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Research Paper

Partial generalized probability weighted moments for exponentiated exponential distribution

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Abstract

The generalized probability weighted moments are widely used in hydrology for estimating parameters of flood distributions from complete sample. The method of partial generalized probability weighted moments was used to estimate the parameters of distributions from censored sample. This article offers new method called partial generalized probability weighted moments (PGPWMs) for the analysis of censored data. The method of PGPWMs is an extended class from partial generalized probability weighted moments. To illustrate the new method, estimation of the unknown parameters from exponentiated exponential distribution based on doubly censored sample is considered. PGPWMs estimators for right and left censored samples are obtained as special cases. Simulation study is conducted to investigate performance of estimates for exponentiated exponential distribution. Comparison between estimators is made through simulation via their biases and mean square errors. An illustration with real data is provided.

Keywords: *Generalized Probability Weighted Moments, Partial Probability Weighted Moments, Partial Generalized Probability Weighted Moments, Generalized Exponential Distribution, and Censored Samples.*

1. Introduction

[1] Introduced the generalized exponential distribution which, also known as exponentiated exponential (EE) distribution. It is observed that the EE distribution can be considered for situation where a skewed distribution for a non- negative random variable is needed. Also, it is observed that it can use quite effectively to analyze lifetime data in place of gamma, Weibull and log-normal distributions.
The probability density function of the two parameter EE distribution is given by;

$$f\left(x;\alpha,\lambda\right)=\alpha\lambda\left(1-e^{-\lambda x}\right)^{\alpha-1}e^{-\lambda x}, \qquad \alpha,\lambda,x>0$$

The corresponding cumulative distribution function is given by:

$$F\left(x;\alpha,\lambda\right)=\left(1-e^{-\lambda x}\right)^{\alpha}, \qquad \alpha,\lambda,x>0 \tag{1}$$

The probability weighted moments are useful for estimating the unknown parameters and quantiles of a distribution from complete sample. [2] Introduced extended class of probability weighted moments which called partial probability weighted moments (PPWMs) to estimate the parameters of a distribution from censored sample. [2], [3] and [4] originally introduced that concept for the purpose of estimating the upper quantiles of flood flows when one's interest is in the right tail of the distribution and there is some benefit to censoring some of the smaller observations in the left tail. The initial idea of PPWMs was to remove the smaller observations from the process of fitting the distribution because such observations are of little interest in flood frequency analysis and such observations can a nuisance to the fitting process. The method is computationally simple, robust and possesses the same merit as the original method of probability weighted moments.
[5] introduced the evaluation of L- moments for a left censored observation from PPWMs. [6] presented a new distribution for estimating quantile function of a non-negative random variable using a censored sample of data, which is based on the principle of partial maximum entropy in which PPWMs are used as constraints.

This article presents new method which is called partial generalized probability weighted moments for estimating the unknown parameters of distributions from censored samples. The PGPWMs will be used to estimate unknown parameters of EE distribution under doubly censored samples. Then, the PGPWMs estimators of the unknown parameters from left and right censored samples will be obtained as special cases. At the same time, the generalized probability weighted moments can be obtained as the special case from PGPWMs. To illustrate the properties of the new estimators, an extensive numerical study will be performed. Analysis of a real data set has been performed.

This article is organized as follows. In Section 2, the definition of PGPWMs and their sample estimators from doubly, right, and left censored samples is introduced. Section 3 discusses the PGPWMs method of estimation for the EE distribution from doubly, left and right censored samples. A simulation study is performed to investigate the properties of the new estimators from censored samples with two special cases in Section 4. Real data example is given in Section 5. Finally, the conclusions are included in Section 6. Tables and some Figures are included in the appendix.

2. The PGPWMs and their sample estimators

The generalized probability weighted moments (GPWMs) was introduced by [7] as a tool for estimating the parameters of probability distribution, for complete sample, expressible in inverse form. The GPWMs can be defined as

$$M_{p,u,v} = E\left(X^{p}\{F(X)\}^{u}\{1-F(X)\}^{v}\right) = \int_{0}^{1}\{x(F)\}^{p}F^{u}(1-F)^{v}\,dF\ , \tag{2}$$

where X is a random variable and $F(x)$ is cumulative distribution function.

We introduce the method of PGPWMs as an extended class of GPWMs and PPWMs methods. The general form of PGPWMs with double bound censoring for the random variable X is defined as follows:

$$W_{p,u,v} = E\left(X^{p}\{F(X)\}^{u}\{1-F(X)\}^{v}\right) = \int_{c}^{d}\{x(F)\}^{p}F^{u}(1-F)^{v}\,dF\ , \tag{3}$$

where $p=1$ u and v to be real values (i.e. exponents u and v of the PGPWMs can be ratio or integers). Also, $c = F(x_{01})$ and $d = F(x_{02})$, x_{01} and x_{02} are lower and upper bound censoring respectively; Therefore, for $p=1$, and $v=0$, $W_{p,r,s}$ becomes;

$$W_{1,u,0} = E\left(X\ \{F(X)\}^{u}\right) = \int_{c}^{d}\{x(F)\}F^{u}dF\ , \tag{4}$$

It is noted when $d=1$ and the lower bound $c=0$ then the PGPWMs with double bound censoring is reduced to GPWMs in complete samples as defined in (2).

The PGPWMs with lower bound (left) censoring, denoted by $LW_{1,u,0}$, can be obtained as a special case from (4) by putting $d=1$. Therefore, the general form of PGPWMs with left censoring for the random variable X takes the following form

$$LW_{1,u,0} = E\left(X\ \{F(X)\}^{u}\right) = \int_{c}^{1}\{x(F)\}F^{u}dF\ . \tag{5}$$

The PGPWMs with upper bound (right) censoring, denoted by $RW_{1,u,0}$, can be defined as a special case from (4) by putting $c=0$. Therefore, the general form of PGPWMs with right censoring for the random variable X takes the following form

$$RW_{1,u,0} = E\left(X\ \{F(X)\}^{u}\right) = \int_{0}^{d}\{x(F)\}F^{u}dF\ . \tag{6}$$

Let $x_{(1)}, x_{(2)}, ..., x_{(n)}$ be a random sample of size n from the distribution function $F(X)$, and $x_{(1)} < x_{(2)}, ... < x_{(n)}$ be the corresponding ordered sample. According to [8] the estimates of $W_{1,u,0}$, $LW_{1,u,0}$ and $RW_{1,u,0}$ is defined as the following

$$\hat{W}_{1,u,0} = n^{-1}\sum_{i} x_{(i)}\left(\frac{i-0.35}{n}\right)^{u}\ , \tag{7}$$

$$L\hat{W}_{1,u,0} = n^{-1}\sum_{i} x_{(i)}\left(\frac{i-0.35}{n}\right)^{u}\ , \tag{8}$$

$$R\hat{W}_{1,u,0} = n^{-1}\sum_{i} x_{(i)}\left(\frac{i-0.35}{n}\right)^{u}\ . \tag{9}$$

Note that, the PGPWM estimators with doubly, left or right censoring is obtained by equating the theoretical PGPWM with the corresponding sample PGPWM.

