About asymmetric noisy chaotic maps

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Abstract

In this paper we will introduce three families of asymmetric maps, and discuss some dynamical properties for these families in the deterministic case, and noisy case. New mixed noisy chaotic map will be suggested and then studied with some details.

Keywords: *Asymmetric Maps, Lyapunov Exponent Mixed Noisy Chaotic Map, Random Dynamical System, And Random Fixed Point.*

1. Basic concepts

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denoted a probability space, and let $(\vartheta^t)_{t \in \mathbb{Z}^+} : \Omega \to \Omega$ be an invertible map such that ϑ^t and ϑ^{-t} are measurable, $\vartheta\mathbb{P} = \mathbb{P}$, $\vartheta^0 = id_\Omega$, $\vartheta^{n+m} = \vartheta^n o \vartheta^m$, and all $B \in \mathcal{F}$ with $\vartheta^{-1}(B) = B$ then $\mathbb{P}(B) \in \{0,1\}$, we will define the ergodic process $\xi_n(\omega) = \xi(\vartheta^n(\omega))$, $n \in \mathbb{Z}^+$, where $\xi: \Omega \to \mathbb{R}$ is measurable function.

Definition 1.1: (random dynamical system – discrete time) [1].
Let $(\Omega, \mathcal{F}, \mathbb{P})$ denoted a probability space then the random dynamical system is define as a random difference equation $x_{t+1}(\omega) = \varphi(t, x_0, \xi_0(\omega)) = \varphi(1, x_t, \xi_t(\omega))$, where $t \in \mathbb{Z}^+$ $\varphi: \mathbb{Z}^+ \times \mathbb{R} \times \Omega \to \mathbb{R}$ and $\xi_t(\omega)$ defined above. Such that
i) $\quad \varphi(0, x_0, \xi_0(\omega)) = x_0(\omega)$
ii) $\quad \varphi(n + m, x_0, \xi_0(\omega)) = \varphi(n, x_m, \xi_m(\omega)) o \varphi(m, x_0, \xi_0(\omega))$

Definition 1.2: (random fixed point) [6].
A random fixed point $p(\omega)$ of a random dynamical system φ on $\mathbb{R}$ is a random variable $p(\omega): \Omega \to \mathbb{R}$ such that. Almost surely $(a.s.)$ $\quad p(\omega) = \varphi(1, p(\omega), \xi_0(\omega))$
The random fixed point $p(\omega)$ is attracting with probability $P_r(p(\omega) \ is \ attracting) = P$ if $P_r(\lim_{t \to \infty} |x_t(\omega) - p(\omega)| = 0) = P$, for all $x_0(\omega) \in (p(\omega) - \varepsilon, p(\omega) + \varepsilon)$ for same $\varepsilon > 0$
If $P = 1$ (i.e. $\lim_{t \to \infty} |x_t(\omega) - p(\omega)| = 0$ $a.s.$) then $p(\omega)$ is called globally attracting [6]**.**

Lemma 1.1: *let $p(\omega)$ be a random fixed point to the random dynamical system φ on $\mathbb{R}$, and φ partial differentiable at $p(\omega)$ then*
$$P_r(p(\omega) \ is \ attracting) = P_r\left(\left| \frac{\partial \varphi(t, x, \xi_t(\omega))}{\partial p(\omega)} \right| < 1 \right)$$

Proof:
Let $\left| \frac{\partial \varphi(t, x, \xi_t(\omega))}{\partial p(\omega)} \right| < 1$ then $\left| \frac{x_1(\omega) - p(\omega)}{x_0(\omega) - p(\omega)} \right| \le A$ for same $0 < A < 1$ and same $\varepsilon > 0$. and any $x_0(\omega) \in (p(\omega) - \varepsilon, p(\omega) + \varepsilon)$ then we have $|x_1(\omega) - p(\omega)| \le A|x_0(\omega) - p(\omega)|$. by mathematical induction we get $|x_t(\omega) - p(\omega)| \le A^t|x_0(\omega) - p(\omega)|$ and hence $\lim_{t \to \infty} |x_t(\omega) - p(\omega)| = 0$, which implies to $P_r\left(\left| \frac{\partial \varphi(t, x, \xi_t(\omega))}{\partial p(\omega)} \right| < 1 \right) = P_r(\lim_{t \to \infty} |x_t(\omega) - p(\omega)| = 0) = P_r(p(\omega) \ is \ attracter)$.

2.   Some dynamical properties in deterministic case

2.1. The family of asymmetric tent map

The dynamical system for the asymmetric tent map can be defined as follows:

$$x_{t+1} = T_{a,\mu}(x_t) = \begin{cases} \mu\dfrac{x_t}{a} & if\ \ 0 \le x_t \le a \\ \mu\dfrac{1-x_t}{1-a} & if\ \ a\ < x_t \le 1 \end{cases} \tag{1}$$

Where $T_{a,\mu}: [0,1] \to [0,1]$ and $0\le \mu \le 1$, $0\le a \le 1$.

Theorem 2.1: For a dynamical system in (1).

1)   If $\mu < a$ then the system has one attracting fixed point $p = 0$.
2)   If $\mu = a$ then all points of the interval $[0, a]$ represent fixed point to the system and all points from the interval $(a, 1]$ represent eventually fixed points and it is orbit have only two points .
3)   If $a < \mu < 1 - a$ then the system has two fixed points $p_1 = 0$ replying and $p_2 = \dfrac{\mu}{\mu+(1-a)}$ attracting.
4)   If $\mu > a$ and $\mu > 1 - a$ then there are no any attracting fixed or periodic point and the system becomes chaotic, see Lyapunov exponent figure (1).

Proof: Clear.

The satisfaction of the above properties can be seen from the bifurcation diagrams in figure (2)
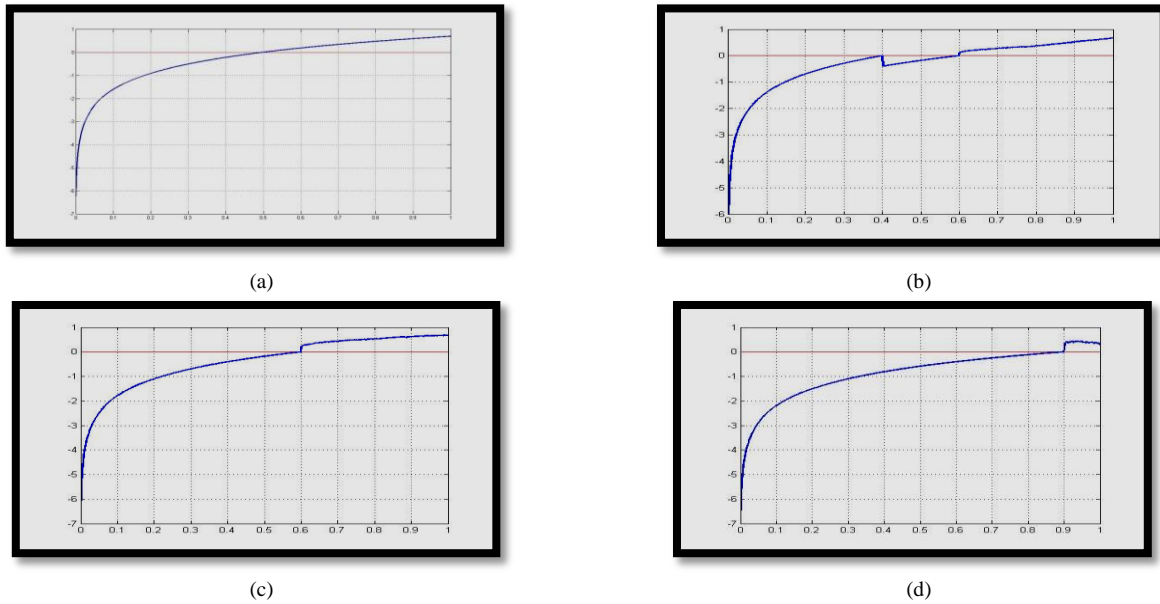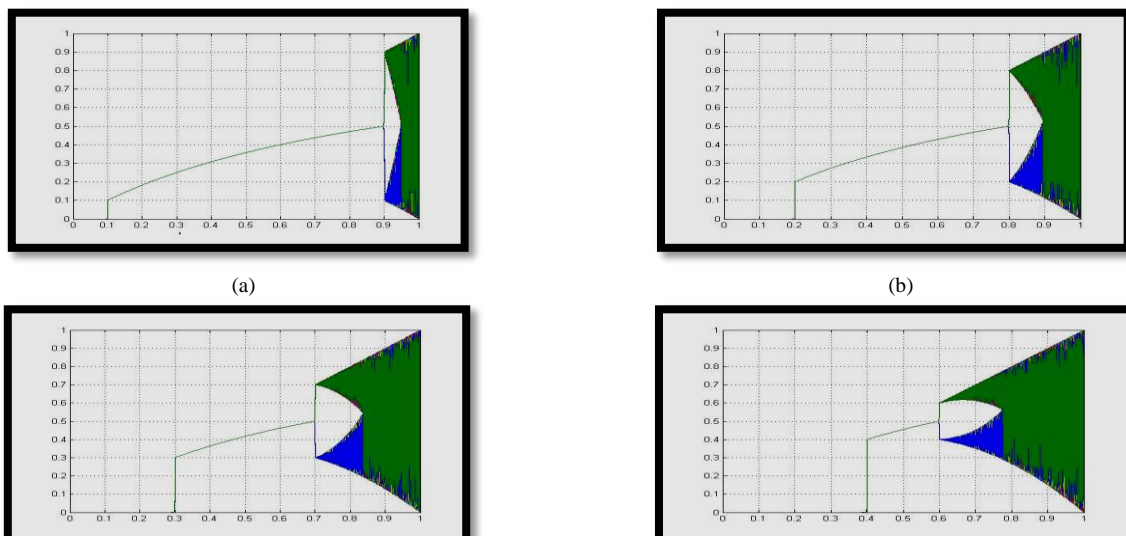

