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Research paper

Exact static spherical symmetric soliton-like solutions to the scalar and electromagnetic nonlinear induction field equations in general relativity

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Abstract

In this paper, we have obtained exact static spherical symmetric soliton-like solutions to the electromagnetic and scalar nonlinear induction field equations taking into account the own gravitational field of the elementary. The results show that the metric tensor functions are regular with localized energy density. Moreover, the total energy of the nonlinear induction fields is bounded and the total charge of elementary particles has a finite value. The importance of the own gravitational field of elementary particles and the role of the nonlinearity of fields in the determination of these solutions have been proved.

Keywords: *Interaction, scalar, electromagnetic, gravitational, fields, description, configuration, elementary particles.*

1. Introduction

The soliton is a particular, spatially localized solution of nonlinear differential equations. Its experimental and mathematical properties and exceptional stability, have amazed the scientific world [1,2].
One of the domains of application of the soliton concept is elementary particle physics, where soliton solutions of nonlinear differential field equations are used as models to describe the configuration of elementary particles [3,4]. These solutions, as suggested by Rajaraman [5], allow to model elementary particles as not being material points but extended objects with a complex spatial configuration. The complete description of elementary particles with all their physical characteristics (charge, spin) is possible only in the framework of the field interaction theory and quantum mechanic [6]. The choice of the field equations is one of the main problems of the nonlinear theory. It has given rise to a lot of work in the particle physics domain. For example, Korteweg et al. [7] gave the theoretical interpretation to the observation of J. S. Russell [8] by solving the KdV equation. Schwarzschild exhibited a spherical symmetric solution describing the static exterior of a gravitational source called Schwarzschild solution which remained for a long time only a mathematical result. His weak field approximation described correctly the mechanic of the solar system. Its global interpretation (especially for the inner solution) in terms of space-time containing a black hole was only understood many years later. Other exact solutions were discovered afterward. For instance, the works of Kerr-Newmann [9], Wiltshire [10], Wald [11] and the famous Carter-Penrose diagram, constitute references for the study of the singularity of static spherical and symmetric solutions. R. Pellicer et al. [12] generalized the solutions to the Born-Infeld nonlinear electrodynamics equation based on the work of Plebanski [13]. They discovered a new class of non-singular static spherical symmetric solutions to the modified Einstein-Maxwell equation, some of whose tensors are close to those of Riessner-Nordström. Bronnikov et al. [14] solved the equation to the nonlinear scalar, electromagnetic and own gravitational fields. They obtained particular static spherical symmetry solutions on the basis of predefined strong and weak criteria relating to singularity, regularity and localization. They concluded the existence of a system of even particle-antiparticles and gave applications. Rybakov et al. [15,16] determined soliton-like solutions of the equation to the nonlinear electromagnetic field interacting with the scalar field in the spherical and/or cylindrical symmetric metric in the presence to the own gravitational field of the elementary particles by briefly exploring the particular case where $S(k,\xi) = \xi$. They found that with the calibrated invariance function $P(I) = P_0(\lambda I - N)^2$, the metric tensor function are regular function and the energy of the field is finite. They

obtained a solution describing a massive system but the total charge of elementary particles was not examined. Recently, A. Adomou et al. [17–20] established spherical symmetric soliton solutions to the spinor and gravitational field equations using several bilinear invariants. In these works related to solitons, only the case where $S(k,\xi)=\xi$ is often addressed. The aim of this paper is to determine the exact static spherical symmetric soliton-like solutions to the electromagnetic and scalar nonlinear induction fields equation taking into account the own gravitational field of the elementary particles by considering all forms of the function $S(k,\xi)$, using the calibrated invariance function $P(I)=P_0(N-\lambda I)^2$, under the condition $N>\lambda I$. To achieve this, Section 2 gives a brief overview of the basic equations. In Section 3, the obtained solutions are presented. A discussion and a comparative study are included in Section 4. The role of the own gravitational field of elementary particles and the influence of the nonlinearity fields in obtained solutions are studied in detail in Section 5. Section 6 is devoted to the conclusion and future work.

2. Basic equations

In this research work, we opt for the static spherical symmetric metric:

$$ds^2 = e^{2\gamma}dt^2 - e^{2\alpha}d\xi^2 - e^{2\beta}\left[d\theta^2 + \sin^2(\theta)\,d\varphi^2\right], \tag{1}$$

where the functions α, β and γ depend only on the radial component $\xi = \dfrac{1}{r}$ and verify the coordinate condition [21]:

$$\alpha = 2\beta + \gamma. \tag{2}$$

In general relativity, Einstein's equation is:

$$G^{\nu}_{\mu} = -\chi T^{\nu}_{\mu}. \tag{3}$$

This equation contains the Einstein tensor (G^{ν}_{μ}); the Einstein gravitational constant (χ) and the energy momentum metric tensor (T^{ν}_{μ}). From Eq.(1), Eq.(2) and Eq.(3), the non-zero components of Einstein's tensor equation are written as [17] :

$$G^0_0 = e^{-2\alpha}(2\beta'' - 2\gamma'\beta' - \beta'^2) - e^{-2\beta} = -\chi T^0_0 \tag{4}$$

$$G^1_1 = e^{-2\alpha}(2\gamma'\beta' + \beta'^2) - e^{-2\beta} = -\chi T^1_1 \tag{5}$$

$$G^2_2 = e^{-2\alpha}(\beta'' + \gamma'' - 2\gamma'\beta' - \beta'^2) = -\chi T^2_2 \tag{6}$$

$$G^2_2 = G^3_3 \tag{7}$$

$$T^2_2 = T^3_3 \tag{8}$$

where ($'$) denotes the first derivative with respect to ξ.
The Lagrangian of the interacting nonlinear electromagnetic, scalar and the own gravitational fields has the form:

$$L = \frac{R}{2\chi} - \frac{1}{4}F_{ij}F^{ij} + \frac{1}{2}\varphi_{,i}\varphi^{,i}\psi(I) \tag{9}$$

where $I = A_iA^i$ is the chronometric invariant; $\psi(I) = 1 + \lambda\,\phi(I)$ is some arbitrary function characterizing the interaction between the nonlinear electromagnetic and scalar fields; $A_i(\,A(\xi),0,0,0)$ is the 4-vector potential; λ represents the parameter of the nonlinearity.
In the absence of a magnetic monopole and a current source, we write the scalar and the electromagnetic field equations corresponding to the Lagrangian Eq.(9) [16]:

