



Nonlinear water waves (KdV) equation and Painlevé technique

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Abstract

The Korteweg-de Vries (KdV) equation which is the third order nonlinear PDE has been of interest since Scott Russell (1844) [5]. In this paper we study this kind of equation by Painlevé equation and through this study, we find that KdV equation satisfies Painlevé property [4], but we could not find a solution directly, so we transformed the KdV equation to the like-KdV equation, therefore, we were able to find four exact solutions to the original KdV equation.

Keywords: Kortewege-de Vrise equation; Painlevé property; Resonance points; Exact solutions.

1. Introduction

Most phenomena in the scientific field and other domains can be described and classified as nonlinear diffusion equation which normally results from natural phenomena that appear in our daily lives such as the water waves at the beach caused by wind or tides, also the movement of a ship, or those waves created by throwing a stone in a pond or by raindrops; the same applies to other physical and mathematical phenomena [2] [6]. In this study, we tried to find a solution to this type of equations although it is normally very difficult to find a clear-cut solution [3]. However, through the use of Painlevé technique it is possible to find an analytic solution which physicists, meteorologists, oceanographers, and others may benefit from in order to explain the result solution and arrive at a better understanding.

2. Painlevé property

In this section we apply Painlevé's property in the KdV equation:

$$u_t + \alpha uu_x + u_{xxx} = 0, \quad \alpha \in \mathbb{R} \setminus \{0\} \quad (1)$$

Let $u = \sum_{j=0}^{\infty} u_j \phi^{j-p}$ be the series solution of the equation (1), where ϕ and u_j are analytic functions in a neighbourhood of the manifold $\phi = \phi(t, x) = 0$. First, we need to find value of p , where p is the equilibrium point in the series solution. Now, to derive u in the series, where $u_t(t, x) = \frac{\partial u(t, x)}{\partial t}$, $u_x(t, x) = \frac{\partial u(t, x)}{\partial x}$ and

$u_{xxx}(t, x) = \frac{\partial^3 u(t, x)}{\partial x^3}$, by substituting them into the equation (1), and by comparing the lowest powers in the produced equation, we find $p = 2$, by associating the summation [7], we get:

$$\begin{aligned} & \sum_{j=3}^{\infty} u_{j-3,t} \phi^{j-5} + \alpha \sum_{j=0}^{\infty} \left[\sum_{k=0}^{j-1} u_k u_{j-1-k,x} + \sum_{i=0}^j u_{j-i} u_i (i-2) \phi_x \right] \phi^{j-5} \\ & + \sum_{j=3}^{\infty} u_{j-3,xxx} \phi^{j-5} + \sum_{j=2}^{\infty} 3(j-4) u_{j-2,xx} \phi_x \phi^{j-5} \\ & + \sum_{j=1}^{\infty} 3(j-3)(j-4) u_{j-1,x} \phi_x^2 \phi^{j-5} + \sum_{j=2}^{\infty} (j-4) u_{j-2} \phi_{xxx} \phi^{j-5} \\ & + \sum_{j=1}^{\infty} 3(j-3)(j-4) u_{j-1} \phi_x \phi_{xx} \phi^{j-5} + \sum_{j=2}^{\infty} 3(j-4) u_{j-2,x} \phi_{xx} \phi^{j-5} \\ & + \sum_{j=2}^{\infty} (j-4) u_{j-2} \phi_t \phi^{j-5} + \sum_{j=0}^{\infty} (j-2)(j-3)(j-4) u_j \phi_x^3 \phi^{j-5} = 0, \end{aligned} \tag{2}$$

To find u_0 , then at $j = 0$ in the equation (2), we get:

$$u_0 = -\frac{12}{\alpha} \phi_x^2, \tag{3}$$

To find u_1 , then at $j = 1$ in the equation (2), we get:

$$u_1 = \frac{12}{\alpha} \phi_{xx}, \tag{4}$$

To find u_2 , then at $j = 2$ in the equation (2), we get:

$$u_2 = -\frac{1}{\alpha} \frac{\phi_t}{\phi_x} - \frac{4}{\alpha} \frac{\phi_{xxx}}{\phi_x} + \frac{3}{\alpha} \left(\frac{\phi_{xx}}{\phi_x} \right)^2, \tag{5}$$

Since $p = 2$, by using the technique of truncation, and let $u_j = 0$, for all $j > 2$.