(a)

(b)

(c)

(d)

Fig. 1: Lyapunov exponent of asymmetric tent map , x_ axes represent the values of μ and y_ axes represent the values of Lyapunov exponent ,
(a)$a = 0.5$, (b) $a = 0.4$, (c) $a = 0.6$, (d) $a = 0.9$

(a)

(b)

(g)   (h)
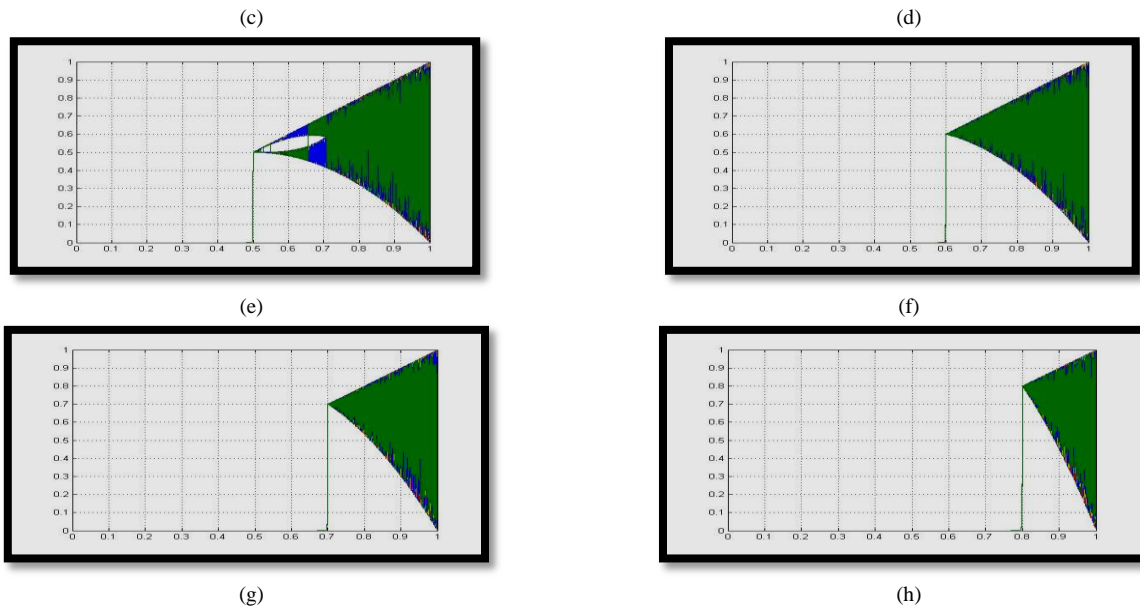
Fig. 2: bifurcation diagram of asymmetric tent map family , x-axes represent the values of μ and y-axes represent the values of x_{t+1} where $x_{t+1} \in [0,1]$, (a) $a = 0.1$, (b) $a = 0.2$, (c) $a = 0.3$,(d) $a = 0.4$, (e) $a = 0.5$, (f)$a = 0.6$,(g) $a = 0.7$, (h) $a = 0.8$

2.2. The family of asymmetric logistic map

The dynamical system for asymmetric logistic map can be defined as follows:

$$x_{t+1} = G_{a,\mu}(x_t) = \begin{cases} \mu \frac{x_t}{2a}\left(1 - \frac{x_t}{2a}\right) & if \quad 0 \le x_t \le a \\ \mu\left(\frac{x_t+1-2a}{2-2a}\right)\left(1 - \frac{x_t+1-2a}{2-2a}\right) & if \quad a < x_t \le 1 \end{cases} \tag{2}$$

Where $G_{a,\mu}: [0,1] \to [0,1]$ and $0 \le a \le 1$, $0 \le \mu \le 4$.

Theorem 2.2: *For a dynamical system in (2)*

1) If $\mu < 2a$ then the system has on attracting fixed point $p = 0$.

2) If $2a < \mu \le 4a$ then the system has two fixed points $p_1 = 0$ is replying and $p_2 = \frac{2a\mu - 4a^2}{\mu}$ is attracting and $p_1, p_2 \in [0,a]$.

3) If $2a < \mu < 2a + \sqrt{4a^2 + 3(2-2a)^2}$ then the second fixed point

$$p_2 = \frac{\left(\frac{2a\mu}{(2-2a)^2} - 1\right) - \sqrt{\left(\frac{2a\mu}{(2-2a)^2}-1\right)^2 + \left(\frac{4(\mu^2 - 2a\mu^2)}{(2-2a)^4}\right)}}{\frac{2\mu}{(2-2a)^2}} \text{ is attracting and } p_2 \in [0,1] .$$

Proof: Clear**.**

(a)   (b)

(c)   (d)
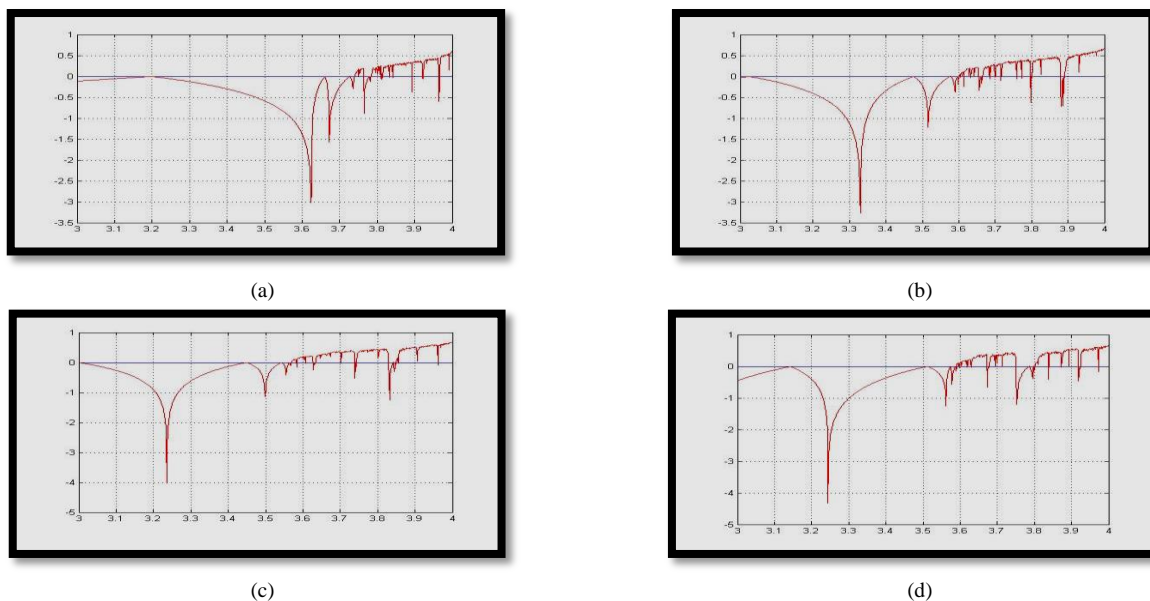
Fig. 3: Lyapunov exponent of asymmetric logistic map , x_ axes represent the values of μ and y_ axes represent the values of Lyapunov exponent , (a) $a = 0.2$, (b) $a = 0.4$, (c) $a = 0.5$, (d) $a = 0.7$.
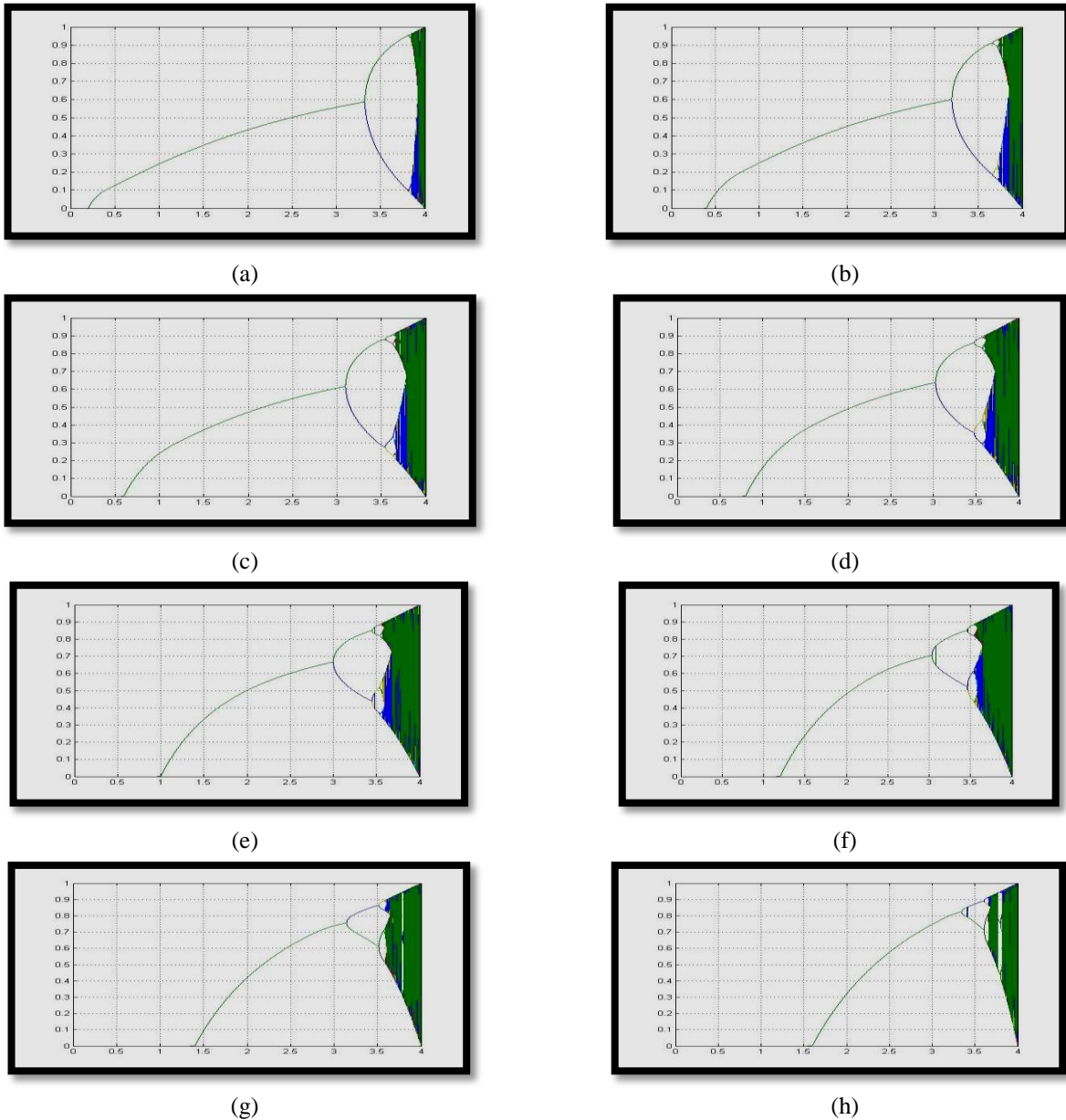
(a)

