in X$, where $0 \leq k \leq \frac{1}{2}$.

Then T_1 and T_2 have a unique fixed point in X. And for any $x \in X$, iterative sequences $\{T_1^{2n+1} x\}$ and $\{T_2^{2n+2} x\}$ converse to the common fixed point.

Proof: For each $x_0 \in X$ and n ≥ 1, set $x_1 = T_1x_0$ and $x_{2n+1} = T_1x_{2n} = T_1^{2n+1}x_0$.
Similarly $x_{2n+1} = T_2x_{2n+1} = T_2^{2n+2} x_0$. Then we have

$$
\begin{aligned}
d(x_{2n+1}, x_{2n}) &= d(T_1x_{2n}, T_2x_{2n-1}) \\
&\leq K[d(T_1x_{2n}, x_{2n}) + d(x_{2n}, x_{2n-1}) + d(T_2x_{2n-1}, x_{2n-1})] \\
&= K[d(x_{2n+1}, x_{2n}) + d(x_{2n}, x_{2n-1}) + d(x_{2n}, x_{2n-1})]
\end{aligned}
$$

So,
$$
\begin{aligned}
d(x_{2n+1}, x_{2n}) &= \frac{2k}{1-k} d(x_{2n}, x_{2n-1}) \\
&= h\, d(x_{2n}, x_{2n-1}) \text{ where } h = \frac{2k}{1-k}.
\end{aligned}
$$

For $n \geq m$

$$
\begin{aligned}
d(x_{2n}, x_{2m}) &\leq d(x_{2n}, x_{2n-1}) + d(x_{2n-1}, x_{2n-2}) + \dots + d(x_{2m+1}, x_{2m}) \\
&\leq (h^{2n-1} + h^{2n-2} + \dots + h^{2m})\, d(x_1, x_0) \\
&\leq \frac{h^{2m}}{1-h} d(x_1, x_0)
\end{aligned}
$$

Let $0 \ll c$ be given, choose a positive integer N_1, such that $\frac{h^{2m}}{1-h} d(x_1, x_0) \ll c$. For all $m \geq N_1$ Thus $d(x_{2n}, x_{2m}) \ll c$, for $n > m$. Therefore $\{x_{2n}\}$ is a Cauchy sequence in (X, d). Since (X, d) be a complete cone metric space, there exist $x^* \in X$ such that $x_{2n} \longrightarrow x^*$. Now choose a positive integer N_2 such that $d(x_{2n+1}, x_{2n}) \ll \frac{c(1-k)}{3k}$, $d(x_{2n}, x^*) \ll \frac{c(1-k)}{3k}$ and $d(x_{2n+1}, x^*) \ll \frac{c(1-k)}{3}$ for all $n \geq N_2$. Hence for $n \geq N_2$ we have,

$$
\begin{aligned}
d(T_1x^*, x^*) &\leq d(T_1x_{2n}, T_1x^*) + d(T_1x_{2n}, x^*) \\
&\leq K[d(T_1x_{2n}, x_{2n}) + d(x_{2n}, x^*) + d(T_1x^*, x^*)] + d(x_{2n+1}, x^*) \\
&\leq K[d(x_{2n+1}, x_{2n}) + d(x_{2n}, x^*) + d(T_1x^*, x^*)] + d(x_{2n+1}, x^*)
\end{aligned}
$$

$$d(T_1x^*,x^*) \leq \frac{1}{1-k}\,[\,k\,d(x_{2n+1},x_{2n}) + kd(x_{2n},x^*) + d(x_{2n+1},x^*)]$$

$$\leq \frac{c}{3}+\frac{c}{3}+\frac{c}{3}.$$

$$= c$$

Thus, $d(T_1x^*,x^*) \ll \frac{c}{m}$ for all $m \geq 1$.so , $\frac{c}{m}$ - $d(T_1x^*,x^*) \in P$ forall$m \geq 1$.
Since $\frac{c}{m} \longrightarrow 0\,(as m \longrightarrow \infty)$ and P is closed, - $d(T_1x^*,x^*) \in P$. But $d(T_1x^*,x^*) \in P$ .Therefore $d(T_1x^*,x^*) = 0$ and so,T_1x^*=x^*. So x^* is a fixed point of T_1.
Now if $y^*$is another fixed point of T_1, Then

$$d(x^*,y^*) \leq k[d(T_1x^*,x^*) + d(T_1y^*,y^*) + d(x^*,y^*)]$$
$$= 0.$$

Hence $x^* = y^*$. Therefore the fixed point of T_1 is unique.
Similarly, it can be established that $T_2x^* = x^*$. Hence $T_1x^* = x^* = T_2x^*$. Thus x^* is the common fixed point of T_1 and T_2 .