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial\xi^{\nu}}\left[\sqrt{-g}g^{\nu\mu}\varphi_{,\mu}\psi(I)\right] = 0, \tag{10}$$

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial\xi^{\mu}}\left[\sqrt{-g}F^{\nu\mu}\right] - \varphi_{,i}\varphi^{,i}\psi_I(I)A^{\nu} = 0. \tag{11}$$

The energy-momentum metric tensor reads:

$$T^{\nu}_{\mu} = \varphi_{,\mu}\varphi^{,\nu}\psi(I) - F_{\mu i}F^{\nu i} + \varphi_{,i}\varphi^{,i}\psi_I(I)A^{\nu}A_{\mu} - \delta^{\nu}_{\mu}\left[-\frac{1}{4}F_{ij}F^{ij} + \frac{1}{2}(\varphi_{,i}\varphi^{,i})\psi(I)\right]. \tag{12}$$

Using Eq.(12), the explicit form of the non-zero components of the energy-momentum metric tensor are:

$$T^0_0 = \frac{1}{2}e^{-2\alpha}[e^{-2\gamma}(A')^2 + C^2P(I) + 2e^{-2\gamma}C^2P_I(I)(A)^2] \tag{13}$$

$$T_1^1 = -T_2^2 = -T_3^3 = \frac{1}{2}e^{-2\alpha}[-C^2P(I) + e^{-2\gamma}(A')^2]. \tag{14}$$

The scalar field equation Eq.(10) has a solution:

$$\frac{d\varphi}{d\xi} = \frac{C}{\psi(I)} = CP(I). \tag{15}$$

The nonlinear electromagnetic field equation Eq.(11) is reduced to:

$$(e^{-2\gamma}A')' - C^2e^{-2\gamma}P_I(I)A = 0. \tag{16}$$

From Eq.(5) and Eq.(6), we find the Liouville equation which takes the form [21]:

$$(\beta+\gamma)'' = e^{2(\beta+\gamma)} \tag{17}$$

with the solution :

$$S(k,\xi) = e^{-(\beta+\gamma)} = \begin{cases} \dfrac{\sinh(k\xi)}{k}, & k>0 \\ \xi \quad , & k=0 \\ \dfrac{\sin(k\xi)}{k}, & k<0 \end{cases} \tag{18}$$

Summation of Eq.(4) and Eq.(5), leads to:

$$\beta'' - e^{2(\beta+\gamma)} = -\frac{\chi}{2}e^{-2\gamma}[-C^2P(I) + e^{-2\gamma}(A')^2]. \tag{19}$$

From Eq.(16), Eq.(17) and Eq.(19), we obtain:

$$\gamma'' = \frac{1}{2}\chi(AA'e^{-2\gamma})'. \tag{20}$$

The first integral of Eq.(20) reads:

$$\gamma'(\xi) = \frac{1}{2}\chi(AA'e^{-2\gamma}) + Y, \tag{21}$$

For $Y = 0$, the solution of Eq.(21) is:

$$e^{2\gamma} = \frac{\chi A^2}{2} + H, \tag{22}$$

where H is the constant of integration, which under the regular condition of the components of metric tensor, we write Eq.(22) in the form:

$$e^{2\gamma} = \frac{\chi A^2}{2} + 1. \tag{23}$$

Putting Eq.(23) into Eq.(16), we get :

$$\pm(\xi+\xi_0) = \int \frac{dA}{\left(\dfrac{\chi A^2}{2}+1\right)\sqrt{C^2P(I)+K}}, \tag{24}$$

where $\xi_0 = const$, $K = const$.
The solution of the Eq.(24) will lead to the expression of the electric scalar potential $A(\xi)$ knowing the concrete form of $P(I)$. Knowing this potential, we could rewrite the relation Eq.(13) which will allow us to obtain the energy density per unit invariant volume $T(\xi)$ and total energy of the nonlinear induction fields interaction E_f :

$$T(\xi) = T_0^0(\xi)\sqrt{{}^3_-g} \tag{25}$$

$$E_f = \int_0^{\xi_c} T_0^0\sqrt{{}^3_-g}\,d\xi. \tag{26}$$

In generally from (11) one gets [22]:

$$j^\nu = -\varphi_{,i}\varphi^{,i}\psi_I(I)A^\nu. \tag{27}$$

Expanding it, we obtain the following components of the 4-vector current density j^ν:

$$j^0 = -e^{-2(\gamma+\alpha)}\,C^2\,P_I(I)A, \tag{28}$$

$$j^1 = j^2 = j^3 = 0. \tag{29}$$

The charge density $\rho_e(\xi)$, the charge density per unit invariant volume $\rho(\xi)$ and the total charge Q of elementary particles verify the relations:

$$\rho_e(\xi) = -C^2 e^{-(2\alpha+\gamma)}\, P_I(I)A, \tag{30}$$

$$\rho(\xi) = -C^2 e^{-2\gamma} A\, P_I(I)\, \sin\theta, \tag{31}$$

$$Q = -C^2 \int_0^{\xi_c} e^{-2\gamma} A \frac{dP(I)}{dI}\, \sin\theta\, d\xi. \tag{32}$$

Let us determine in Sec.3, the exact static spherical symmetric solutions to the Einstein's equation, the electromagnetic and scalar nonlinear induction field equations by choosing $P(I)$ in the form:

$$P(I) = P_0 (N - \lambda I)^2, \tag{33}$$

where P_0, N are some dimensionless constants satisfy $N > \lambda I$ and $P(I) = 1$ in $\xi = 0$.