3. PGPWMs estimation of the EE

In this section, the PGPWMs estimation method will be used to estimate the unknown parameters from EE distribution with doubly censored samples. Furthermore, the PGPWMs estimators with left and right censored samples are obtained as special cases.

3.1. PGPWMs estimation of the EE from doubly censored sample

The theoretical PGPWMs with double bound censored, $W_{1,u,0}$ for the EE distribution can be found from (4) after setting $u = u_1$ and $u = u_2$ as the following;

$$W_{1,u_1,0} = \int_c^d \{x(F)\}F^{u_1}dF = \frac{1}{\lambda}\sum_{j=1}^{\infty}\left[\frac{d^{\frac{j}{\alpha}+u_1+1} - c^{\frac{j}{\alpha}+u_1+1}}{j(\frac{j}{\alpha}+u_1+1)}\right] \tag{10}$$

$$W_{1,u_2,0} = \int_c^d \{x(F)\}F^{u_2}dF = \frac{1}{\lambda}\sum_{j=1}^{\infty}\left[\frac{d^{\frac{j}{\alpha}+u_2+1} - c^{\frac{j}{\alpha}+u_2+1}}{j(\frac{j}{\alpha}+u_2+1)}\right] \tag{11}$$

From Equations (10) and (11), the two parameters, α and λ, can be expressed as follows:

$$\lambda = \frac{\sum_{j=1}^{\infty}\left[\frac{d^{\frac{j}{\alpha}+u_1+1} - c^{\frac{j}{\alpha}+u_1+1}}{j(\frac{j}{\alpha}+u_1+1)}\right]}{W_{1,u_1,0}}. \tag{12}$$

$$W_{1,u_2,0}\sum_{j=1}^{\infty}\left[\frac{d^{\frac{j}{\alpha}+u_1+1} - c^{\frac{j}{\alpha}+u_1+1}}{j(\frac{j}{\alpha}+u_1+1)}\right] - W_{1,u_1,0}\sum_{j=1}^{\infty}\left[\frac{d^{\frac{j}{\alpha}+u_2+1} - c^{\frac{j}{\alpha}+u_2+1}}{j(\frac{j}{\alpha}+u_2+1)}\right] = 0. \tag{13}$$

The PGPWMs estimators of α and λ from doubly censored samples, can be obtained by solving equations (12) and (13) in terms of α and λ; where $W_{1,u_1,0}$ and $W_{1,u_2,0}$ are replaced by their sample estimators, $\hat{W}_{1,u_1,0}$ and $\hat{W}_{1,u_2,0}$ given by Equation (7) by setting $u = u_1, u_2$, therefore

$$\hat{\lambda} = \frac{\sum_{j=1}^{\infty}\left[\frac{d^{\frac{j}{\hat{\alpha}}+u_1+1} - c^{\frac{j}{\hat{\alpha}}+u_1+1}}{j(\frac{j}{\hat{\alpha}}+u_1+1)}\right]}{\hat{W}_{1,u_1,0}}. \tag{14}$$

$$\hat{W}_{1,u_2,0}\sum_{j=1}^{\infty}\left[\frac{d^{\frac{j}{\hat{\alpha}}+u_1+1} - c^{\frac{j}{\hat{\alpha}}+u_1+1}}{j(\frac{j}{\hat{\alpha}}+u_1+1)}\right] - \hat{W}_{1,u_1,0}\sum_{j=1}^{\infty}\left[\frac{d^{\frac{j}{\hat{\alpha}}+u_2+1} - c^{\frac{j}{\hat{\alpha}}+u_2+1}}{j(\frac{j}{\hat{\alpha}}+u_2+1)}\right] = 0. \tag{15}$$

Solving numerically Equation (15) based on some iteration technique to obtain $\hat{\alpha}$. Once the estimate of α is determined, an estimate of λ is obtained by substitute $\hat{\alpha}$ in Equation (14).
Some special cases can be obtained from the above the PGPWMs

i. For $u_1 = 0$ and $u_2 = 1$, the PPWMs can be obtained from PGPWMs with doubly censored as obtained by[9].
ii. For $c = 0$ and $d = 1$, the GPWMs can be obtained from PGPWMs with doubly censored as obtained by [10].
iii. For $u_1 = 0$, $u_2 = 1$, $c = 0$ and $d = 1$, the PWMs can be obtained from PGPWMs with doubly censored as obtained by [11]

3.2. PGPWMs estimation of the EE from left censored sample

The theoretical PGPWMs with left bound censored, $LW_{1,u,0}$ for the EE distribution can be found from (5) after setting $u = u_1$ and $u = u_2$ as the following;

$$LW_{1,u_1,0} = \int_c^1 \{x(F)\} F^{u_1} dF \quad = \frac{1}{\lambda} \sum_{j=1}^{\infty} \left[\frac{1 - c^{\frac{j}{\alpha} + u_1 + 1}}{j(\frac{j}{\alpha} + u_1 + 1)} \right] \tag{16}$$

$$LW_{1,u_2,0} = \int_c^1 \{x(F)\} F^{u_2} dF \quad = \frac{1}{\lambda} \sum_{j=1}^{\infty} \left[\frac{1 - c^{\frac{j}{\alpha} + u_2 + 1}}{j(\frac{j}{\alpha} + u_2 + 1)} \right] \tag{17}$$

From Equations (16) and (17), the two parameters, α and λ, can be expressed as follows:

$$\lambda = \frac{\sum_{j=1}^{\infty} \left[\frac{1 - c^{\frac{j}{\alpha} + u_1 + 1}}{j(\frac{j}{\alpha} + u_1 + 1)} \right]}{LW_{1,u_1,0}} . \tag{18}$$

$$LW_{1,u_2,0} \sum_{j=1}^{\infty} \left[\frac{1 - c^{\frac{j}{\alpha} + u_1 + 1}}{j(\frac{j}{\alpha} + u_1 + 1)} \right] - LW_{1,u_1,0} \sum_{j=1}^{\infty} \left[\frac{1 - c^{\frac{j}{\alpha} + u_2 + 1}}{j(\frac{j}{\alpha} + u_2 + 1)} \right] = 0 . \tag{19}$$

To compute the PGPWMs estimators of α and λ, denoted by $\hat{\alpha}_L$ and $\hat{\lambda}_L$, from left censored samples, $LW_{1,u_1,0}$ and $LW_{1,u_2,0}$ are replaced by their sample estimators, $L\hat{W}_{1,u_1,0}$ and $L\hat{W}_{1,u_2,0}$ by setting $u = u_1, u_2$ in Equation (8), hence

$$\hat{\lambda}_L = \frac{\sum_{j=1}^{\infty} \left[\frac{1 - c^{\frac{j}{\hat{\alpha}_L} + u_1 + 1}}{j(\frac{j}{\hat{\alpha}_L} + u_1 + 1)} \right]}{L\hat{W}_{1,u_1,0}} , \tag{20}$$

$$L\hat{W}_{1,u_2,0} \sum_{j=1}^{\infty} \left[\frac{1 - c^{\frac{j}{\hat{\alpha}_L} + u_1 + 1}}{j(\frac{j}{\hat{\alpha}_L} + u_1 + 1)} \right] - L\hat{W}_{1,u_1,0} \sum_{j=1}^{\infty} \left[\frac{1 - c^{\frac{j}{\hat{\alpha}_L} + u_2 + 1}}{j(\frac{j}{\hat{\alpha}_L} + u_2 + 1)} \right] = 0 \tag{21}$$

Solving numerically by iteration Equation (20) to obtain $\hat{\alpha}_L$. Once the estimate of α is determined, an estimate of λ is obtained by substitute $\hat{\alpha}_L$ in Equation (21).