(b)

(c)

(d)

(e)

(f)

(g)

(h)

Fig. 4: bifurcation diagram of asymmetric logistic map family , x-axes represent the values of μ and y-axes represent the values of x_{t+1} with $x_{t+1} \in [0,1]$, (a) $a = 0.1$, (b) $a = 0.2$, (c) $a = 0.3$,(d) $a = 0.4$, (e) $a = 0.5$, (f) $a = 0.6$,(g) $a = 0.7$, (h) $a = 0.8$.

2.3. The family of mixed logistic-tent map

The dynamical system for the mixed logistic-tent map can be defined as follows:

$$x_{t+1} = TG_{a,\mu}(x_t) = \begin{cases} \mu \frac{x_t}{2a}\left(1 - \frac{x_t}{2a}\right) & if \quad 0 \le x_t \le a \\ \mu\left(\frac{1-x_t}{4(1-a)}\right) & if \quad a < x_t \le 1 \end{cases} \tag{3}$$

Where $TG_{a,\mu}: [0,1] \to [0,1]$ and $0 \le a \le 1$, $0 \le \mu \le 4$.

Theorem 2.3: *For a dynamical system in (3)*
1) If $\mu < 2a$ then the system has one attracting fixed point $p = 0$.
2) If $2a < \mu < 4a$ then the system has two fixed points $p_1 = 0$ is replying and $p_2 = \frac{2a\mu - 4a^2}{\mu}$ is attracting, and $p_1, p_2 \in [0, a]$.
3) If $\mu = 4a$ then the second fixed point $p_2 = a$ is attracting.
4) If $4a < \mu < 4(1-a)$ then $p_2 = \frac{\mu}{\mu + 4(1-a)}$ is attracting , and p_2 replying if $\mu > \max\{4a , a(1-a)\}$.

Proof:

The proofs of (1), (2) and (4) are clear. To prove (3), $p_2 = \frac{2a\mu - 4a^2}{\mu} = \frac{8a^2 - 4a}{4a} = a$

To show that p_2 is attracter we must find open interval $J = (p_2 - \varepsilon, p_2 + \varepsilon)$, $\varepsilon > 0$, such that for any $x_0 \in J$ then the orbit $\{x_t\} \to p_2$, where $\{x_t\}$ the orbit of x_0 .

Claim : $J = (0,1)$: At first let $x_0 \in (0, a)$ since $GT_{a,4a}(x_t) \leq GT_{a,4a}(a) = a$, for every $x_t \in (0, a)$, then $\{x_t\}$ bounded from above and a represent $sup(\{x_t\})$, and since $GT'_{a,4a}(x_t) = \frac{4a}{2a}\left(1 - \frac{x}{a}\right) > 0$ for every $x_t \in (0, a)$ then $\{x_t\}$ increasing , that implies $\{x_t\}$ converge to a . To complete the proof we need to show for any $x_0 \in (a, 1)$ then $x_1 \in (0, a)$.

$x_1 = GT_{a,4a}(x_0) = a\left(\frac{1 - x_0}{1 - a}\right)$, since $0 < a < x_0 < 1$ then $0 < 1 - x_0 < 1 - a < 0$, and hence $\frac{1 - x_0}{1 - a} < 1$ which implies to $0 < x_1 = a\frac{1 - x_0}{1 - a} < a$.

(a)

(b)

(c)

(d)

Fig. 5: Lyapunov exponent of mixed logistic-tent map , x_ axes represent the values of μ and y_ axes form value of Lyapunov exponent , (a)$a = 0.2$, (b) $a = 0.4$, (c) $a = 0.5$, (d) $a = 0.7$.

Table 1: Some properties of the asymmetric tent , asymmetric logistic and mixed logistic-tent map

No.	property	map	value of "a"								
			0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	Value of "μ" which Fixed point becomes unstable .	A.L	3.3240	3.2000	3.0980	3.0270	3.0000	3.0330	3.1430	3.3430	3.6330
		A.T	0.9000	0.8000	0.7000	0.6000	0.5000	0.6000	0.7000	0.8000	0.9000
		M.L-T	3.6000	3.2000	2.8000	2.4000	2.0000	2.4000	2.8000	3.2000	3.6000
2	Value of "μ" which Chaotic region begins .	A.L	3.8510	3.7400	3.6580	3.6020	3.5700	3.5660	3.5910	3.6570	3.7810
		A.T	0.9000	0.8000	0.7000	0.6000	0.5000	0.6000	0.7000	0.8000	0.9000
		M. L-T	3.6560	3.4370	3.3160	3.2680	3.2460	3.2520	3.3010	3.4180	3.6480
3	Value of "μ" which cycle with period 3 appears .	A. L	3.9905	3.9645	3.9255	3.8786	3.8285	3.7819	3.7496	3.7490	3.8102
		A. T	0.995	0.9688	0.9280	0.8748	0.8091	0.7294	0.7001	0.8002	0.9004
		M. L-T	3.9860	3.9340	3.8520	3.7520	3.6410	3.5282	3.4489	3.4670	3.6550
4	Value of "μ" which chaotic region ends .	A. L.	4	4	4	4	4	4	4	4	4
		A. T.	1	1	1	1	1	1	1	1	1
		M. L-T	4	4	4	4	4	4	4	4	4
5	Are there stable cycles in the chaotic region ?	A. L.	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
		A. T.	No	No	No	No	No	No	No	No	No
		M. L-T	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes

(a)

(b)

(c)