3. Exact static spherical symmetric solutions of the Einstein, the electromagnetic and scalar nonlinear induction field equations

Substituting of Eq.(33) in Eq.(24), the electric scalar potential, solution of the nonlinear electromagnetic field equation Eq.(16) is given:

$$A(\xi) = \sqrt{\frac{N}{\lambda(1-\sigma^2)}}\tanh[b(\xi+\xi_0)], \tag{34}$$

where $b = C\sqrt{NP_0\lambda(1-\sigma^2)}$ and $\sigma^2 = \dfrac{\chi N}{2\lambda}$
From Eq.(2), Eq.(18), Eq.(23) and Eq.(34), the solutions of Einstein's equation are established as:

$$g_{00} = \frac{\cosh^2[b(\xi+\xi_0)] - \sigma^2}{(1-\sigma^2)\cosh^2[b(\xi+\xi_0)]}, \tag{35}$$

$$g_{11} = -\frac{(1-\sigma^2)}{S^4}\frac{\cosh^2[b(\xi+\xi_0)]}{\cosh^2[b(\xi+\xi_0)] - \sigma^2}, \tag{36}$$

$$g_{22} = \frac{g_{33}}{\sin^2(\theta)} = -\frac{(1-\sigma^2)}{S^2}\frac{\cosh^2[b(\xi+\xi_0)]}{\cosh^2[b(\xi+\xi_0)] - \sigma^2}, \tag{37}$$

where $S = S(k,\xi)$.
From Eq.(13), Eq.(18), Eq.(25), Eq.(26), Eq.(34), Eq.(35), Eq.(36) and Eq.(37), the chronometric invariant $I(\xi)$, the energy density $T_0^0(\xi)$, the energy density per unit invariant volume $T(\xi)$ and the total energy E_f of the nonlinear induction fields verify the relations:

$$I = \frac{N\sinh^2[b(\xi+\xi_0)]}{\lambda\left[\cosh^2[b(\xi+\xi_0)] - \sigma^2\right]}, \tag{38}$$

$$T_0^0(\xi) = \frac{S^4N^2P_0C^2}{2\cosh^2[b(\xi+\xi_0)]}\left[\frac{1-\sigma^2}{\cosh^2[b(\xi+\xi_0)]-\sigma^2} + \frac{1}{\cosh^2[b(\xi+\xi_0)]} - \frac{4\sinh^2[b(\xi+\xi_0)]}{\cosh^2[b(\xi+\xi_0)]-\sigma^2}\right], \tag{39}$$

$$T(\xi) = M\left[(1-\sigma^2)\frac{\cosh[b(\xi+\xi_0)]}{\left[\cosh^2[b(\xi+\xi_0)]-\sigma^2\right]^{5/2}} + \frac{1}{\cosh[b(\xi+\xi_0)]\left[\cosh^2[b(\xi+\xi_0)]-\sigma^2\right]^{3/2}} - \frac{4\sinh^2[b(\xi+\xi_0)]\cosh[b(\xi+\xi_0)]}{\left[\cosh^2[b(\xi+\xi_0)]-\sigma^2\right]^{5/2}}\right], \tag{40}$$

$$E_f = M\left[(1-\sigma^2)E_{f_1} + E_{f_2} - 4E_{f_3}\right], \tag{41}$$

where $M = \dfrac{N^2P_0C^2}{2}\left(1-\sigma^2\right)^{3/2}\sin\theta$, $E_{f_1} = \int_0^{\xi_c}\dfrac{\cosh[b(\xi+\xi_0)]}{\left[\cosh^2[b(\xi+\xi_0)]-\sigma^2\right]^{5/2}}d\xi$, $E_{f_2} = \int_0^{\xi_c}\dfrac{1}{\cosh[b(\xi+\xi_0)]\left[\cosh^2[b(\xi+\xi_0)]-\sigma^2\right]^{3/2}}d\xi$
$E_{f_3} = \int_0^{\xi_c}\dfrac{\sinh[b(\xi+\xi_0)]^2\cosh[b(\xi+\xi_0)]}{\left[\cosh^2[b(\xi+\xi_0)]-\sigma^2\right]^{5/2}}d\xi.$

After a series of mathematical transformations, we obtain E_{f_1} and E_{f_2} in the form:

$$E_{f_1} = R_1\left[\frac{\sinh\left[b(\xi_c+\xi_0)\right]}{\left[\cosh^2\left[b(\xi_c+\xi_0)\right]-\sigma^2\right]^{3/2}} - \frac{\sinh(b\xi_0)}{\left[\cosh^2(b\xi_0)-\sigma^2\right]^{3/2}}\right] + 2R_2\left[\frac{\sinh\left[b(\xi_c+\xi_0)\right]}{\sqrt{\cosh^2\left[b(\xi_c+\xi_0)\right]-\sigma^2}} - \frac{\sinh(b\xi_0)}{\sqrt{\cosh^2(b\xi_0)-\sigma^2}}\right]; \tag{42}$$