3.3 PGPWMs estimation of the EE from right censored sample

The theoretical PGPWMs with right bound censored, $RW_{1,u,0}$ for the EE distribution can be obtained from (6) after setting $u = u_1$ and $u = u_2$ as the following

$$RW_{1,u_1,0} = \int_0^d \{x(F)\} F^{u_1} dF \quad = \frac{1}{\lambda} \sum_{j=1}^{\infty} \left[\frac{d^{\frac{j}{\alpha} + u_1 + 1}}{j(\frac{j}{\alpha} + u_1 + 1)} \right] \tag{22}$$

$$RW_{1,u_2,0} = \int_0^d \{x(F)\} F^{u_2} dF \quad = \frac{1}{\lambda} \sum_{j=1}^{\infty} \left[\frac{d^{\frac{j}{\alpha} + u_2 + 1}}{j(\frac{j}{\alpha} + u_2 + 1)} \right] \tag{23}$$

From Equations (22) and (23), the two parameters, α and λ, can be expressed as follows:

$$\lambda = \frac{\sum_{j=1}^{\infty} \left[\frac{d^{\frac{j}{\alpha} + u_1 + 1}}{j(\frac{j}{\alpha} + u_1 + 1)} \right]}{RW_{1,u_1,0}} . \tag{24}$$

$$RW_{1,u_2,0}\sum_{j=1}^{\infty}\left[\frac{d^{\frac{j}{\alpha}+u_1+1}}{j\,(\frac{j}{\alpha}+u_1+1)}\right]-RW_{1,u_1,0}\sum_{j=1}^{\infty}\left[\frac{d^{\frac{j}{\alpha}+u_2+1}}{j\,(\frac{j}{\alpha}+u_2+1)}\right]=0 \tag{25}$$

The PGPWMs estimators of α and λ, denoted by $\hat{\alpha}_R$ and $\hat{\lambda}_R$, from right censored samples, can be obtained by solving Equations (24) and (25); where $RW_{1,u_1,0}$ and $RW_{1,u_2,0}$ are replaced by their sample estimators, $R\hat{W}_{1,u_1,0}$ and $R\hat{W}_{1,u_2,0}$ by setting $u=u_1,u_2$, in Equation (9) as follows

$$\hat{\lambda}_R=\frac{\sum_{j=1}^{\infty}\left[\frac{d^{\frac{j}{\hat{\alpha}_R}+u_1+1}}{j\,(\frac{j}{\hat{\alpha}_R}+u_1+1)}\right]}{R\hat{W}_{1,u_1,0}}, \tag{26}$$

$$R\hat{W}_{1,u_2,0}\sum_{j=1}^{\infty}\left[\frac{d^{\frac{j}{\hat{\alpha}_R}+u_1+1}}{j\,(\frac{j}{\hat{\alpha}_R}+u_1+1)}\right]-R\hat{W}_{1,u_1,0}\sum_{j=1}^{\infty}\left[\frac{d^{\frac{j}{\hat{\alpha}_R}+u_2+1}}{j\,(\frac{j}{\hat{\alpha}_R}+u_2+1)}\right]=0. \tag{27}$$

A numerical procedure is required to estimate α based on iterative technique. Thus the value of $\hat{\lambda}_R$ can be obtained by substituting the value of $\hat{\alpha}_R$ in Equation (27).

4. Numerical experiments and discussions

An extensive numerical study is carried out to investigate the properties of the PGPWMs method of estimation for the EE distribution from censored samples. The investigated properties are biases and mean square errors (MSEs) of the PGPWMs estimators of the two parameters α and λ. The simulation procedure can be summarized as the following steps:

Step (1): 1000 random sample, $x_1,x_2,...,x_n$, of sizes n = 15, 20, 25, 30, 35and 50 are generated from the EE distribution.

Step (2): The true parameter selected values for the shape parameter α are 0.5, 0.6, and 0.7 with scale parameter $\lambda=1$. Choosing the exponent values of PGPWMs as $u_1=0, 0.1, 0.5$ and $u_2=1, 0.4, 1.5$. Levels of censoring are considered, $c=0.2$ and $d=0.8$.

Step (3): For each combination of values of sample size n, α and λ, the parameters of distribution are estimated using three different estimation methods; PGPWMs with doubly, left and right censored samples.

Step (4): PGPWMs of the unknown parameter α for EE distribution with doubly censored samples are obtained by solving numerically the non-linear Equation (15). The estimate of the scale parameter λ is obtained by substituting the value of $\hat{\alpha}$ in (14).

Step (5): Based on left censored samples, PGPWMs of the shape parameter α for EE distribution is obtained by solving iteratively the non-linear Equation (21). Hence the $\hat{\lambda}_L$ is obtained by substituting the value of $\hat{\alpha}_L$ in (20)

Step (6): Based on right censored samples, PGPWMs of the shape parameter α for EE distribution is obtained by solving numerically the non-linear Equation (27). To compute $\hat{\lambda}_R$, substitute $\hat{\alpha}_R$ in (26)

Step (7): PPWMs estimators with doubly, left and right censored samples for α ,for EE distribution is obtained by solving numerically the non-linear Equations (15),(21) and (27) after setting $u_1=0$ and $u_2=1$. Then, the estimates of λ is obtained by substituting the estimate of α in Equations (14), (20) and (26).

Step (8): The MSEs for the different estimators of the two parameters and for all sample sizes are tabulated and represented through some figures.

All simulation studies presented here are obtained via the MathCAD14 software. The MSEs of the different estimators of α and λ are reported in Tables (1-6) and described in Figures (1-18). From simulation study many observations can be made on the performance of PGPWMs estimators with doubly, left and right censored samples. These observations are summarized as follows:

1) It is observed from Tables (1), (2) and (3) that the MSE of estimators for α decreases as the sample size increases. This indicate that PGPWMs based on doubly, left and right censoring provided consistent estimators for α [see Figures(1-9)]
2) It is observed from Tables (4), (5) and (6) that the MSE of estimators for λ decreases as the sample size increases. This indicates that PGPWMs with doubly, left and right censoring provided consistent estimators for λ [see Figures (10-18)]

3) When the exponents of PGPWMs u_1 and u_2 decrease, the MSE of estimators for α and λ decreases [see Figures (1-18) and Tables (1-5)]
4) Considering the MSE of the different estimators of α, it is clear from Tables (1) (2)and (3) that the PGPWMs under right censoring estimator has the minimum MSE in almost all of the cases followed by PGPWMs under doubly censoring and the PGPWMs under left censoring [see Figures(1-9)]
5) Considering the MSE of the different estimators of λ, it is clear from Tables (4) ,(5)and (6) that the PGPWMs under right censoring estimator has the minimum MSE in almost all of the cases followed by PGPWMs under doubly censoring and PGPWMs under left censoring [see Figures (10-18)].