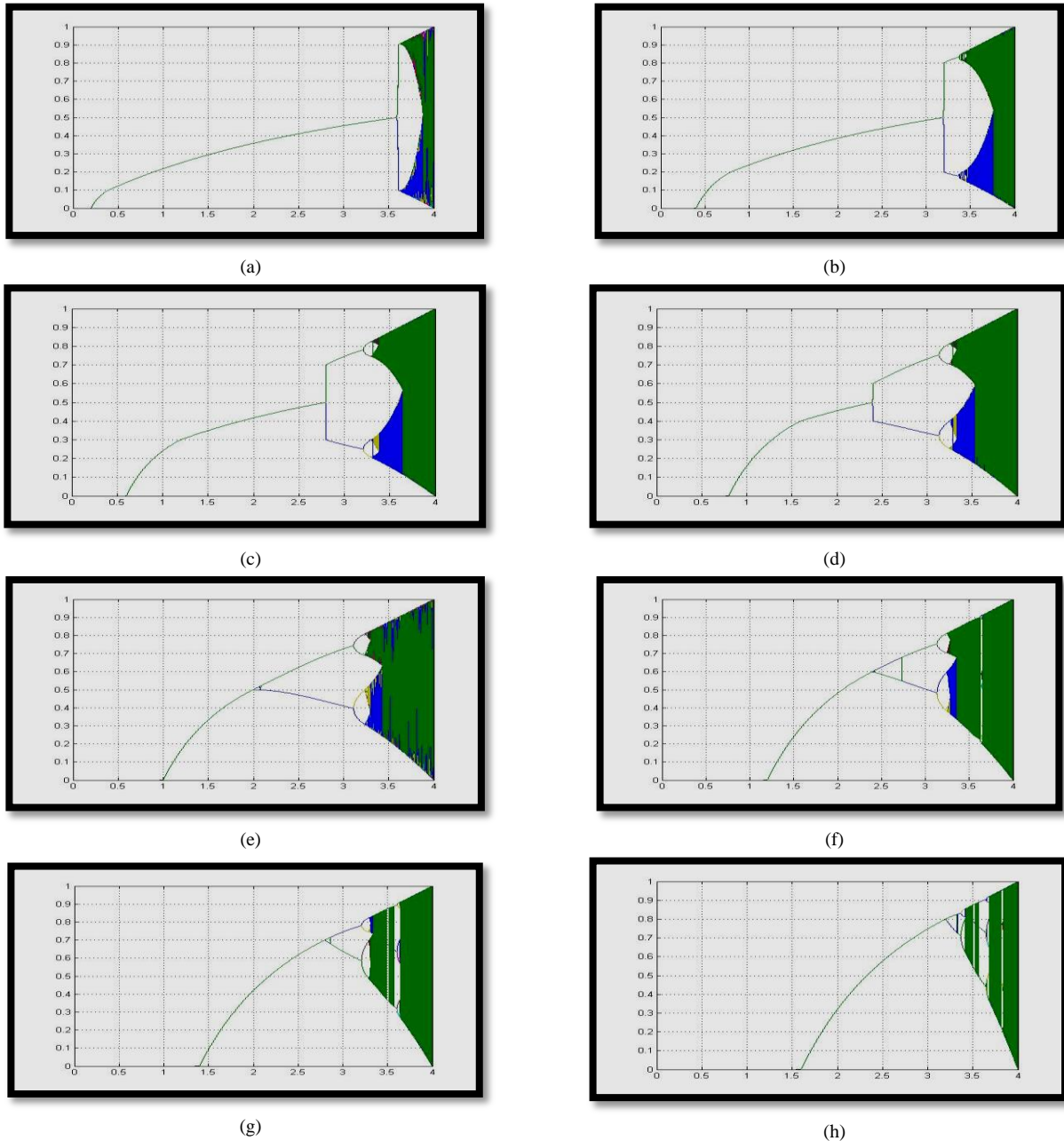(d)

(e)

(f)

(g)

(h)

Fig. 6: bifurcation diagram of mixed logistic-tent map family , x-axes represent the values of μ and y-axes represent the values of x_{t+1} with $x_{t+1}{\in}[0,1]$, (a) $a = 0.1$, (b) $a = 0.2$, (c) $a = 0.3$,(d) $a = 0.4$, (e) $a = 0.5$, (f) $a = 0.6$,(g) $a = 0.7$, (h) $a = 0.8$

3. Some dynamical properties in noisy case

All maps treated with in previous section were defined on closed interval [0, 1], but in the noisy case it is not true generally.

3.1. The family of noisy asymmetric tent map.

The noisy dynamical system for asymmetric tent map with additive noise is given by.

$$x_{t+1}(\omega) = \tilde{T}_{a,\mu}(x_t , \xi_t(\omega)) = \begin{cases} \mu \frac{x_t}{a} + \xi_t(\omega) & if \ 0 \leq x_t \leq a \\ \mu \frac{1-x_t}{1-a} + \xi_t(\omega) & if \ a < x_t \leq 1 \end{cases} \quad (4)$$

Where $\tilde{T}_{a,\mu}: \mathbb{R} \times \Omega \rightarrow \mathbb{R}$, $\{\xi_t(\omega)\}$ is a sequence of independent identically distributed (iid) random variables and $0 \leq a \leq 1$, $0 \leq \mu \leq 1$

Theorem 3.1: *For the dynamical system (4), where* $\xi_t(\omega) \sim Gauss(0, \sigma^2)$ *,*

1) If $\mu < a$ then the system has one random fixed point $p(\omega) \sim Gauss(0, \sigma^2)$, and $P_r(p(\omega)$ *is attracting*$) = P_r(\xi(\omega) < a)$, if $\mu > 1 - a$, and $P_r(p(\omega)$ *is attracting*$) = 1$ if $\mu < 1 - a$

2) If $\mu = a$ then there are infinite number of random fixed points $p(\omega) \sim Gauss(p^*, \sigma^2)$, $p^* \in [0, a]$.

3) If $a < \mu < 1 - a$ then the system has two random fixed points $p_1(\omega) \sim Gauss(0, \sigma^2)$ and $p_2(\omega) \sim Gauss(\frac{\mu}{\mu + (1-a)}, \sigma^2)$, where $P_r(p_1(\omega)$ *is attracting* $) = P_r(\xi(\omega) > a)$ and $P_r(p_2(\omega)$ *is attracting* $) = P_r\left(\xi(\omega) > a - \frac{\mu}{\mu + (1-a)}\right)$.

4) If $\mu > a$ and $\mu > 1 - a$ then for any random fixed (periodic) point $p(\omega)$, $P_r(p(\omega)$ *is attracting*$) = 0$, and the system become noisy chaotic.

Proof: *The proofs of (1), (2) and (3) are clear.*

To prove (4), since $\left|\frac{\partial \tilde{T}_{a,\mu}(x_t, \xi_t(\omega))}{\partial x_t}\right| = \begin{cases} \dfrac{\mu}{a} & if \ x_t < a \\ \dfrac{\mu}{1-a} & if \ x_t > a \end{cases}$

Then by lemma 1 any random fixed (periodic) point $p(\omega)$ is attracter with probability 0.

To show that the system is noisy chaotic we will calculate stochastic Lyapunov exponent (SLE) as follows.

$$\lambda_s(x_0) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \left|\frac{\partial \tilde{T}_{a,\mu}(x_j, \xi_j(\omega))}{\partial x_0}\right|$$

And since $\left|\frac{\partial \tilde{T}_{a,\mu}(x_j, \xi_j(\omega))}{\partial x_0}\right| > 1$ then $\lambda_s(x_0) > 0$ and the system is noisy chaotic.

Example 3.1: Let $\xi_t(\omega) \sim Gauss(0, 0.0025)$, for system (4)

Case 1: let $\mu = 0.5$, $a = 0.6$ then by theorem (3.1)-part 1, the system has one random fixed point $p(\omega) \sim Gauss(0, 0.0025)$ and $P_r(p(\omega)$ *is attracting*$) = P_r(\xi(\omega) < 0.6) = P_r(z(\omega) < 12) \cong 1$, where $z(\omega) \sim Gauss(0, 1)$, and the SLE of the system is about $\lambda_s = -0.1822$.

Case 2: let $\mu = a = 0.7$ then by theorem (3.1)-part 2 , then the system has infinity number of random fixed points $p(\omega) \sim Gauss(p^*, 0.0025)$, for all points $p^* \in [0, a]$, and the SLE of the system is about $\lambda_s = -4.0547 \times 10^{-5}$.

Case 3: let $\mu = 0.5$, $a = 0.4$ then by theorem (3.1)-part 3, the system has two random fixed points $p_1(\omega) \sim Gauss(0, 0.0025)$, $p_2(\omega) \sim Gauss(0.4545, 0.0025)$, and $P_r(p_1(\omega)$ *is attracting*$) = P_r(\xi(\omega) > 0.4) = P_r(z(\omega) > 8) \cong 0$, $P_r(p_2(\omega)$ *is attracting*$) = P_r(\xi(\omega) > -0.0545) = P_r(z(\omega) > -1.09) = 0.86214$, and the SLE of the system is about $\lambda_s = -0.1822$.