Then the series solution $u = \sum_{j=0}^{\infty} u_j \phi^{j-p}$, becomes:

$$u = \frac{u_0}{\phi^2} + \frac{u_1}{\phi} + u_2, \tag{6}$$

This is the relation between u and u_2 .

Now, in the equation (2), we have to find all coefficients of u_j , where $u_j \equiv 0$ for all $j < 0$.

$$If : i = 0 \Rightarrow \alpha \sum_{i=0}^j u_{j-i} u_i (i-2) \phi_x = 2\alpha \phi_x^3 u_j,$$

and,

$$If : i = j \Rightarrow \alpha \sum_{i=0}^j u_{j-i} u_i (i-2) \phi_x = -\alpha \phi_x^3 (j-2) u_j.$$

Thus, the recursion relation is:

$$\begin{aligned} & (j-4)[j^2 - 5j + (6-\alpha)] \phi_x^3 u_j = -u_{j-3,t} - (j-4) u_{j-2} \phi_t - 3(j-4) u_{j-2,x} \phi_{xx} \\ & - \alpha \sum_{i=1}^{j-1} u_{j-i} u_i (i-2) \phi_x - \alpha \sum_{k=0}^{j-1} u_k u_{j-1-k,x} - u_{j-3,xxx} - 3(j-4) u_{j-2,xx} \phi_x \\ & - 3(j-3)(j-4) u_{j-1,x} \phi_x^2 - (j-4) u_{j-2} \phi_{xxx} - 3(j-3)(j-4) u_{j-1} \phi_x \phi_{xx}, \end{aligned} \tag{7}$$

We note that the coefficients of u_j in the equation (7) are $(j-4)$ and $[j^2 - 5j + (6-\alpha)]$, then, in the general of the integer resonance point is $j = 4$. The other resonance points depend on the value of α . For example, if $\alpha = 6$, the

resonance points are $j = 0, 4, 5$, and if $\alpha = 12$, the resonance points are $j = -1, 4, 6$.
Now at $j = 3$ in the equation (7) and using the equations (3) and (4), we have,

$$u_3 = \frac{1}{\alpha} \frac{\phi_{xt}}{\phi_x^2} + \frac{\phi_{xx}u_2}{\phi_x^2} + \frac{1}{\alpha} \frac{\phi_{xxxx}}{\phi_x^2}, \quad (8)$$

When $j = 4$ in the equation (7), since $u_j = 0$ for all $j > 2$, we get $u_4 = 0$.
Then the equation (1) satisfies the Painlevé property [7].

When $j = 5$ in the equation (7), since $u_j = 0$ for all $j > 2$, we get,

$$u_{2,t} + \alpha u_2 u_{2,x} + u_{2,xxx} = 0. \quad (9)$$

Then u_2 is also a solution of the KdV equation (1).

3. Analytic solution

In this section, we follow the idea to derive analytic solution. They are invariant under this transformation [1],

$$H : \phi \longrightarrow \frac{a\phi + b}{c\phi + d} \quad \text{where} \quad ad - bc \neq 0.$$

They are the Schwartzian derivative [3],

$$S(\phi) = \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2} \left(\frac{\phi_{xx}}{\phi_x} \right)^2, \quad (10)$$

and dimension of velocity,

$$C(\phi) = -\frac{\phi_t}{\phi_x}. \quad (11)$$

Furthermore, we define,

$$L(\phi) = -\frac{\phi_{xx}}{2\phi_x}. \quad (12)$$

The relations,

$$L_t = CL_x - LC_x + \frac{1}{2}C_{xx} \quad \text{and} \quad L_x = -L^2 - \frac{1}{2}S. \quad (13)$$