5. Data analysis

This section presents a real data set for illustrative purposes. [12] provided real data, which represents the numberof million revolutions before failure for each of 23 ball bearings in a life test: 17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.80, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, and 173.40.

Data analysis is presented to illustrate different PGWMs estimators from doubly, left and right censored samples. The data set is carried out to investigate the properties of the PGPWMs method of estimation for the EE distribution. The investigated properties of estimators for the two parameters α and λ are biases and MSEs. The results are displayed in Tables 7 and 8 and some observations are summarized as follows:

i. Considering the MSE of the estimator for α, it is clear from Table (7) that the PGPWMs under doubly censoring estimator has the minimum MSE in almost all of the cases considered for estimating α follow it PGPWMs under right censoring and the final PGPWMs under left censoring.
ii. Considering the MSE of the estimator for λ, it is clear from Table (8) that the PGPWMs under doubly censoring estimator has the minimum MSE in almost all of the cases considered for estimating λ follow it PGPWMs under right censoring and the final PGPWMs under left censoring.

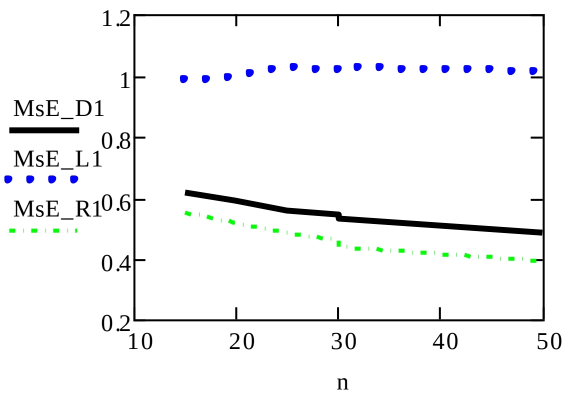

Fig. 1: The MSE Of $\hat{\alpha}, \hat{\alpha}_L, \hat{\alpha}_R$ When u_1=0.1 u_2=0.4for α =0.5

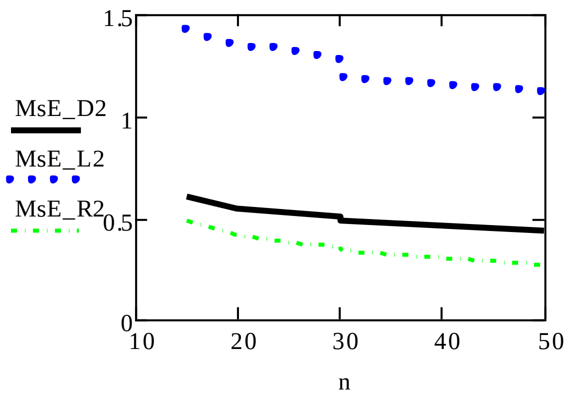

Fig. 2: The MSE Of $\hat{\alpha}, \hat{\alpha}_L, \hat{\alpha}_R$ When u_1=0.1, u_2=0.4 For α =0.6

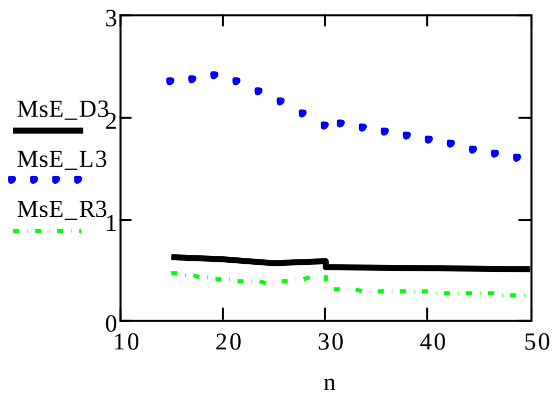

Fig. 3: The MSE Of $\hat{\alpha}, \hat{\alpha}_L, \hat{\alpha}_R$ When u_1=0.1, u_2=0.4 For α =0.7

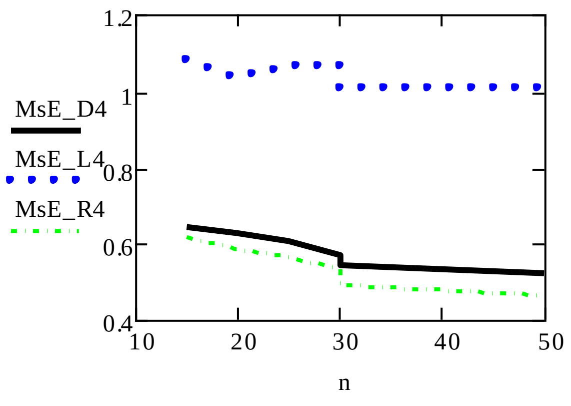

Fig. 4: The MSE Of $\hat{\alpha}, \hat{\alpha}_L, \hat{\alpha}_R$ When u_1=0.5 u_2=1.5for α =0.5

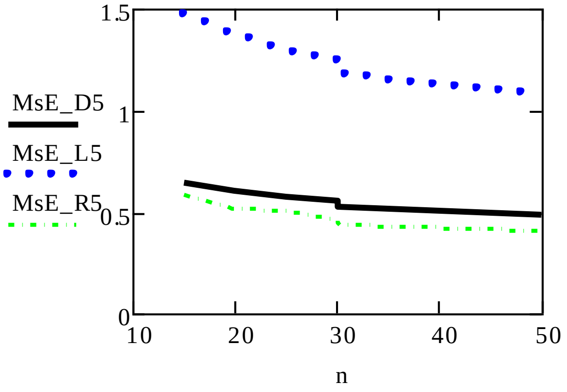

Fig. 5: The MSE Of $\hat{\alpha},\hat{\alpha}_L,\hat{\alpha}_R$ When u_1=0.5 u_2=1.5for α =0.6

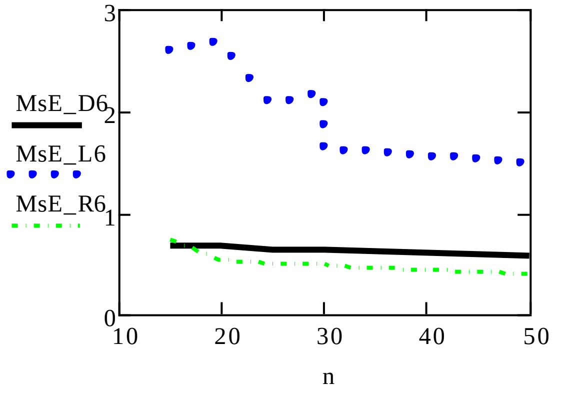

Fig. 6: The MSE Of $\hat{\alpha},\hat{\alpha}_L,\hat{\alpha}_R$ When u_1=0.5 u_2=1.5for α =07