Case 4: let $= 0.7$, $a = 0.4$, then by theorem (3.1)-part 4, all fixed or periodic points in the system are replying with probability 1 , and the SLE of the system is about $\lambda_s = 0.2691$.
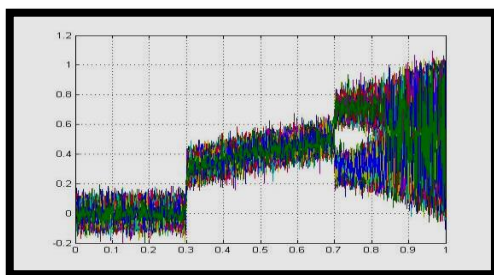
Not that Stochastic Lyapunov exponent of the system (4) and deterministic Lyapunov exponent of the system (1) are the same, because of the partial derivative of system (4) is not dependent on $x_t(\omega)$. The satisfaction of the above properties can be seen from the bifurcation diagrams figure (7) .
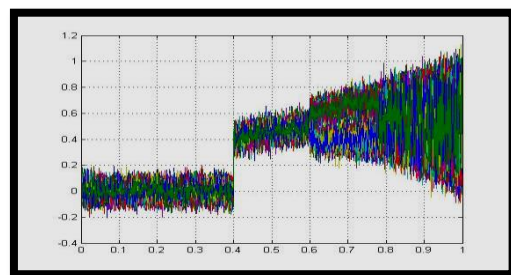

(a)

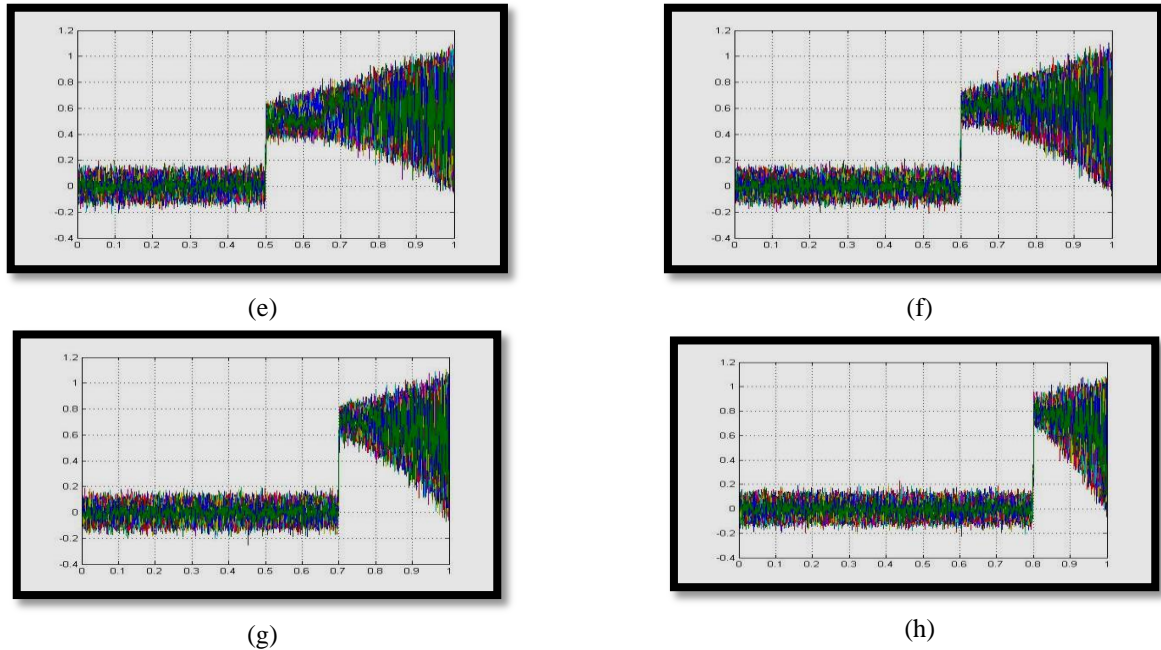(b)

(c)

(d)

(e)

(f)

(g)

(h)
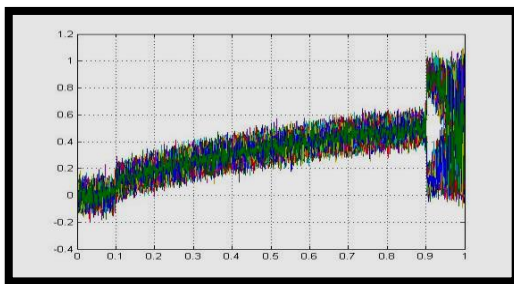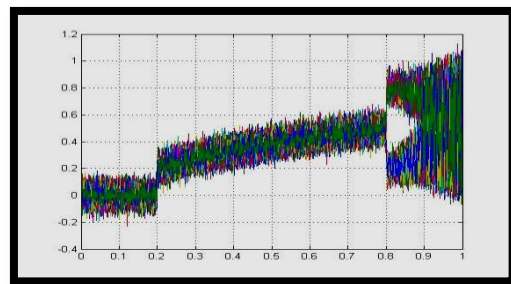
Fig. 7: bifurcation diagram of noisy asymmetric tent map family with additive noise Gauss (0 , 0.0025) , x-axes represent the values of μ and y-axes represent the values of $x_{t+1(\omega)}$, (a) $a = 0.1$, (b) $a = 0.2$, (c) $a = 0.3$,(d) $a = 0.4$, (e) $a = 0.5$, (f) $a = 0.6$, (g) $a = 0.7$, (h) $a = 0.8$.

3.5. The family of noisy asymmetric logistic map.

The noisy dynamical system for asymmetric logistic map with additive noise is given by.

$$x_{t+1}(\omega) = \tilde{G}_{a,\mu}(x_t , \xi_t(\omega)) = \begin{cases} \mu \frac{x_t}{2a}\left(1 - \frac{x_t}{2a}\right) + \xi_t(\omega) & if\ 0 \le x_t \le a \\ \mu\left(\frac{x_t+1-2a}{2-2a}\right)\left(1 - \left(\frac{x_t+1-2a}{2-2a}\right)\right) + \xi_t(\omega) & if\ a < x_t \le 1 \end{cases} \tag{5}$$

Where $\tilde{G}_{a,\mu} \colon \mathbb{R} \times \Omega \to \mathbb{R}$, $\{\xi_t(\omega)\}$ is a sequence of iid random variables and $0 \le a \le 1$, $0 \le \mu \le 4$.

Theorem 3.2: *For the dynamical system (5), where $\xi_t(\omega) \sim Gauss(0, \sigma^2)$* .

1) If $\mu < 2a$, then the system has one random fixed point $p_1(\omega) \sim Gauss(0, \sigma^2)$, and $P_r(p_1(\omega)\ is\ attracting) = P_r(\xi(\omega) < a) * P_r\left(a - \frac{2a^2}{\mu} < p_1(\omega) < a + \frac{2a^2}{\mu}\right) + P_r(\xi(\omega) > a) * P_r\left(a - \frac{(2-2a)^2}{\mu} < p_1(\omega) < a + \frac{(2-2a)^2}{\mu}\right)$.

2) If $2a < \mu < 4a$ then the system has another random fixed point $p_2(\omega) \sim Gauss\left(\frac{2a\mu-4a^2}{\mu}, \sigma^2\right)$, where $P_r(p_2(\omega)\ is\ attracting) = P_r\left(\xi(\omega) < a - \frac{2a\mu-4a^2}{\mu}\right) * P_r\left(a - \frac{2a^2}{\mu} < p_2(\omega) < a + \frac{2a^2}{\mu}\right) + P_r\left(\xi(\omega) > a - \frac{2a\mu-4a^2}{\mu}\right) * P_r\left(a - \frac{(2-2a)^2}{\mu} < p_2(\omega) < a + \frac{(2-2a)^2}{\mu}\right)$.

3) If $4a < \mu$ then the second random fixed point

$$p_2(\omega) \sim Gauss\left(\frac{\left(\frac{2a\mu}{(2-2a)^2}-1\right) - \sqrt{\left(\frac{2a\mu}{(2-2a)^2}-1\right)^2 + \left(\frac{4(\mu^2-2a\mu^2)}{(2-2a)^4}\right)}}{\frac{2\mu}{(2-2a)^2}}, \sigma^2\right)$$

Is attracting with probability

$$P_r(p_2(\omega)\ is\ attracting\ ) = P_r\left(\xi(\omega) < a - \frac{\left(\frac{2a\mu}{(2-2a)^2}-1\right) - \sqrt{\left(\frac{2a\mu}{(2-2a)^2}-1\right)^2 + \left(\frac{4(\mu^2-2a\mu^2)}{(2-2a)^4}\right)}}{\frac{2\mu}{(2-2a)^2}}\right) * P_r\left(a - \frac{2a^2}{\mu} < p_2(\omega) < a + \right.$$

$$\left. \frac{2a^2}{\mu}\right) + P_r\left(\xi(\omega) > a - \frac{\left(\frac{2a\mu}{(2-2a)^2}-1\right) - \sqrt{\left(\frac{2a\mu}{(2-2a)^2}-1\right)^2 + \left(\frac{4(\mu^2-2a\mu^2)}{(2-2a)^4}\right)}}{\frac{2\mu}{(2-2a)^2}}\right) * P_r\left(a - \frac{(2-2a)^2}{\mu} < p_2(\omega) < a + \frac{(2-2a)^2}{\mu}\right) .$$

Proof: *Clear, by definition 1.2 and lemma 1.1.*
Example 3.2: Let $\xi_t(\omega) \sim Gauss(0 , 0.0025)$, for system (4)

<u>Case 1</u>: let $= 1$, $a = 0.6$, by theorem (3.2)-part 1 , the system has one random fixed point $p_1(\omega) \sim Gauss(0, 0.0025)$, and it is attracting with probability $P_r(p(\omega)\, is\, attracting) = P_r(\xi(\omega) < 0.6) * P_r(-0.12 < p_1(\omega) < 1.32) + P_r(\xi(\omega) > 0.6) * P_r(-0.04 < p_1(\omega) < 1.24) = P_r(z(\omega) < 12) * P_r(-2.4 < z(\omega) < 26.4) + P_r(z(\omega) > 12) * P_r(-0.8 < z(\omega) < 24.8) \cong 0.9918$, where $z(\omega) \sim Gauss(0,1)$, and the SLE of system is about $\lambda_s = -0.1863$.