The compatibility of S and C given by,

$$S_t + C_{xxx} + 2C_x S + CS_x = 0. \quad (14)$$

Now, by using the equations (5) and (8), we obtain:

$$\alpha \phi_x u_3 = \frac{\phi_{xt}}{\phi_x} + \frac{\alpha \phi_{xx}}{\phi_x} \left(-\frac{1}{\alpha} \frac{\phi_t}{\phi_x} - \frac{4}{\alpha} \frac{\phi_{xxx}}{\phi_x} + \frac{3}{\alpha} \left(\frac{\phi_{xx}}{\phi_x} \right)^2 \right) + \frac{\phi_{xxxx}}{\phi_x}.$$

Since, $u_j = 0$ for all $j > 2$, we get:

$$\frac{\phi_t \phi_{xx}}{\phi_x^2} - \frac{\phi_{xt}}{\phi_x} = \frac{\phi_{xxxx}}{\phi_x} - \frac{4\phi_{xx}\phi_{xxx}}{\phi_x^2} + 3 \left(\frac{\phi_{xx}}{\phi_x} \right)^3. \quad (15)$$

Then, by comparing both sides of the equation (15) with equations (10) and (11), we observe:

$$C_x = S_x. \quad (16)$$

Now, by using the equations (10), (11) and (12), then the equation (5), becomes:

$$u_2 = \frac{1}{\alpha} C - \frac{4}{\alpha} S - \frac{12}{\alpha} L^2. \quad (17)$$

We derive the equation (16), to find $u_{2,t}$, $\alpha u_2 u_{2,x}$ and $u_{2,xxx}$ and substitute them into the equation (9), then:

$$\begin{aligned} & \frac{1}{\alpha} C_t - \frac{4}{\alpha} S_t - \frac{24}{\alpha} LL_t - \frac{1}{\alpha} CC_x - \frac{4}{\alpha} CS_x - \frac{24}{\alpha} CLL_x - \frac{4}{\alpha} SC_x + \frac{16}{\alpha} SS_x \\ & + \frac{96}{\alpha} SLL_x - \frac{12}{\alpha} L^2 C_x + \frac{48}{\alpha} L^2 S_x + \frac{288}{\alpha} L^3 L_x + \frac{1}{\alpha} C_{xxx} - \frac{4}{\alpha} S_{xxx} \\ & - \frac{24}{\alpha} LL_{xxx} - \frac{72}{\alpha} L_x L_{xx} = 0, \end{aligned} \tag{18}$$

To eliminate L , by using the relations (13), and by the equation (16) then, $L^2 C_x - L^2 S_x = 0$ and $LS_{xx} - LC_{xx} = 0$. And by using (14), then the equation (18), becomes:

$$C_t + C_{xxx} + 2C_x S + CC_x = 0, \tag{19}$$

By comparing the equations (14) with (19), and using the equation (16), we get:

$$C_t = S_t, \tag{20}$$

Then by the equations (16) and (20), we get, $C = S + K$ where K is constant. For $K = 0$, we get:

$$C = S, \tag{21}$$

By substituting $C = S$ into the equation (14), we get:

$$S_t + 3SS_x + S_{xxx} = 0, \tag{22}$$

This is Korteweg-de Vries like equation (KdV).

4. Exact solution

Solution for constant S .

The constant functions $S = \pm 2\lambda^2$ where λ is constant, are solutions of the KdV like equation (22).

Lemma 4.1 [1] Let ψ_1 and ψ_2 be two linearly independent solutions of the equation,

$$\frac{d^2\psi}{dz^2} + P(z) = 0, \tag{23}$$

which are defined and holomorphic on some simply connected domain D in complex plane, then $W(z) = \psi_1(z)/\psi_2(z)$ satisfies the equation,

$$\{W, Z\} = 2P(z), \tag{24}$$

Conversely, if $W(z)$ is a solution of (24) at all point of D , then one can find two linearly holomorphic independent solutions ψ_1 and ψ_2 of (23) such that $W(z) = \psi_1(z)/\psi_2(z)$ in some neighborhood of $z_0 \in D$.

Lemma 4.2 [3] The Schwartzian derivative is invariant under fractional linear transformation acting on the first argument, namely,

$$\left\{ \frac{aW + b}{cW + d}; z \right\} = \{W; z\} \quad \text{where} \quad ad - bc \neq 0,$$

where a, b, c and d are constants.