$$E_{f_2} = \frac{2\sigma P}{\sigma^2-1}\left[\frac{\sinh\left[b(\xi_c+\xi_0)\right]}{\sqrt{\cosh^2\left[b(\xi_c+\xi_0)\right]-\sigma^2}} - \frac{\sinh(b\xi_0)}{\sqrt{\cosh^2(b\xi_0)-\sigma^2}}\right] + P\ln\left|\frac{\sigma\sinh\left[b(\xi_c+\xi_0)\right]-\sqrt{\cosh^2\left[b(\xi_c+\xi_0)\right]-\sigma^2}}{\sigma\sinh\left[b(\xi_c+\xi_0)\right]+\sqrt{\cosh^2\left[b(\xi_c+\xi_0)\right]-\sigma^2}}\right| + P\ln\left|\frac{\sigma\sinh(b\xi_0)-\sqrt{\cosh^2(b\xi_0)-\sigma^2}}{\sigma\sinh(b\xi_0)+\sqrt{\cosh^2(b\xi_0)-\sigma^2}}\right|, \tag{43}$$

where $R_1 = \dfrac{1}{3b(1-\sigma^2)}$, $R_2 = \dfrac{1}{3b(1-\sigma^2)^2}$, $P = -\dfrac{1}{2b\sigma^3}$.
The integral E_{f_3} being difficult to obtain, we will make its extension on $[0,+\infty]$ then deduce its localization on $[0,\xi_c]$. To do this, we consider the following formula from the standard table of integrals [23] :

$$\int_0^{\infty}\frac{\sinh^{\mu-1}x\cosh^{\nu-1}x}{\left(\cosh^2 x-\beta\right)^{\rho}}dx = 2B\left(\frac{\mu}{2},1+\rho-\frac{\mu+\nu}{2}\right)\times F\left(\rho,1+\rho-\frac{\mu+\nu}{2};1+\rho-\frac{\nu}{2},\beta\right);\beta\notin(1,\infty),\ Re(\mu)>0,\ 2Re(1+\rho)>Re(\nu+\mu). \tag{44}$$

In this expression, $B(a,b)$ is the Bêta function and $F(a,b;c,m)$ is the Gauss hyper-geometric function. So, we obtain:

$$E_{f_3} < \int_0^{\infty}\frac{\sinh\left[b(\xi+\xi_0)\right]^2\cosh\left[b(\xi+\xi_0)\right]}{\left[\cosh^2\left[b(\xi+\xi_0)\right]-\sigma^2\right]^{5/2}}d\xi, \tag{45}$$

$$E_{f_3} < F\left(\frac{5}{2};1;\frac{5}{2};\sigma^2\right)\times 2B\left(\frac{3}{2},1\right), \tag{46}$$

$$E_{f_3} < \frac{4}{3(1-\sigma^2)}. \tag{47}$$

Thus, we have:

$$E_f < \infty. \tag{48}$$

Moreover, the expressions Eq.(30), Eq.(31) and Eq.(32) are transformed into:

$$\rho_e(\xi) = \frac{2P_0C^2S^4N^{3/2}\sqrt{\lambda}\sinh\left[b(\xi+\xi_0)\right]}{\cosh^2\left[b(\xi+\xi_0)\right]\sqrt{\cosh^2\left[b(\xi+\xi_0)\right]-\sigma^2}}, \tag{49}$$

$$\rho(\xi) = \frac{2C^2P_0\sqrt{\lambda}\left[N(1-\sigma^2)\right]^{3/2}\sinh\left[b(\xi+\xi_0)\right]}{\cosh\left[b(\xi+\xi_0)\right]^{-1}\left[\cosh^2\left[b(\xi+\xi_0)\right]-\sigma^2\right]^2}\sin\theta, \tag{50}$$

$$Q = -CNP_0^{1/2}\left[\frac{(1-\sigma^2)}{\cosh^2\left[b(\xi_c+\xi_0)\right]-\sigma^2} - \frac{(1-\sigma^2)}{\cosh^2\left[b\xi_0\right]-\sigma^2}\right]\sin\theta. \tag{51}$$

In Sec.4, we will discuss the influence of the function $S(k,\xi)$ on the obtained solutions.

4. Discussion

The electric scalar potential $A(\xi)$, the component $g_{00}(\xi)$ of the metric tensor, the energy and the charge densities per unit invariant volume $(T(\xi),\rho(\xi))$ are regular functions, independent of the concrete form of the function $S(k,\xi)$.
Fig.1 and Fig.2 give a graphic illustration:
In Fig.1(a), $A(0)=0$, the solution of the equation to the electromagnetic and scalar fields of nonlinear induction and taking into account the own gravitational field of elementary particles describes a massless system contrary to the massive system obtained by [16].
In Fig.1 (b), the regularity condition of the component of the metric tensor g_{00} in infinite space imposes the nullity of the integration constant ξ_0.
In Fig.2 (a), the energy density per unit invariant volume is an asymptotic and localized function. Its depth and width of localization depend on the value of the integration constants.
In Fig.2 (b), the charge density per unit invariant volume is also an asymptotic and localized function, with depth and width of localization varying with the values of the constants integration.
Let us point out from Eq.(48) and Eq.(51) that, the total charge of elementary particles is a finite quantity and the total energy of fields is limited.
On the other hand, the components g_{11}, g_{22}, g_{33} of the metric tensor, the energy and charge densities $\left(T_0^0,\rho_e\right)$ depend on the concrete form of the function $S(k,\xi)$.
In order to respect the regularity conditions [16], the obvious and trivial form $S(k,\xi)=\xi$ is often used. Let us analyze the solutions of field and Einstein equations of Sec.3 using the limited development in $k\xi$ of all concrete forms of $S(k,\xi)$ in the following different cases:

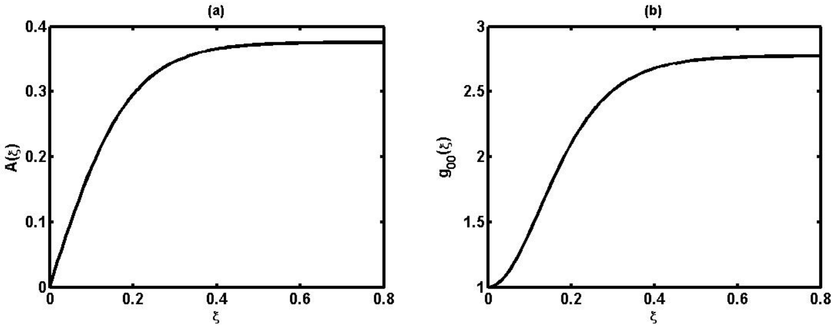

Figure 1: (a)-Electric scalar potential $A(\xi)$, (b)-Component g_{00} of the metric tensor. The parameter values used for these simulation are: $\lambda = 39$; $\xi_0 = 0$; $C = N = 2$; $\chi = 8\pi$ and $\theta = \dfrac{\pi}{2}$.