<u>Case 2</u> : let $\mu = 2$, $a = 0.6$ by theorem (3.2)-part 2 , the system has two random fixed point $p_1(\omega) \sim Gauss(0,0.0025)$, $p_2(\omega) \sim Gauss(0.48, 0.0025)$, and $P_r(p_1(\omega)\, is\, attracting) = P_r(\xi(\omega) < 0.6) * P_r(0.24 < p_1(\omega) < 0.96) + P_r(\xi(\omega) > 0.6) * P_r(0.28 < p_1(\omega) < 0.92) = P_r(z(\omega) < 12) * P_r(4.8 < z(\omega) < 19.2) + P_r(\xi(\omega) > 12) * P_r(5.6 < z(\omega) < 18.4) \cong 0$, $P_r(p_2(\omega)\, is\, attracting) = P_r(\xi(\omega) < 0.12) * P_r(0.24 < p_2(\omega) < 0.96) + P_r(\xi(\omega) > 0.12) * P_r(0.28 < p_2(\omega) < 0.92) = P_r(z(\omega) < 2.4) * P_r(-4.8 < z(\omega) < 9.6) + P_r(z(\omega) > 2.4) * P_r(-4 < p_2(\omega) < 8.8) \cong 1$, where $z(\omega) \sim Gauss(0,1)$ and the SLE of the system is about $\lambda_s = -1.2356$.

<u>Case 3</u>: let $= 2.5$, $a = 0.6$, by theorem (3.2)-part 3 , the second fixed point $p_2(\omega) \sim Gauss(0.624, 0.0025)$, and $P_r(p_2(\omega)\, is\, attracting\,) = P_r(\xi(\omega) < -0.024) * P_r(0.312 < p_2(\omega) < 0.888) + P_r(\xi(\omega) > -0.024) * P_r(0.344 < p_2(\omega) < 0.856) = P_r(z(\omega) < -0.48) * P_r(-6.24 < z(\omega) < 5.28) + P_r(z(\omega) > -0.48) * P_r(-5.6 < z(\omega) < 4.64) \cong 1$, where $z(\omega) \sim Gauss(0,1)$, and the SLE of the system is about $\lambda_s = -1.7190$.

In this example the approximate value of $\mu_{nc} = 3.523$.
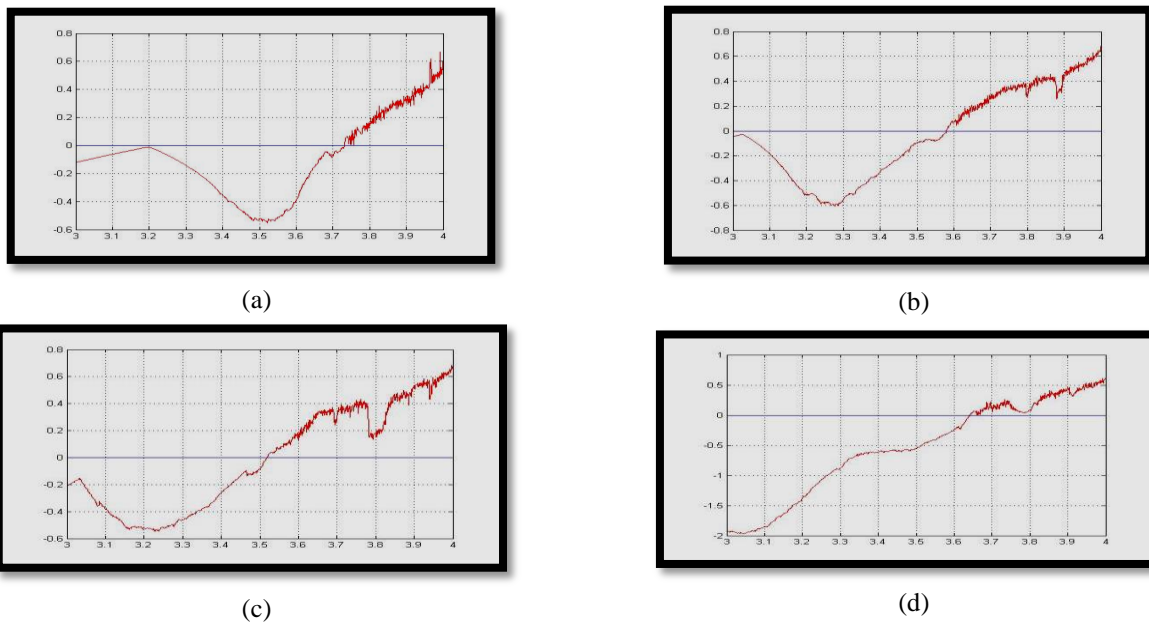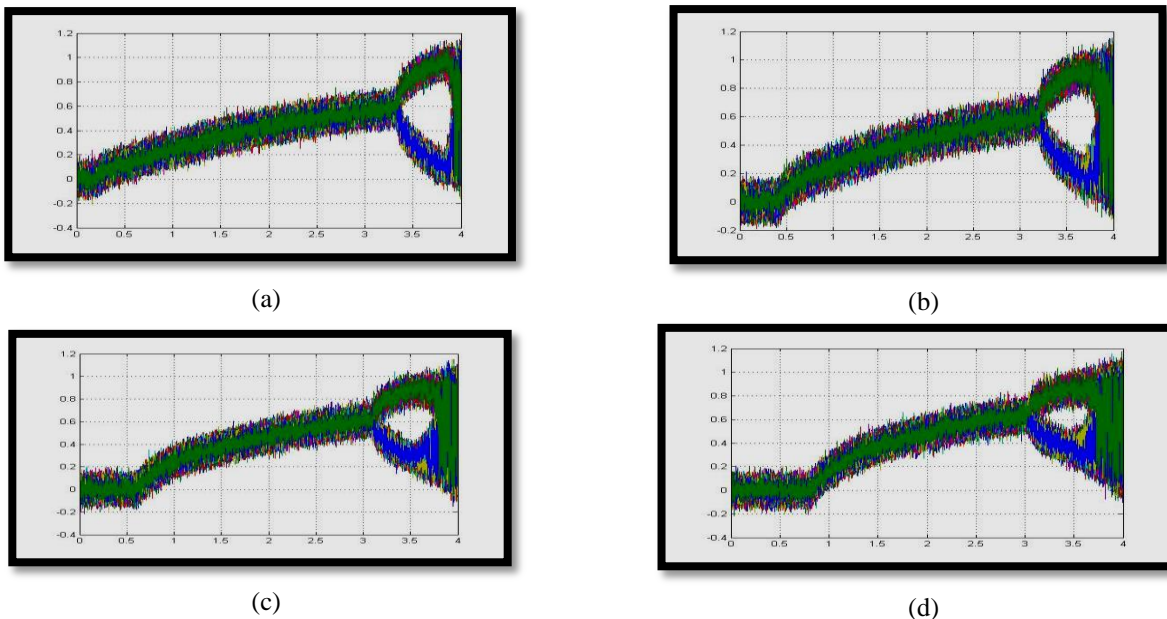

(a)

(b)

(c)

(d)

Fig. 8: S.L.E. for noisy asymmetric logistic map with additive Gauss (0 , 0.0025) noise , x_ axes represent the values of μ and y_ axes represent the values of S.L.E. , (a) $a = 2$, (b) $a = 0.4$, (c) $a = 0.6$, (d) $a = 0.8$.