Case A:

For $S = -2\lambda^2$, we have, $S = \{\phi, x\} = -2\lambda^2$.

Hence $P(x) = -\lambda^2$ in (24), and two linearly independent solutions are:

$$\Psi_1 = E(t)e^{\lambda x} + F(t)e^{-\lambda x} \quad , \quad \Psi_2 = G(t)e^{\lambda x} + H(t)e^{-\lambda x}$$

Therefore by Lemma 4.1 and Lemma 4.2, we obtain:

$$\phi(t, x) = \frac{E(t)e^{\lambda x} + F(t)e^{-\lambda x}}{G(t)e^{\lambda x} + H(t)e^{-\lambda x}} \quad \text{where} \quad EH - FG \neq 0, \quad (25)$$

By using the equations (11) and (21), then:

$$C = S = -\frac{\phi_t}{\phi_x} = -2\lambda^2, \quad (26)$$

Now, to find the equation of coefficients $E(t)$, $F(t)$, $G(t)$ and $H(t)$, we derive $\phi(t, x)$ in the equation (25), once respect to t and once respect to x and substituting them into the equation(26), we obtain:

$$\begin{aligned} \phi_t &= \frac{[G(t)E'(t) - E(t)G'(t)]e^{2\lambda x} + [H(t)F'(t) - F(t)H'(t)]e^{-2\lambda x}}{[G(t)e^{\lambda x} + H(t)e^{-\lambda x}]^2} \\ &+ \frac{G(t)F'(t) - F(t)G'(t) + H(t)E'(t) - E(t)H'(t)}{[G(t)e^{\lambda x} + H(t)e^{-\lambda x}]^2}, \end{aligned}$$

and,

$$\phi_x = \frac{2\lambda[H(t)E(t) - G(t)F(t)]}{[G(t)e^{\lambda x} + H(t)e^{-\lambda x}]^2},$$

Then, the equation (26) becomes:

$$\begin{aligned} C &= \frac{[G(t)E'(t) - E(t)G'(t)]e^{2\lambda x} + [H(t)F'(t) - F(t)H'(t)]e^{-2\lambda x}}{-2\lambda[H(t)E(t) - G(t)F(t)]} \\ &+ \frac{G(t)F'(t) - F(t)G'(t) + H(t)E'(t) - E(t)H'(t)}{-2\lambda[H(t)E(t) - G(t)F(t)]} = -2\lambda^2. \end{aligned}$$

Then,

$$\begin{aligned} &[G(t)E'(t) - E(t)G'(t)]e^{2\lambda x} + [H(t)F'(t) - F(t)H'(t)]e^{-2\lambda x} + G(t)F'(t) \\ &- F(t)G'(t) + H(t)E'(t) - E(t)H'(t) = 4\lambda^3(H(t)E(t) - G(t)F(t)). \end{aligned}$$

This leads to a system of nonlinear ordinary differential equation in coefficients $E(t)$, $F(t)$, $G(t)$ and $H(t)$, then:

$$\begin{aligned} \text{(I)} \quad &GE' - EG' = 0 \\ \text{(II)} \quad &HF' - FH' = 0 \\ \text{(III)} \quad &(GF' - FG') + (HE' - EH') = 4\lambda^3(HE - GF) \end{aligned}$$

Particular solutions of (I) and (II) are:

$$E(t) = AG(t) \quad \text{and} \quad F(t) = BH(t)$$

where A and B are real arbitrary constants. By substituting these into (III), we get:

$$B(G(t)H'(t) - H(t)G'(t)) + A(H(t)G'(t) - G(t)H'(t)) = 4\lambda^3H(t)G(t)(A - B),$$

then:

$$\frac{H'(t)}{H(t)} - \frac{G'(t)}{G(t)} = -4\lambda^3,$$

By integrating the above, we get:

$$\frac{H(t)}{G(t)} = \exp(-4\lambda^3 t),$$

Then the equation (25), becomes:

$$\phi(t, x) = \frac{AG(t) \exp(\lambda x) + BG(t) \exp(-4\lambda^3 t - \lambda x)}{G(t) \exp(\lambda x) + G(t) \exp(-4\lambda^3 t - \lambda x)},$$