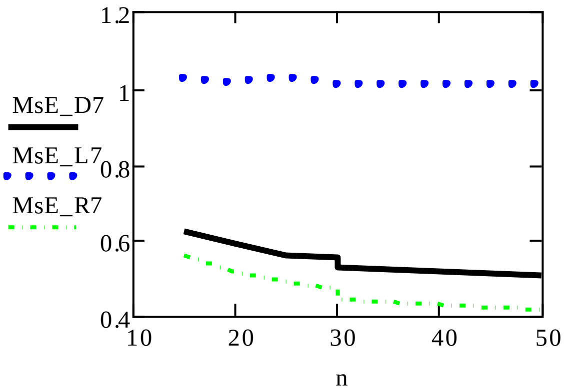

Fig. 7: The MSE Of $\hat{\alpha},\hat{\alpha}_L,\hat{\alpha}_R$ When u_1=0 u_2=1for α =0.5

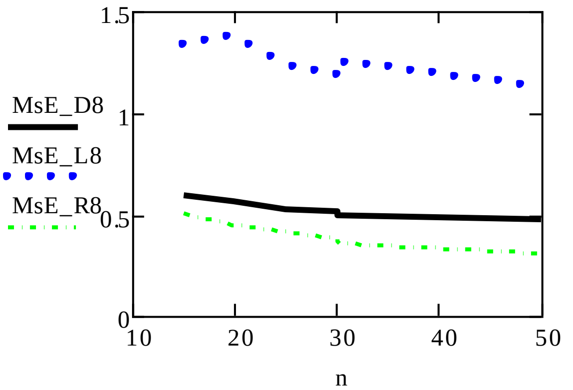

Fig. 8: The MSE Of $\hat{\alpha},\hat{\alpha}_L,\hat{\alpha}_R$ When u_1=0 u_2=1for α =0.6

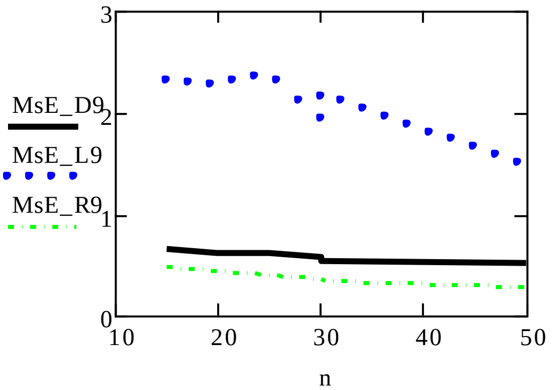

Fig. 9: The MSE Of $\hat{\alpha},\hat{\alpha}_L,\hat{\alpha}_R$ When u_1=0 u_2=1for α =0.5

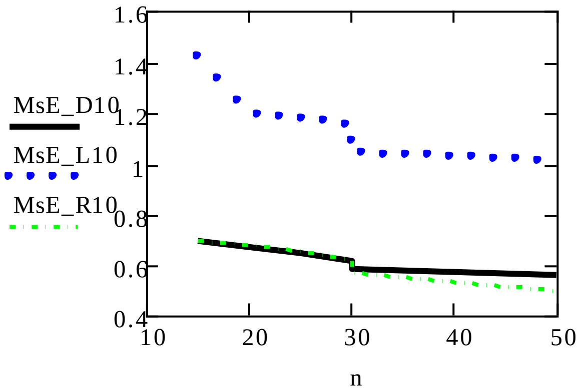

Fig. 10: The MSE Of $\hat{\lambda},\hat{\lambda}_L,\hat{\lambda}_R$ When u_1=0.1 u_2=1 For α =0.5

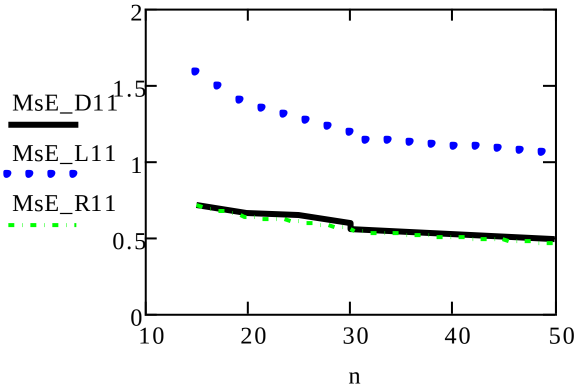

Fig. 11: The MSE Of $\hat{\lambda},\hat{\lambda}_L,\hat{\lambda}_R$ When u_1=0.1 u_2=1 For α =0.6

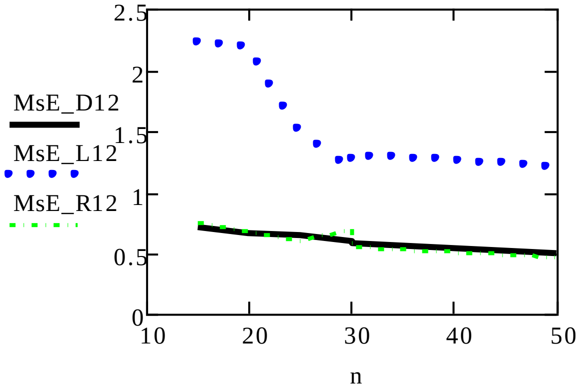

Fig. 12: The MSE Of $\hat{\lambda},\hat{\lambda}_L,\hat{\lambda}_R$ When u_1=0.1 u_2=1 For α =0.7

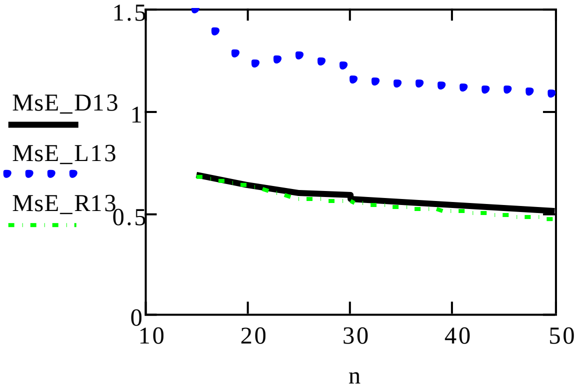

Fig. 13: The MSE Of $\hat{\lambda}, \hat{\lambda}_L, \hat{\lambda}_R$ When u_1=0.1 u_2=0.4 For α =0.5

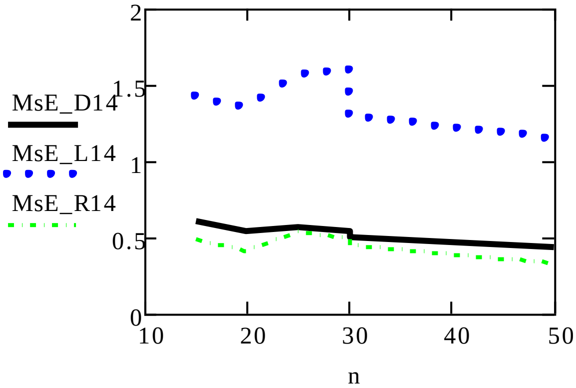

Fig. 14: The MSE of $\hat{\lambda}, \hat{\lambda}_L, \hat{\lambda}_R$ when u_1=0.1 u_2=0.4 for α =0.6