The satisfaction of the above properties can be seen from the bifurcation diagrams figure (9).
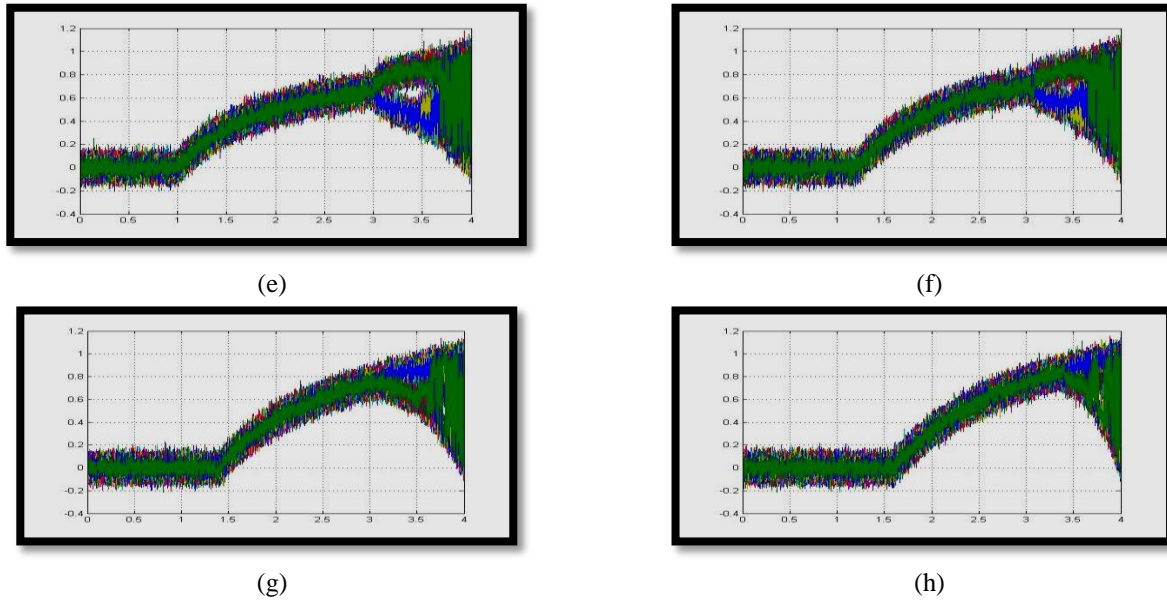

(a)

(b)

(c)

(d)

(e)   (f)

(g)   (h)

Fig. 9: bifurcation diagram of noisy asymmetric logistic map family with additive noise Gauss (0 , 0.0025) , x-axes represent the values of μ and y-axes represent the values of $x_{t+1}(\omega)$ (a) $a = 0.1$, (b) $a = 0.2$, (c) $a = 0.3$,(d) $a = 0.4$, (e) $a = 0.5$, (f) $a = 0.6$,(g) $a = 0.7$, (h) $a = 0.8$.

3.3. The family of noisy mixed logistic-tent map.

The noisy dynamical system for mixed logistic-tent map with additive noise is given by:

$$x_{t+1}(\omega) = \widetilde{GT}_{a,\mu}\big(x_t \,, \xi_t(\omega)\big) = \begin{cases} \mu \frac{x_t}{2a}\left(1 - \frac{x_t}{2a}\right) + \xi_t(\omega) & if\ 0 \le x_t \le a \\ \mu\left(\frac{1-x_t}{4(1-a)}\right) + \xi_t(\omega) & if\ a < x_t \le 1 \end{cases} \tag{6}$$

Where $\widetilde{GT}_{a,\mu}: \mathbb{R} \times \Omega \ \to \mathbb{R}$, $\{\xi_t(\omega)\}$ sequence of (iid) random variables and $0 \le \ a \ \le 1$, $0 \le \mu \le 4$

Theorem 3.3: *For the dynamical system (6), if $\xi_t(\omega) \sim Gauss(0, \sigma^2)$.*
1) If $\mu < 2a$, then the system has one random fixed point $p_1(\omega) \sim Gauss(0, \sigma^2)$, and $P_r(p_1(\omega)\ is\ attracting) = $
    $P_r(\xi(\omega) < a) * P_r\left(a - \frac{2a^2}{\mu} < p_1(\omega) < a + \frac{2a^2}{\mu}\right) + P_r(\xi(\omega) > a)$, if $< 4(1 - a)$, and if $\mu > 4(1 - a)$ then
    $P_r(p_1(\omega)\ is\ attracting) = P_r(\xi(\omega) < a) * P_r\left(a - \frac{2a^2}{\mu} < p_1(\omega) < a + \frac{2a^2}{\mu}\right)$.

2) If $2a < \mu \le 4a$ then the system has another random fixed points $p_2(\omega) \sim Gauss\left(\frac{2a\mu - 4a^2}{\mu}, \sigma^2\right)$, and
    $P_r(p_2(\omega)\ is\ attracting) = P_r\left(\xi(\omega) < a - \frac{2a\mu - 4a^2}{\mu}\right) * P_r\left(a - \frac{2a^2}{\mu} < p_2(\omega) < a + \frac{2a^2}{\mu}\right) + P_r\left(\xi(\omega) > a - \frac{2a\mu - 4a^2}{\mu}\right)$, if $\mu < 4(1 - a)$ and if $\mu > 4(1 - a)$, then $P_r(p_2(\omega)\ is\ attracting) = P_r\left(\xi(\omega) < a - \frac{2a\mu - 4a^2}{\mu}\right) * P_r\left(a - \frac{2a^2}{\mu} < p_2(\omega) < a + \frac{2a^2}{\mu}\right)$.

3) If $4a < \mu < 4(1 - a)$ then the second random fixed point $p_2(\omega) \sim Gauss\left(\frac{\mu}{\mu + 4(1-a)}, \sigma^2\right)$ is attracting with
    probability $P_r(p_2(\omega)\ is\ attracting) = P_r\left(\xi(\omega) < a - \frac{\mu}{\mu + 4(1-a)}\right) * P_r\left(a - \frac{2a^2}{\mu} < p_2(\omega) < a + \frac{2a^2}{\mu}\right) + P_r\left(\xi(\omega) > a - \frac{\mu}{\mu + 4(1-a)}\right)$, and $P_r(p_2(\omega)\ is\ attracting) = P_r\left(\xi(\omega) < a - \frac{\mu}{\mu + 4(1-a)}\right) * P_r\left(a - \frac{2a^2}{\mu} < p_2(\omega) < a + \frac{2a^2}{\mu}\right)$ if $\mu > \max\{4a\, , 4(1 - a)\}$.

Proof: *The proofs of (1) and (3) is in the similar way of (2)*
To prove (2), let $p_2(\omega) = \widetilde{GT}_{a,\mu}\big(p \,, \xi(\omega)\big) = \mu \frac{p}{2a}\left(1 - \frac{p}{2a}\right) + \xi(\omega) = \frac{2a\mu - 4a^2}{\mu} + \xi(\omega)$, since

$2a < \mu \le 4a$ Implies $p_2(\omega) \sim Gauss\left(\frac{2a\mu - 4a^2}{\mu}, \sigma^2\right)$.
Now to find probability of $p_2(\omega)$ is attracter
$P_r(p_2(\omega)\ is\ attracting) = P_r(p_2(\omega) < a) * P_r(p_2(\omega)\ is\ attracting \mid p_2(\omega) < a) + P_r(p_2(\omega) > a) * P_r(p_2(\omega)\ is\ attracting \mid p_2(\omega) > a)$
We know that if $\mu < 4(1 - a)$ then $P_r(p_2(\omega)\ is\ attracting \mid p_2(\omega) > a) = 1$
Then if $\mu < 4(1 - a)$

$P_r(p_2(\omega) \text{ is attracting }) = P_r\left(\xi(\omega) < a - \frac{2a\mu - 4a^2}{\mu}\right) * P_r\left(a - \frac{2a^2}{\mu} < p_2(\omega) < a + \frac{2a^2}{\mu}\right) + P_r\left(\xi(\omega) > \quad a - \frac{2a\mu - 4a^2}{\mu}\right)$, by lemma 1 and if $\mu > 4(1-a)$ $P_r(p_2(\omega) \text{ is attracting } | p_2(\omega) > a) = 0$ and hence

$P_r(p_2(\omega) \text{ is attracting })= P_r\left(\xi(\omega) < a - \frac{2a\mu - 4a^2}{\mu}\right) * P_r\left(a - \frac{2a^2}{\mu} < p_2(\omega) < a + \frac{2a^2}{\mu}\right)$

Example 3.3: Let $\xi_t(\omega) \sim Gauss(0, 0.0025)$, for system (6)

<u>Case 1</u>: let $= 1$, $a = 0.6$, by theorem (3.3)-part 1, the system has on random fixed point $p(\omega) \sim Gauss(0, 0.0025)$, and it is attracting with probability $P_r(p(\omega) \text{is attracting}) = P_r(\xi(\omega) < 0.6) * P_r(-0.12 < p_1(\omega) < 1.32) + P_r(\xi(\omega) > a) = P_r(z(\omega) < 12) * P_r(-2.4 < z(\omega) < 26.4) + P_r(z(\omega) > 12) \cong 0.9918$, where $z(\omega) \sim Gauss(0,1)$ and the SLE of system is about $\lambda_s = -0.1869$.

<u>Case 2</u>: let $\mu = 2$, $a = 0.6$ by theorem (3.3)-part 2, the system has another random fixed point $p_2(\omega) \sim Gauss(0.48, 0.0025)$, and $P_r(p_2(\omega) \text{is attracting}) = P_r(\xi(\omega) < 0.12) * P_r(0.24 < p_2(\omega) < 0.96) = P_r(z(\omega) < 2.4) * P_r(-4.8 < z(\omega) < 9.6) \cong 0.9918$, where $z(\omega) \sim Gauss(0,1)$ and the SLE of the system is about $\lambda_s = -1.2148$.