Which leads to:

$$\begin{aligned} \phi(t, x) &= \frac{Ae^{\lambda\xi_1} + Be^{-\lambda\xi_1}}{e^{\lambda\xi_1} + e^{-\lambda\xi_1}}, \quad \text{where } \xi_1 = x + 2\lambda^2 t \\ &= \frac{(A + B) \cosh \lambda\xi_1 + (A - B) \sinh \lambda\xi_1}{2 \cosh \lambda\xi_1} \end{aligned}$$

Then:

$$\phi(t, x) = K_1 + K_2 \tanh \lambda\xi_1, \tag{27}$$

where K_1 and K_2 are arbitrary constants, and $K_1 = (A + B)/2$ and $K_2 = (A - B)/2$. For $K_1 = 0$, and by substituting the equation (27) into the equation (5), we obtain:

$$\begin{aligned} u_2 &= -\frac{1}{\alpha} \frac{2K_2 \lambda^3 \operatorname{sech}^2 \lambda\xi_1}{K_2 \lambda \operatorname{sech}^2 \lambda\xi_1} - \frac{4 - 2K_2 \lambda^3 \operatorname{sech}^4 \lambda\xi_1 + 4K_2 \lambda^3 \operatorname{sech}^2 \lambda\xi_1 \tanh^2 \lambda\xi_1}{\alpha K_2 \lambda \operatorname{sech}^2 \lambda\xi_1} \\ &+ \frac{3}{\alpha} \frac{4K_2^2 \lambda^4 \operatorname{sech}^4 \lambda\xi_1 \tanh^2 \lambda\xi_1}{K_2^2 \lambda^2 \operatorname{sech}^4 \lambda\xi_1} \end{aligned}$$

Then:

$$u_2 = \frac{12\lambda^2}{\alpha} \left(\operatorname{sech}^2 \lambda\xi_1 - \frac{1}{2} \right), \quad \text{where } \xi_1 = x + 2\lambda^2 t,$$

Hence $u_2(t, x)$ is the first exact solution for KdV equation (1). Now, by the equations (3), (4), (6) and (27), we obtain:

$$\begin{aligned} u &= \frac{-12}{\alpha} \frac{\phi_x^2}{\phi^2} + \frac{12}{\alpha} \frac{\phi_{xx}}{\phi} + u_2, \\ &= \frac{-12}{\alpha} \frac{K_2^2 \lambda^2 \operatorname{sech}^4 \lambda\xi_1}{K_2^2 \tanh^2 \lambda\xi_1} - \frac{24}{\alpha} \frac{K_2 \lambda^2 \operatorname{sech}^2 \lambda\xi_1 \tanh \lambda\xi_1}{K_2 \tanh \lambda\xi_1} + u_2, \end{aligned}$$

Then:

$$u = -\frac{12\lambda^2}{\alpha} \left(\operatorname{csech}^2 \lambda\xi_1 + \frac{1}{2} \right), \quad \text{where } \xi_1 = x + 2\lambda^2 t,$$

Hence $u(t, x)$ is the second exact solution for KdV equation (1).

Case B:

For $S = 2\lambda^2$, we have: $S = \{\phi, x\} = 2\lambda^2$.

Hence $P(x) = -\lambda^2$ in (24), and two linearly independent solutions are:

$$\Psi_3 = E(t)e^{\lambda ix} + F(t)e^{-\lambda ix}, \quad \Psi_4 = G(t)e^{\lambda ix} + H(t)e^{-\lambda ix}$$

Therefore, from Lemma 4.1 and Lemma 4.2, one obtains:

$$\phi(t, x) = \frac{E(t)e^{\lambda ix} + F(t)e^{-\lambda ix}}{G(t)e^{\lambda ix} + H(t)e^{-\lambda ix}} \quad \text{where } EH - FG \neq 0, \tag{28}$$