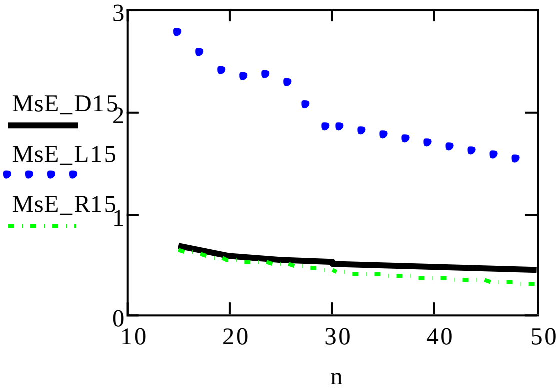

Fig. 15: The MSE of $\hat{\lambda}, \hat{\lambda}_L, \hat{\lambda}_R$ when u_1=0.1 u_2=0.4 for α =0.7

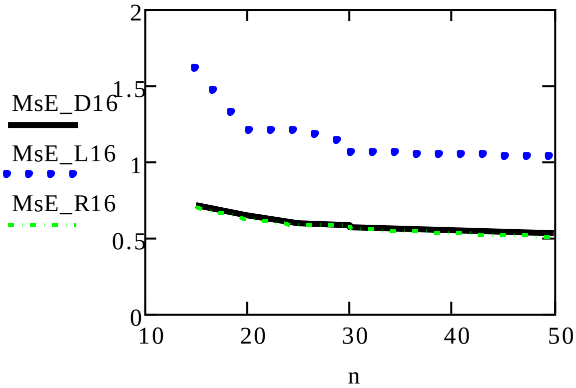

Fig. 16: The MSE of $\hat{\lambda}, \hat{\lambda}_L, \hat{\lambda}_R$ when u_1=0.5 u_2=1.5 for α =0.5

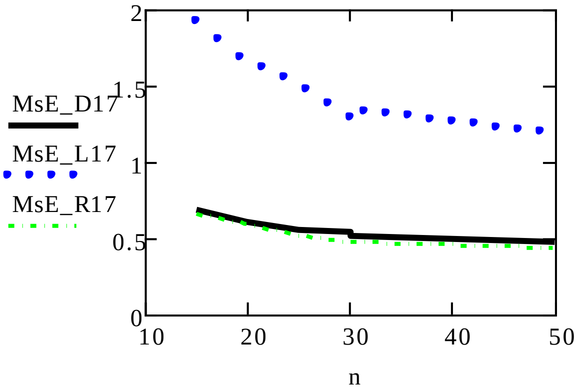

Fig. 17: The MSE of $\hat{\lambda}, \hat{\lambda}_L, \hat{\lambda}_R$ when u_1=0.5 u_2=1.5 for α =0.6

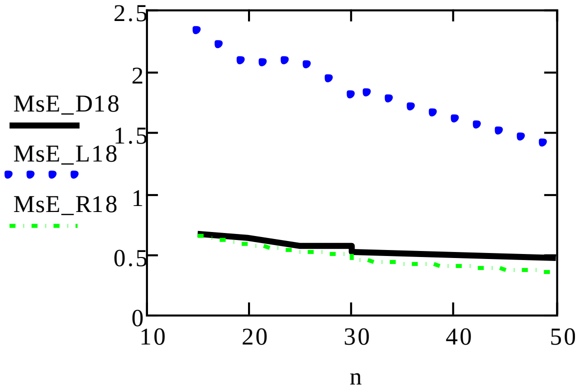

Fig. 18: The MSE of $\hat{\lambda}, \hat{\lambda}_L, \hat{\lambda}_R$ when u_1=0.5 u_2=1.5 for α =0.7

6. Concluding remarks

In this study, the method of PGPWMs is introduced as a new method for estimating the parameters of distribution from censored sample. This framework covers the methods; PWMs, PPWMs and GPWMs. The GPWMs can be obtained via PGPWMs by setting the lower bound $c = 0$ and upper bound $d = 1$. Also, the PPWMs can be obtained from PGPWMs by setting the exponents of methods u equal integer numbers. Finally, the PWMs can be obtained from PGPWMs by setting the exponents of methods u equal integer numbers and the lower and upper bound equal 0 and 1 respectively.

The method of PGPWMs has been applied for estimating the unknown parameters of EE distribution from doubly censored sample. This method has two special cases the left and right censored samples. All methods have provided consistent estimators for shape and scale parameters of EE distribution in terms of MSE. Comparing the performance of estimators from doubly, left and right censored samples the PGPWMs with right censored have the smallest MSE.

Table 1: Mean, Bias and MSE of PGPWMs Parameter Estimators for EE Distribution from Doubly, Left and Right Censoring for $u_1 = 0.5$ and $u_2 = 1.5$.

Sample size n	Properties of α	PGPWMs of $\hat{\alpha}$			LPPWMs of $\hat{\alpha}_L$			RPPWMs of $\hat{\alpha}_R$		
		$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$
15	Mean	0.277	0.349	0.425	0.137	0.281	0.464	0.290	0.369	0.436
	Bias	-0.723	-0.651	-0.575	-0.863	-0.719	-0.536	-0.710	-0.631	-0.564
	MSE	0.644	0.637	0.664	1.078	1.466	2.581	0.614	0.582	0.725
20	Mean	0.296	0.417	0.489	0.078	0.196	0.458	0.318	0.440	0.519
	Bias	-0.704	-0.583	-0.511	-0.922	-0.804	-0.542	-0.682	-0.560	-0.481
	MSE	0.625	0.598	0.662	1.034	1.369	2.683	0.586	0.517	0.537
25	Mean	0.316	0.437	0.555	0.077	0.136	0.278	0.340	0.439	0.571
	Bias	-0.684	-0.563	-0.445	-0.923	-0.864	-0.722	-0.660	-0.561	-0.429
	MSE	0.606	0.574	0.627	1.061	1.292	2.053	0.561	0.508	0.494
30	Mean	0.348	0.489	0.601	0.058	0.102	0.265	0.361	0.505	0.597
	Bias	-0.652	-0.511	-0.399	-0.942	-0.898	-0.735	-0.639	-0.495	-0.403
	MSE	0.570	0.555	0.626	1.063	1.240	2.172	0.532	0.456	0.484
35	Mean	0.388	0.490	0.601	0.018	0.069	0.144	0.407	0.525	0.617
	Bias	-0.612	-0.510	-0.399	-0.982	-0.931	-0.856	-0.593	-0.475	-0.383
	MSE	0.540	0.521	0.627	1.004	1.172	1.625	0.489	0.433	0.464
50	Mean	0.405	0.533	0.688	0.003	0.024	0.098	0.442	0.544	0.724
	Bias	-0.595	-0.467	-0.312	-0.997	-0.976	-0.902	-0.558	-0.456	-0.276
	MSE	0.519	0.482	0.579	1.003	1.074	1.482	0.462	0.400	0.389

Table 2: Mean, Bias and MSE of PGPWMs Parameter Estimators for EE distribution from Doubly, Left and Right Censoring for $u_1 = 0.1$ and $u_2 = 0.4$.