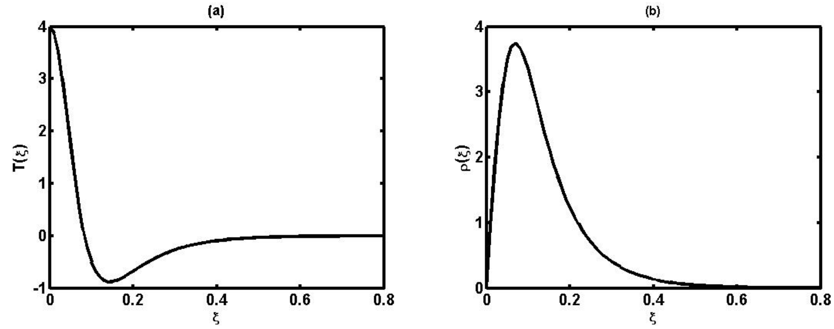

Figure 2: (a)-Energy density per unit invariant volume $T(\xi)$ and (b)- Charge density per unit invariant volume. The parameter values used for these simulation are: $\lambda = 39$; $\xi_0 = 0$; $C = N = 2$; $\chi = 8\pi$ and $\theta = \dfrac{\pi}{2}$.

Case 1: k > 0

The concrete form of $S(k,\xi)$ is:

$$S(k,\xi) = \frac{\sinh(k\xi)}{k}. \tag{52}$$

The limited development of Eq.(52) to low order in $k\xi$, introduced in Eq.(36), Eq.(37), Eq.(39) and Eq.(49) lead respectively to:

$$g_{11} = -\frac{\left(1-\sigma^2\right)}{\left[\xi + \dfrac{k^2\xi^3}{6}\right]^4} \frac{\cosh^2\left[b(\xi+\xi_0)\right]}{\cosh^2\left[b(\xi+\xi_0)\right]-\sigma^2}, \tag{53}$$

$$g_{22} = \frac{g_{33}}{\sin^2(\theta)} = -\frac{\left(1-\sigma^2\right)}{\left[\xi + \dfrac{k^2\xi^3}{6}\right]^2} \frac{\cosh^2\left[b(\xi+\xi_0)\right]}{\cosh^2\left[b(\xi+\xi_0)\right]-\sigma^2}, \tag{54}$$

$$T_0^0(\xi) = \frac{N^2 P_0 C^2 \left[\xi + \dfrac{k^2\xi^3}{6}\right]^4}{2\cosh^2\left[b(\xi+\xi_0)\right]} \left[\frac{1-\sigma^2}{\cosh^2\left[b(\xi+\xi_0)\right]-\sigma^2} + \frac{1}{\cosh^2\left[b(\xi+\xi_0)\right]} - \frac{4\sinh^2\left[b(\xi+\xi_0)\right]}{\cosh^2\left[b(\xi+\xi_0)\right]-\sigma^2}\right], \tag{55}$$

$$\rho_e(\xi) = \frac{2P_0C^2N^{3/2}\sqrt{\lambda}\left[\xi + \dfrac{k^2\xi^3}{6}\right]^4 \sinh\left[b(\xi+\xi_0)\right]}{\cosh^2\left[b(\xi+\xi_0)\right]\sqrt{\cosh^2\left[b(\xi+\xi_0)\right]-\sigma^2}}. \tag{56}$$

Case 2: k < 0

$$S(k,\xi) = \frac{\sin(k\xi)}{k}. \tag{57}$$

The relations Eq.(57) at low order in $k\xi$ in those Eq.(36), Eq.(37), Eq.(39) and Eq.(49) verify the equalities:

$$g_{11} = -\frac{\left(1-\sigma^2\right)}{\left[\xi - \frac{k^2\xi^3}{6}\right]^4}\frac{\cosh^2\left[b(\xi+\xi_0)\right]}{\cosh^2\left[b(\xi+\xi_0)\right]-\sigma^2}, \tag{58}$$

$$g_{22} = \frac{g_{33}}{\sin^2(\theta)} = -\frac{\left(1-\sigma^2\right)}{\left[\xi - \frac{k^2\xi^3}{6}\right]^2}\frac{\cosh^2\left[b(\xi+\xi_0)\right]}{\cosh^2\left[b(\xi+\xi_0)\right]-\sigma^2}, \tag{59}$$

$$T_0^0(\xi) = \frac{N^2 P_0 C^2\left[\xi - \frac{k^2\xi^3}{6}\right]^4}{2\cosh^2\left[b(\xi+\xi_0)\right]}\left[\frac{1-\sigma^2}{\cosh^2\left[b(\xi+\xi_0)\right]-\sigma^2}+\frac{1}{\cosh^2\left[b(\xi+\xi_0)\right]}-\frac{4\sinh^2\left[b(\xi+\xi_0)\right]}{\cosh^2\left[b(\xi+\xi_0)\right]-\sigma^2}\right], \tag{60}$$

$$\rho_e(\xi) = \frac{2\, P_0\, C^2\, N^{3/2}\sqrt{\lambda}\left[\xi - \frac{k^2\xi^3}{6}\right]^4}{\cosh^2\left[b(\xi+\xi_0)\right]}\frac{\sinh\left[b(\xi+\xi_0)\right]}{\sqrt{\cosh^2\left[b(\xi+\xi_0)\right]-\sigma^2}}. \tag{61}$$