Theorem3.2: Let (X, d) be a complete cone metric space and suppose the mapping $T_1, T_2 : X \longrightarrow X$ **satisfy** the contractive condition,
$d(T_1x, T_2y) \leq K[\,(T_1\,x, y) + d(x\,,y) + d(\,x\,,T_2\,y)$ for $x, y \in X$, where $0 \leq k \leq \frac{1}{2}$.
Then T_1 and T_2have a unique fixed point in X. And for any $x \in X$, iterative sequences $\{T_1^{2n+1}\,x\}$ and $\{T_2^{2n+2}\,x\}$ converse to the common fixed point.

Proof: For each $x_0 \in X$ and $n \geq 1$, set $x_1 = T_1x_0$ and $x_{2n+1} = T_1x_{2n} = T_1^{2n+1}x_0$.
Similarly $x_{2n+1} = T_2x_{2n+1} = T_2^{2n+2}x_0$. Then we have

$$d\left(x_{2n+1},x_{2n}\right) = d\left(T_1x_{2n},T_2x_{2n-1}\right)$$

$$\leq K\left[d\left(T_1x_{2n},x_{2n-1}\right) + d(x_{2n},x_{2n-1}) + d(x_{2n},T_2x_{2n-1})\right]$$

$$= K\left[d\left(x_{2n+1},x_{2n-1}\right) + d(x_{2n},x_{2n-1})\right]$$

$$\leq K[d(x_{2n+1}\,,x_{2n}) + d(x_{2n},x_{2n-1}) + d(x_{2n},x_{2n-1})]$$

So, $d\left(x_{2n+1},x_{2n}\right) \leq \frac{2k}{1-k}d\,(x_{2n},x_{2n-1})$
$= h\,d(x_{2n},x_{2n-1})$, where $h = \frac{2k}{1-k}$.

For $n > m$

$$d\left(x_{2n},x_{2m}\right) \leq d\left(x_{2n},x_{2n-1}\right) + d(x_{2n-1},x_{2n-2}) + ... + d\left(x_{2m+1},x_{2m}\right)$$

$$\leq (h^{2n-1} + h^{2n-2} + \cdots + h^{2m})\,d\left(x_1,x_0\right)$$

$$\leq \frac{h^{2m}}{1-h}\,d(x_1,x_0)$$

Let $0 \ll c$ be given, choose a positive integer N_1, such that $\frac{h^{2m}}{1-h}\,d(x_1,x_0) \ll c$. for all$m \geq N_1$.Thus, $d(x_{2n}\,,x_{2m}) \ll c$, for $n > m$ .Therefore $\{x_{2n}\}$ is a Cauchy sequence in(X,d).
Since (X,d) be a complete cone metric space, there exist $x^* \in X$ such that $x_{2n} \longrightarrow x^*$. Now choose a positive integer N_2 such that $d\,(x_{2n}\,,x^*) \ll \frac{c(1-k)}{4}$ and for all $n \geq N_2$. Hence for $n \geq N_2$ we have,

$$d(T_1x^*,x^*) \leq d\left(T_1x_{2n},T_1x^*\right) + d\left(T_1x_{2n},x^*\right)$$

$$\leq K\,[d\left(T_1x_{2n},x^*\right) + d(x_{2n},x^*) + d(T_1x^*,x^*) + d\left(x_{2n+1},x^*\right)$$

$$\leq K\,[d\left(x_{2n+1},x^*\right) + d(x_{2n},x^*) + d(T_1x^*,x^*) + d(x_{2n},x^*) + d\left(x_{2n+1},x^*\right)\,]$$

$$d(T_1x^* , x^*) \le \frac{1}{1-k} [2\, k\, d(x_{2n}, x^*) + kd(x_{2n+1}, x^*)] + d(x_{2n+1}, x^*)$$
$$\le \frac{2c}{4} + \frac{c}{4} + \frac{c}{4}.$$
$$= c$$

Thus, $d\,(T_1x^*, x^*) \ll \frac{c}{m}$ for all $m \ge 1$.so, $\frac{c}{m}$ - $d(T_1x^*, x^*) \in P$ for all $m \ge 1$.
Since $\frac{c}{m} \longrightarrow 0(as m \longrightarrow \infty)$and P is closed, - $d(T_1x^*, x^*) \in P$. But $d(T_1x^*, x^*) \in P$.
Therefore $d(T_1x^*, x^*) = 0$ and so, T_1x^*=x^*.
Now if $y^*$is another fixed point of T_1, Then

$$d(x^*, y^*) \le k[d(T_1x^*, x^*) + d(T_1y^*, y^*) + d(x^*, y^*)]$$
$$= 0.$$

Hence $x^* = y^*$. Therefore the fixed point of T_1 is unique.
Similarly, it can be established that$T_2x^* = x^*$. Hence $T_1x^* = x^* = T_2x^*$. Thus x^* is the common fixed point of T_1 and T_2 .

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