<u>Case 3</u>: let $= 2.5$, $a = 0.6$, by theorem (3.3)-part 3 in above the second random fixed point $p_2(\omega) \sim Gauss(0.6098, 0.0025)$, and $P_r(p_2(\omega) \text{is attracting}) = P_r(\xi(\omega) < -0.0098) * P_r(0.312 < p_2(\omega) < 0.888) = P_r(z(\omega) < -0.196) * P_r(-5.956 < z(\omega) < 5.564) = 0.422695$, and the SLE of the system is about $\lambda_s = -0.9515$.

In this example the approximate value of $\mu_{nc} = 3.1836$.

The satisfaction of the above properties can be seen from the bifurcation diagrams figure (11).
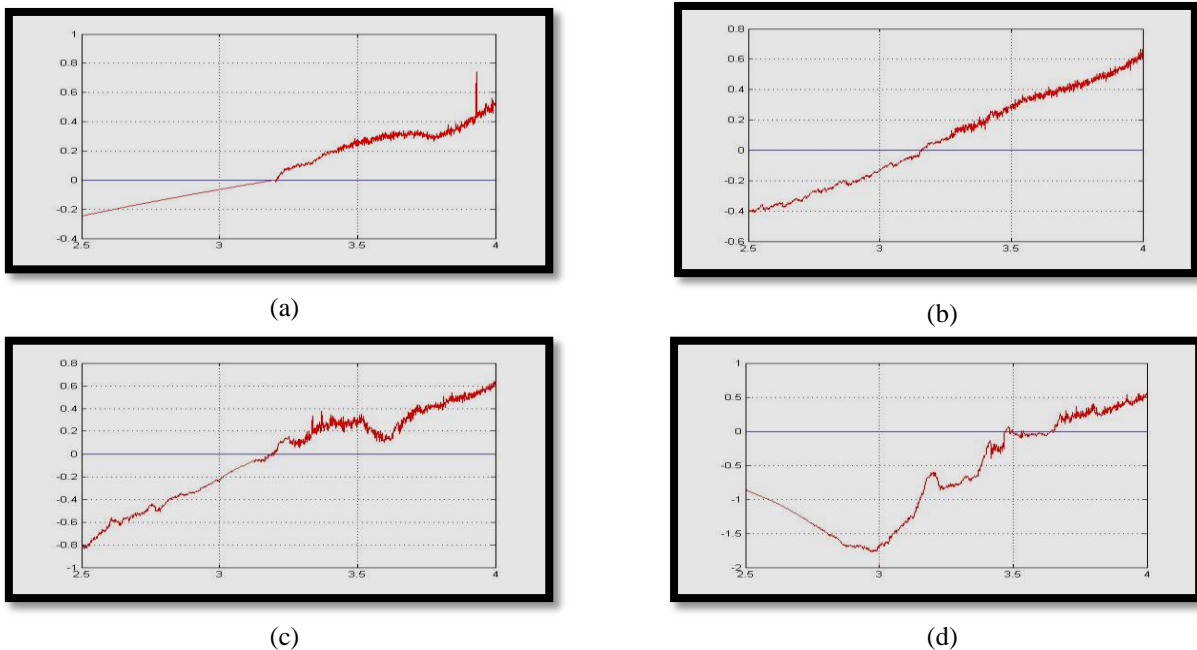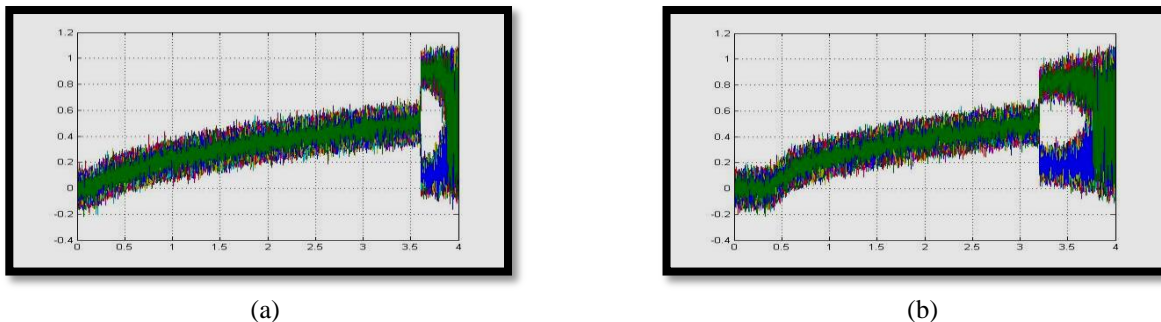

(a)

(b)

(c)

(d)

Fig. 10: SLE for noisy mixed logistic-tent map with additive Gauss (0 , 0.0025) noise , x_ axes represent the values of value of μ and y_ axes represent the values value of S.L.E. , (a)$a = 2$, (b) $a = 0.4$, (c) $a = 0.6$, (d) $a = 0.8$.

(a)

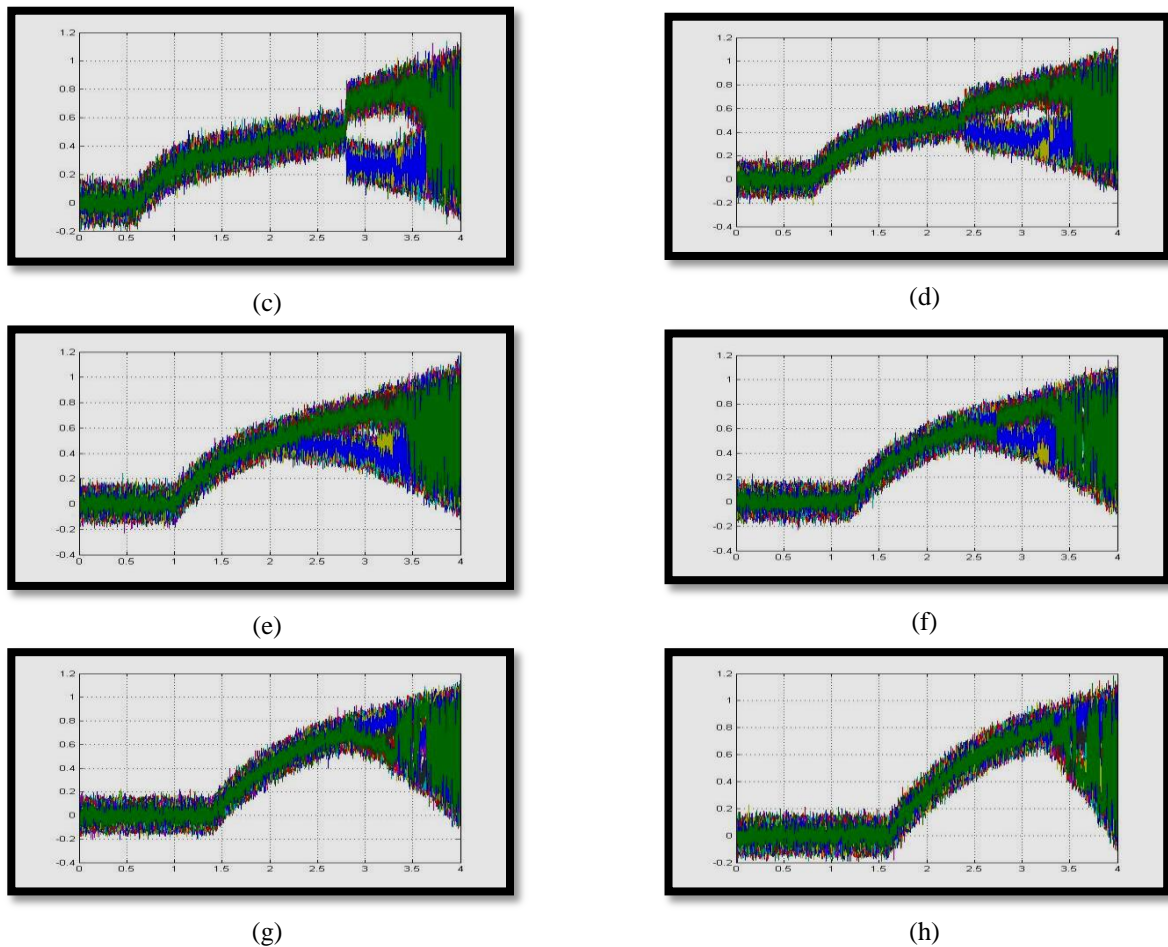

(b)

(c)

(d)

(e)

(f)

(g)

(h)

Fig. 11: bifurcation diagram of noisy mixed logistic-tent map family with additive noise Gauss (0 , 0.0025) , x-axes represent the values of μ and y-axes represent the values of $x_{t+1}(\omega)$, (a) $a = 0.1$, (b) $a = 0.2$, (c) $a = 0.3$,(d) $a = 0.4$, (e) $a = 0.5$, (f) $a = 0.6$, (g) $a = 0.7$, (h) $a = 0.8$.

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