Case 3: k = 0

$$S(k,\xi) = \xi. \tag{62}$$

Substituting Eq.(62) into Eq.(36), Eq.(37), Eq.(39), Eq.(49), we obtain:

$$g_{11} = -\frac{\left(1-\sigma^2\right)}{\xi^4}\frac{\cosh^2\left[b(\xi+\xi_0)\right]}{\cosh^2\left[b(\xi+\xi_0)\right]-\sigma^2}, \tag{63}$$

$$g_{22} = \frac{g_{33}}{\sin^2(\theta)} = -\frac{\left(1-\sigma^2\right)}{\xi^2}\frac{\cosh^2\left[b(\xi+\xi_0)\right]}{\cosh^2\left[b(\xi+\xi_0)\right]-\sigma^2}, \tag{64}$$

$$T_0^0(\xi) = \frac{N^2\, P_0\, C^2\xi^4}{2\cosh^2\left[b(\xi+\xi_0)\right]}\left[\frac{1-\sigma^2}{\cosh^2\left[b(\xi+\xi_0)\right]-\sigma^2}+\frac{1}{\cosh^2\left[b(\xi+\xi_0)\right]}-\frac{4\sinh^2\left[b(\xi+\xi_0)\right]}{\cosh^2\left[b(\xi+\xi_0)\right]-\sigma^2}\right], \tag{65}$$

$$\rho_e(\xi) = \frac{2\, P_0\, C^2\, N^{3/2}\sqrt{\lambda}\xi^4}{\cosh^2\left[b(\xi+\xi_0)\right]}\frac{\sinh\left[b(\xi+\xi_0)\right]}{\sqrt{\cosh^2\left[b(\xi+\xi_0)\right]-\sigma^2}}. \tag{66}$$

Fig.3 and Fig.4 are a summary illustration of the properties of all the solutions obtained in each of the different cases above:

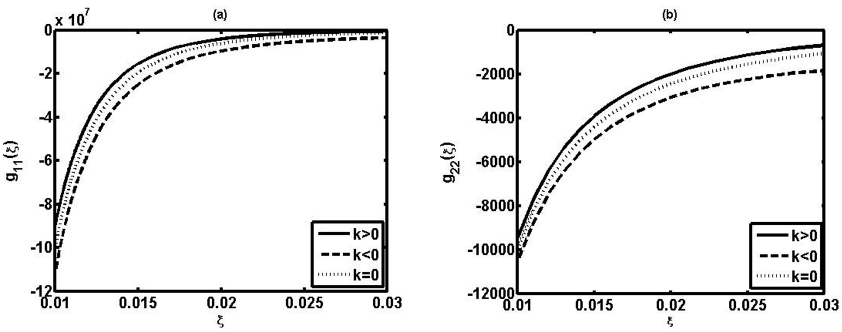

Figure 3: (a)-Components g_{11} of the metric tensor , (b)-Components g_{22} of the metric tensor, using the values for the parameters as in Fig.1 and $k = 0;\ \pm 1.6;\ \pm 40$.

In Fig.3 (a) and Fig.3(b), the components g_{11}, g_{22} and g_{33} of the metric tensor present a gravitational singularity not controllable by change of variable in infinite space whatever the form of the function $S(k,\xi)$ as stated in [24].
In Fig.4(a), the energy densities $T_0^0(\xi)$ are asymptotic, localized functions, all canceling in infinite space. The width of the location and the depth vary according to the values of the integration constants, especially that of k.
In Fig.4(b), the charge densities $\rho_e(\xi)$ exhibit the same properties as the energy densities in Fig.4(a) but are positive defined.
We will focus in Sec.5 on the influence to the own gravitational field of the elementary particles and on the role of the nonlinearity fields in obtained soliton-like solutions.

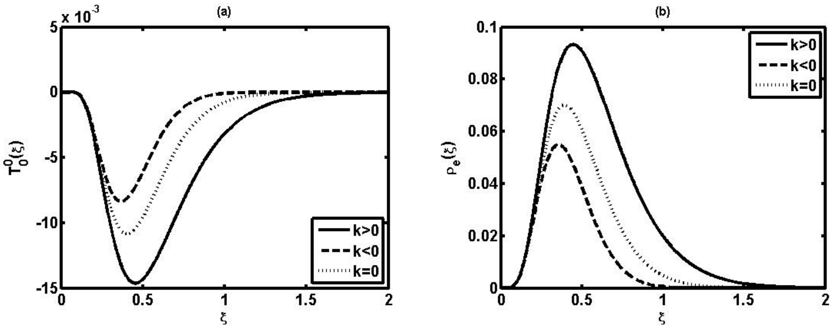

Figure 4: (a)- Energy densities $T_0^0(\xi)$ and (b)- Charge densities $\rho_e(\xi)$ using the values for the parameters as in Fig.1 and $k = 0;\ \pm 1.6;\ \pm 40$.