By using the equations (11) and (21), then:

$$C = S = -\frac{\phi_t}{\phi_x} = 2\lambda^2. \quad (29)$$

Now to find the equation of coefficients $E(t)$, $F(t)$, $G(t)$ and $H(t)$, we derive $\phi(t, x)$ in the equation (28), once respect to t and once respect to x and by substituting them into the equation(29), we obtain:

$$C = \frac{[G(t)E'(t) - E(t)G'(t)]e^{2\lambda ix} + [H(t)F'(t) - F(t)H'(t)]e^{-2\lambda ix}}{-2i\lambda[H(t)E(t) - G(t)F(t)]} + \frac{G(t)F'(t) - F(t)G'(t) + H(t)E'(t) - E(t)H'(t)}{-2i\lambda[H(t)E(t) - G(t)F(t)]} = 2\lambda^2.$$

This leads to a system of nonlinear ordinary differential equations with coefficients $E(t)$, $F(t)$, $G(t)$ and $H(t)$, then:

$$\begin{aligned} \text{(I)} \quad & GE' - EG' = 0, \\ \text{(II)} \quad & HF' - FH' = 0, \\ \text{(III)} \quad & GF' - FG' + (HE' - EH') = -4i\lambda^3(HE - GF), \end{aligned}$$

Particular solutions of (I) and (II) are:

$$E(t) = MG(t) \quad \text{and} \quad F(t) = NH(t)$$

where M and N are real arbitrary constants. By substituting these into (III), we get:

$$\frac{H(t)}{G(t)} = \exp(4i\lambda^3 t),$$

Then the equation (28), becomes:

$$\phi(t, x) = \frac{MG(t) \exp(\lambda ix) + NG(t) \exp(4\lambda^3 it - \lambda ix)}{G(t) \exp(\lambda ix) + G(t) \exp(4\lambda^3 it - \lambda ix)},$$

Which leads to:

$$\begin{aligned} \phi(t, x) &= \frac{Me^{\lambda i \xi_2} + Ne^{-\lambda i \xi_2}}{e^{\lambda i \xi_2} + e^{-\lambda i \xi_2}}, \quad \text{where} \quad \xi_2 = x - 2\lambda^2 t \\ &= \frac{(M + N) \cos \lambda \xi_2 + (M - N) \sin \lambda \xi_2}{2 \cos \lambda \xi_2} \end{aligned}$$

Then:

$$\phi(t, x) = K_3 + K_4 \tan \lambda \xi_2, \quad (30)$$

where $K_3 = (M + N)/2$ and $K_4 = (M - N)/2$ are arbitrary constants. For $K_3 = 0$, by substituting the equation (30) into the equation (5), we get:

$$\begin{aligned} \hat{u}_2 &= \frac{2 K_4 \lambda^3 \sec^2 \lambda \xi_2}{\alpha K_4 \lambda \sec^2 \lambda \xi_2} - \frac{4 K_4 \lambda^3 \sec^4 \lambda \xi_2 + 4 K_4 \lambda^3 \sec^2 \lambda \xi_2 \tan^2 \lambda \xi_2}{K_4 \lambda \sec^2 \lambda \xi_2} \\ &+ \frac{3 K_4^2 \lambda^4 \sec^4 \lambda \xi_2 \tan^2 \lambda \xi_2}{\alpha K_4^2 \lambda^2 \sec^4 \lambda \xi_2}, \end{aligned}$$

Then:

$$\hat{u}_2 = -\frac{12\lambda^2}{\alpha} \left(\sec^2 \lambda \xi_2 - \frac{1}{2} \right), \quad \text{where} \quad \xi_2 = x - 2\lambda^2 t.$$

Hence $\hat{u}_2(t, x)$ is the third exact solution for KdV equation (1). Now, by the equations (3), (4), (6) and (30), we get:

$$\hat{u} = \frac{-12}{\alpha} \frac{\phi_x^2}{\phi^2} + \frac{12}{\alpha} \frac{\phi_{xx}}{\phi} + \hat{u}_2.$$

Then:

$$\hat{u} = -\frac{12\lambda^2}{\alpha} \left(\operatorname{csec}^2 \lambda \xi_2 - \frac{1}{2} \right), \quad \text{where} \quad \xi_2 = x - 2\lambda^2 t.$$

Hence $\hat{u}(t, x)$ is the fourth exact solution for KdV equation (1).

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