Sample size n	Properties of α	PGPWMs of $\hat{\alpha}$			LPPWMs of $\hat{\alpha}_L$			RPPWMs of $\hat{\alpha}_R$		
		$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$
15	Mean	0.300	0.366	0.473	0.196	0.400	0.639	0.330	0.431	0.503
	Bias	0.700	-0.634	-0.527	-0.804	-0.600	-0.361	-0.670	-0.569	-0.497
	MSE	0.614	0.600	0.617	0.981	1.422	2.320	0.550	0.481	0.455
20	Mean	0.320	0.421	0.543	0.108	0.294	0.557	0.367	0.497	0.568
	Bias	-0.680	-0.579	-0.457	-0.892	-0.706	-0.443	-0.633	-0.503	-0.432
	MSE	0.588	0.543	0.588	0.992	1.340	2.389	0.519	0.410	0.394
25	Mean	0.361	0.467	0.584	0.096	0.269	0.455	0.390	0.525	0.629
	Bias	-0.639	-0.533	-0.416	-0.904	-0.731	-0.545	-0.610	-0.475	-0.371
	MSE	0.556	0.520	0.549	1.02	1.321	2.178	0.482	0.379	0.347
30	Mean	0.376	0.500	0.640	0.070	0.230	0.346	0.413	0.553	0.660
	Bias	-0.624	-0.500	-0.360	-0.93	-0.770	-0.654	-0.587	-0.447	-0.340
	MSE	0.540	0.503	0.564	1.018	1.276	1.890	0.455	0.351	0.430
35	Mean	0.383	0.506	0.631	0.067	0.128	0.368	0.429	0.573	0.661
	Bias	-0.617	-0.494	-0.369	-0.933	-0.872	-0.632	-0.571	-0.427	-0.339
	MSE	0.528	0.484	0.510	1.022	1.186	1.956	0.434	0.331	0.302
50	Mean	0.433	0.627	0.730	0.020	0.070	0.192	0.467	0.631	0.728
	Bias	-0.567	-0.373	-0.270	-0.980	-0.930	-0.808	-0.533	-0.369	-0.272
	MSE	0.486	0.432	0.484	1.012	1.111	1.560	0.391	0.262	0.236

Table 3: Mean, Bias and MSE of the PPWMs Parameter Estimators for EE distribution from Doubly, Left and Right Censoring when $u_1 = 0$ and $u_2 = 1$.

Sample size n	Properties of α	PGPWMs of $\hat{\alpha}$			LPPWMs of $\hat{\alpha}_L$			RPPWMs of $\hat{\alpha}_R$		
		$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$
15	Mean	0.298	0.387	0.508	0.197	0.352	0.524	0.330	0.409	0.519
	Bias	-0.702	-0.613	-0.492	-0.803	-0.648	-0.476	-0.670	-0.591	-0.481
	MSE	0.619	0.593	0.650	1.022	1.337	2.316	0.557	0.506	0.473
20	Mean	0.326	0.423	0.543	0.103	0.268	0.430	0.367	0.468	0.552
	Bias	-0.674	-0.577	-0.457	-0.897	-0.732	-0.570	-0.633	-0.532	-0.448
	MSE	0.589	0.561	0.618	1.011	1.372	2.276	0.515	0.448	0.432
25	Mean	0.358	0.485	0.599	0.079	0.185	0.455	0.388	0.503	0.604
	Bias	-0.642	-0.515	-0.401	-0.921	-0.815	-0.545	-0.612	-0.497	-0.396
	MSE	0.559	0.521	0.609	1.024	1.233	2.374	0.488	0.413	0.388
30	Mean	0.366	0.506	0.610	0.043	0.130	0.322	0.409	0.546	0.644
	Bias	-0.634	-0.494	-0.390	-0.957	-0.817	-0.678	-0.591	-0.454	-0.356
	MSE	0.553	0.515	0.572	1.007	1.183	1.941	0.470	0.378	0.358
35	Mean	0.389	0.525	0.653	0.026	0.148	0.346	0.431	0.565	0.675
	Bias	-0.611	-0.475	-0.347	-0.974	-0.852	-0.654	-0.569	-0.435	-0.325
	MSE	0.525	0.491	0.542	1.004	1.251	2.180	0.441	0.351	0.331
50	Mean	0.417	0.547	0.736	0.008	0.067	0.142	0.456	0.627	0.72
	Bias	-0.583	-0.453	-0.264	-0.992	-0.933	-0.858	-0.544	-0.373	-0.280
	MSE	0.503	0.469	0.515	1.007	1.123	1.480	0.416	0.301	0.273

Table 4: Mean, Bias And MSE of PGPWMs Parameter Estimators for EE Distribution from Doubly, Left and Right Censoring for $u_1 = 0.5$ and $u_2 = 1.5$.

Sample size n	Properties of λ	PGPWMs of $\hat{\lambda}$			LPPWMs of $\hat{\lambda}_L$			RPPWMs of $\hat{\lambda}_R$		
		$\alpha=0.5$	$\alpha=0.6$	$\alpha=0.7$	$\alpha=0.5$	$\alpha=0.6$	$\alpha=0.7$	$\alpha=0.5$	$\alpha=0.6$	$\alpha=0.7$
15	Mean	0.259	0.331	0.394	0.211	0.334	0.451	0.279	0.367	0.446
	Bias	-0.741	-0.669	-0.606	-0.789	-0.666	-0.549	-0.741	-0.669	-0.606
	MSE	0.696	0.712	0.704	1.419	1.585	2.227	0.695	0.706	0.733
20	Mean	0.281	0.395	0.436	0.124	0.214	0.428	0.311	0.435	0.501
	Bias	-0.719	-0.605	-0.564	-0.874	-0.786	-0.572	-0.719	-0.605	-0.564
	MSE	0.672	0.660	0.66	1.194	1.366	2.181	0.675	0.634	0.676
25	Mean	0.295	0.418	0.504	0.101	0.142	0.237	0.328	0.439	0.555
	Bias	-0.705	-0.582	-0.496	-0.899	-0.858	-0.763	-0.705	-0.582	-0.496
	MSE	0.648	0.641	0.640	1.170	1.280	1.487	0.651	0.599	0.599
30	Mean	0.322	0.452	0.545	0.075	0.097	0.215	0.345	0.494	0.578
	Bias	-0.678	-0.548	-0.455	-0.925	-0.903	-0.785	-0.678	-0.548	-0.455
	MSE	0.615	0.589	0.588	1.153	1.186	1.596	0.611	0.558	0.693
35	Mean	0.372	0.465	0.544	0.029	0.068	0.115	0.404	0.524	0.600
	Bias	-0.628	-0.535	-0.456	-0.971	-0.932	-0.885	-0.628	-0.535	-0.456
	MSE	0.578	0.549	0.583	1.039	1.147	1.304	0.566	0.538	0.545
50	Mean	0.388	0.500	0.619	0.003	0.021	0.071	0.33	0.535	0.706
	Bias	-0.612	-0.500	-0.381	-0.997	-0.979	-0.929	-0.612	-0.500	-0.381
	MSE	0.555	0.492	0.499	1.005	1.048	1.206	0.493	0.465	0.468