5. Solutions in flat space-time and in linear case

5.1. Solutions in flat space-time

In the absence to the own gravitational field of elementary particles, the metric Eq.(1)becomes:

$$ds^2 = dt^2 - d\xi^2 - \left[d\theta^2 + \sin^2(\theta)\, d\varphi^2\right]. \tag{67}$$

The nonlinear electromagnetic field equation Eq.(16) becomes :

$$A'' - C^2 P_I(I) A = 0, \tag{68}$$

which has the solution:

$$\pm C(\xi + \xi_0) = \int \frac{dA}{\sqrt{P(I)}}. \tag{69}$$

Using Eq.(33), the relation Eq.(69) gives the expression of the electric scalar potential:

$$A(\xi) = \sqrt{\frac{N}{\lambda}} \tanh\left[u(\xi + \xi_0)\right], \tag{70}$$

where $u = C\sqrt{NP_0\lambda}$.
The energy and charge densities per unit invariant volume $(T(\xi), \rho(\xi))$ verify the following expressions:

$$T(\xi) = \frac{P_0 N^2\, C^2}{\cosh^4\left[u(\xi + \xi_0)\right]} \left[1 - 2\sinh^2\left[u(\xi + \xi_0)\right]\right] \sin\theta, \tag{71}$$

$$\rho(\xi) = 2C^2\, P_0\, N^{3/2} \sqrt{\lambda}\, \frac{\sinh\left[u(\xi + \xi_0)\right]}{\cosh^3\left[u(\xi + \xi_0)\right]} \sin\theta. \tag{72}$$

Fig.5 below, shows the influence of the own gravitational field of the elementary particles on the solutions established in Sec.4.

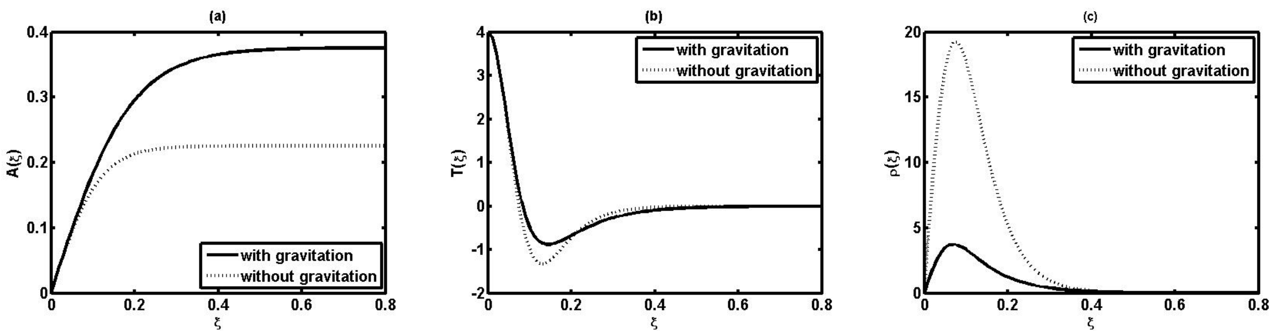

Figure 5: (a)-Electric scalar potential $A(\xi)$, (b)- Energy density per unit invariant volume $T(\xi)$ and (c)- Charge density per unit invariant volume $\rho(\xi)$ keeping the values for the parameters as in Fig.1 and Fig.2.

In Fig.5 (a), we see that in the absence of the own gravitational field of the elementary particles, the electric scalar potential Eq.(70) remains a regular but of smaller amplitude than the one obtained in Fig.1(a).
In Fig.5 (b), the energy density per unit invariant volume has the same properties as those obtained by taking into account the own gravitational field of the elementary particles but with almost equal depth.
In Fig.5 (c), the charge density per unit invariant volume has a more extensive depth in the absence of the elementary particle gravitational field, is asymptotic and localized in an interval whose width depends on the integration constants .

The total energy of fields E_f and the total charge Q of elementary particles are given by the expressions:

$$E_f = R_3\left[\tanh^3(u\xi_0) - \tanh^3[u(\xi_c+\xi_0)] - \tanh(u\xi_0) + \tanh[u(\xi_c+\xi_0)]\right], \tag{73}$$

$$Q = -\frac{2C\,N\,P_0^{1/2}}{3}\left[\frac{\sin\theta}{\cosh^3[u(\xi_c+\xi_0)]} - \frac{\sin\theta}{\cosh^3[u\xi_0]}\right] \tag{74}$$

where $R_3 = \dfrac{C\,N^{3/2}\,P_0^{1/2}}{\lambda^{1/2}}\sin\theta$.

5.2. Linear solutions

In the absence of the non-linearity of the fields, the coupling is minimal :

$$\psi(I) = P(I) = P_0\,N^2 = 1. \tag{75}$$

The linear equations of the electromagnetic and scalar fields are:

$$(e^{-2\gamma}A')' = 0, \tag{76}$$

$$\frac{d\varphi}{d\xi} = C. \tag{77}$$

Their solutions are:

$$A(\xi) = K\int e^{2\gamma}d\xi,\; K = const, \tag{78}$$

$$\varphi(\xi) = C\xi + D,\; C = const,\, D = const. \tag{79}$$

The non-zero components of the energy-momentum metric tensor are:

$$T_0^0(\xi) = \frac{1}{2}e^{-2\alpha}[e^{-2\gamma}(A')^2 + C^2], \tag{80}$$

$$T_1^1 = -T_2^2 = -T_3^3 = \frac{1}{2}e^{-2\alpha}[e^{-2\gamma}(A')^2 - C^2]. \tag{81}$$

The sum of the Einstein tensor $\begin{pmatrix}1\\1\end{pmatrix}$ + $\begin{pmatrix}2\\2\end{pmatrix}$ gives the equation Eq.(17). On the other hand that $\begin{pmatrix}0\\0\end{pmatrix}$ + $\begin{pmatrix}1\\1\end{pmatrix}$ leads to:

$$\beta'' - e^{2(\beta+\gamma)} = -\frac{\chi\,K^2}{2}e^{2\gamma}. \tag{82}$$

From Eq.(17) and Eq.(82), we obtain:

$$\gamma'' = \frac{\chi\,K^2}{2}e^{2\gamma}. \tag{83}$$

For $K^2 = -1$, the relation Eq.(83) becomes:

$$\gamma'' = -\frac{\chi}{2}e^{2\gamma}, \tag{84}$$

which has the solution:

$$e^{2\gamma} = \frac{2\eta^2}{\chi}\frac{1}{\cosh^2(\eta\xi)}, \quad \eta^2 = const. \tag{85}$$