Table 5: Mean, Bias and MSE of Pgpwms Parameter Estimators for EE Distribution from Doubly, Left and Right Censoring for $u_1 = 0.1$ And $u_2 = 0.4$

Sample size n	Properties of λ	PGPWMs of $\hat{\lambda}$			LPPWMs of $\hat{\lambda}_L$			RPPWMs of $\hat{\lambda}_R$		
		$\alpha=0.5$	$\alpha=0.6$	$\alpha=0.7$	$\alpha=0.5$	$\alpha=0.6$	$\alpha=0.7$	$\alpha=0.5$	$\alpha=0.6$	$\alpha=0.7$
15	Mean	0.290	0.352	0.443	0.352	0.400	0.782	0.329	0.431	0.514
	Bias	-0.710	-0.634	-0.557	-0.648	-0.600	-0.218	-0.710	-0.569	-0.557
	MSE	0.685	0.600	0.672	1.493	1.422	2.773	0.669	0.481	0.637
20	Mean	0.307	0.421	0.496	0.179	0.294	0.594	0.361	0.497	0.572
	Bias	-0.693	-0.579	-0.504	-0.821	-0.706	-0.406	-0.697	-0.503	-0.504
	MSE	0.634	0.543	0.582	1.210	1.340	2.329	0.631	0.410	0.535
25	Mean	0.353	0.447	0.538	0.157	0.332	0.504	0.391	0.539	0.638
	Bias	-0.647	-0.553	-0.462	-0.843	-0.668	-0.496	-0.647	-0.553	-0.462
	MSE	0.592	0.565	0.542	1.261	1.563	2.339	0.563	0.536	0.488
30	Mean	0.366	0.489	0.588	0.114	0.300	0.353	0.414	0.576	0.669
	Bias	-0.634	-0.511	-0.412	-0.886	-0.700	-0.647	-0.634	-0.511	-0.412
	MSE	0.578	0.542	0.517	1.200	1.597	1.770	0.549	0.483	0.430
35	Mean	0.371	0.494	0.585	0.099	0.151	0.376	0.432	0.586	0.669
	Bias	-0.629	-0.506	-0.415	-0.901	-0.849	-0.624	-0.629	-0.506	-0.415
	MSE	0.560	0.505	0.493	1.145	1.286	1.862	0.538	0.449	0.413
50	Mean	0.420	0.595	0.681	0.032	0.078	0.192	0.466	0.644	0.743
	Bias	-0.580	-0.405	-0.319	-0.968	-0.922	-0.808	-0.580	-0.405	-0.319
	MSE	0.508	0.43	0.436	1.072	1.142	1.485	0.465	0.328	0.305

Table 6: Mean, Bias And MSE Of Ppwms Parameter Estimators For EE Distribution From Doubly, Left And Right Censoring For $u_1 = 0$ And $u_2 = 1$.

Sample size n	Properties of λ	PGPWMs of $\hat{\lambda}$			LPPWMs of $\hat{\lambda}_L$			RPPWMs of $\hat{\lambda}_R$		
		$\alpha=0.5$	$\alpha=0.6$	$\alpha=0.7$	$\alpha=0.5$	$\alpha=0.6$	$\alpha=0.7$	$\alpha=0.5$	$\alpha=0.6$	$\alpha=0.7$
15	Mean	0.286	0.367	0.472	0.343	0.497	0.562	0.328	0.415	0.535
	Bias	-0.714	-0.633	-0.528	-0.657	-0.503	-0.437	-0.714	-0.633	-0.528
	MSE	0.707	0.686	0.662	1.601	1.923	2.219	0.694	0.659	0.644
20	Mean	0.311	0.400	0.502	0.162	0.332	0.442	0.359	0.469	0.561
	Bias	-0.689	-0.600	-0.498	-0.838	-0.668	-0.558	-0.689	-0.600	-0.498
	MSE	0.641	0.606	0.633	1.193	1.653	2.043	0.625	0.593	0.571
25	Mean	0.337	0.454	0.531	0.124	0.243	0.442	0.381	0.505	0.600
	Bias	-0.663	-0.546	-0.469	-0.876	-0.757	-0.558	-0.663	-0.546	-0.469
	MSE	0.596	0.553	0.564	1.201	1.514	2.085	0.583	0.519	0.517
30	Mean	0.350	0.481	0.566	0.070	0.158	0.328	0.409	0.547	0.656
	Bias	-0.650	-0.519	-0.434	-0.093	-0.842	-0.672	-0.650	-0.519	-0.434
	MSE	0.585	0.542	0.556	1.104	1.291	1.796	0.578	0.480	0.489
35	Mean	0.371	0.494	0.600	0.040	0.168	0.326	0.428	0.566	0.685
	Bias	-0.629	-0.506	-0.400	-0.960	-0.832	-0.674	-0.629	-0.506	-0.400
	MSE	0.562	0.515	0.514	1.050	1.345	1.834	0.547	0.468	0.452
50	Mean	0.405	0.519	0.682	0.012	0.077	0.135	0.457	0.637	0.733
	Bias	-0.595	-0.481	-0.318	-0.988	-0.923	-0.865	-0.595	-0.481	-0.318
	MSE	0.527	0.476	0.463	1.025	1.183	1.367	0.499	0.434	0.341

Table 7: Mean, Bias and MSE of the Shape Parameter Estimators for Scale Parameters Estimators for EE Distribution by the Methods of Pgpwms With Doubly, Left Pgpwms with Doubly, Left and Right Censored Based on Real Data.

Sample size n	Properties of α	u_1	u_2	$\alpha = 5.28$		
				PGPWMs of $\hat{\alpha}$	LPGPWMs of $\hat{\alpha}$	RPGPWMs of $\hat{\alpha}$
23	Mean	0.5	1.5	0.8948	1.1401	0.9034
	Bias			0.3943	0.6401	0.4034
	MSE			0.1554	0.4098	0.1627
	Mean	0.1	0.4	0.8308	1.0817	0.9472
	Bias			0.3308	0.5817	0.4472
	MSE			0.1094	0.3384	0.2000
	Mean	0	1	0.9383	1.2248	1.0088
	Bias			0.4383	0.7248	0.5088
	MSE			0.1921	0.5254	0.2589

Table 8: Mean, Bias and MSE of the EE Distribution by the Methods of and Right Censored Based on Real Data

Sample size n	Properties Of α	u_1	u_2	$\lambda = 1$		
				PGPWMs of $\hat{\lambda}$	LPGPWMs of $\hat{\lambda}$	RPGPWMs of $\hat{\lambda}$
23	Mean	0.5	1.5	0.0320	0.0263	0.0314
	Bias			-0.9680	-0.9688	-0.9686
	MSE			0.9370	0.9386	0.9382
	Mean	0.1	0.4	0.1547	0.0279	0.0282
	Bias			-0.8453	-0.9664	-0.9659
	MSE			0.7145	0.9339	0.9300
	Mean	0	1	0.0261	0.0098	0.0211
	Bias			-0.9690	-0.9902	-0.9761
	MSE			0.9390	0.9805	0.9527

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