The regularity of Eq.(85) in the infinite space, allows to have:

$$e^{2\gamma} = \frac{1}{\cosh^2(\eta\xi)}, \quad \eta^2 = \frac{\chi}{2}. \tag{86}$$

From Eq.(2), Eq.(18) and Eq.(86), we establish the solutions of Einstein's equation:

$$g_{00} = \frac{1}{\cosh^2(\eta\xi)}, \tag{87}$$

$$g_{11} = -\frac{\cosh^2(\eta\xi)}{S^4}, \tag{88}$$

$$g_{22} = \frac{g_{33}}{\sin^2(\theta)} = -\frac{\cosh^2(\eta\xi)}{S^2}. \tag{89}$$

From Eq.(87), Eq.(88) and Eq.(89), the electric scalar potential $A(\xi)$,the energy density $T_0^0(\xi)$, the energy density per unit invariant volume $T(\xi)$, the total energy of field interaction E_f, the components of 4-vector current density (j^0, j^1, j^2, j^3), the charge density $\rho_e(\xi)$ and the total charge of elementary particles Q verify the relations:

$$A(\xi) = \frac{i}{\eta}\tanh(\eta\xi), \tag{90}$$

$$T_0^0(\xi) = \frac{S^4}{2\cosh^2(\eta\xi)}\left[C^2 - \frac{1}{\cosh^2(\eta\xi)}\right], \tag{91}$$

$$T(\xi) = \frac{1}{2}\left[C^2\cosh(\eta\xi) - \frac{1}{\cosh(\eta\xi)}\right]\sin\theta, \tag{92}$$

$$E_f = \frac{1}{2\eta}\left[\frac{\pi}{2} + C^2\sinh(\eta\xi_c) - 2\arctan(e^{\eta\xi_c})\right]\sin\theta, \tag{93}$$

$$j^0 = j^1 = j^2 = j^3 = 0, \tag{94}$$

$$\rho_e(\xi) = \rho(\xi) = 0, \tag{95}$$

$$Q = 0. \tag{96}$$

The component of the metric tensor $g_{00}(\xi)$ is a regular function while the other components of this tensor present a singularity in infinite space as obtained in the nonlinear case. The energy density is a regular and localized function whatever the form of the function $S(k,\xi)$. The total energy of the interaction fields is bounded but the electric scalar potential is imaginary. This solution is a soliton-like solution.
For $K^2 \neq -1$, the relation Eq.(83) has a solution:

$$e^{2\gamma} = -\frac{2\eta^2}{\chi K^2}\frac{1}{\cosh^2(\eta\xi)}, \quad \eta^2 = const. \tag{97}$$

Its regularity in the infinite space, imposed:

$$e^{2\gamma} = \frac{1}{\cos^2(\xi)}, \quad K^2 = \frac{2}{\chi}, \quad \eta^2 = -1. \tag{98}$$

From Eq.(2), Eq.(18) and Eq.(98), the solutions of the Einstein equation are as follows:

$$g_{00} = \frac{1}{\cos^2(\xi)}, \tag{99}$$

$$g_{11} = -\frac{\cos^2(\xi)}{S^4}, \tag{100}$$

$$g_{22} = \frac{g_{33}}{\sin^2(\theta)} = -\frac{\cos^2(\xi)}{S^2}. \tag{101}$$

From Eq.(99), Eq.(100) and Eq.(101), the electric scalar potential $A(\xi)$,the energy density $T_0^0(\xi)$, the energy density per unit invariant volume $T(\xi)$, the total energy of field interaction E_f, the components of 4-vector current density (j^0, j^1, j^2, j^3), the charge density $\rho_e(\xi)$ and the total charge of elementary particles Q verify the relations:

$$A(\xi) = \sqrt{\frac{2}{\chi}}\tan(\xi), \tag{102}$$

$$T_0^0(\xi) = \frac{S^4}{2\cos^2(\xi)}\left[C^2 + \frac{2}{\chi\cos^2(\xi)}\right], \tag{103}$$

$$T(\xi) = \frac{1}{2}\left[C^2\cos(\xi) + \frac{2}{\chi\cos(\xi)}\right]\sin\theta, \tag{104}$$

$$E_f = \frac{1}{2}\left[\frac{2}{\chi}\ln\left[\tan\left(\frac{\pi}{4} + \frac{\xi_c}{2}\right)\right] + C^2\sin(\xi_c)\right]\sin\theta, \tag{105}$$

$$j^0 = j^1 = j^2 = j^3 = 0, \tag{106}$$

$$\rho_e(\xi) = \rho(\xi) = 0, \tag{107}$$

$$Q = 0. \tag{108}$$

For $\xi \neq \frac{\pi}{2}$, solutions obtained have the same properties as those established in the case where $K^2 = -1$.

6. Conclusion

It has been proved that with the limited development to low order in $k\xi$ of the different forms of the function $S(k,\xi)$, we can solve the Einstein equation, the electromagnetic and scalar nonlinear induction equation by taking into account the own gravitational field of the elementary particles as in the case most often approached $S(k,\xi) = \xi$. The obtained results show that taking into account or not the own gravitational field of the elementary particles, the metric tensor function are regular. All energy densities are localized, have a depth and a width of localization whose interval varies according to the values of the integration constants. The total energy of nonlinear induction fields and the total charge of the elementary particles are finite. These solutions describe massless systems contrary to the massive ones obtained by Rybakov et al. [16]. They give a new orientation to Heisenberg's prediction that the determination of the masses of elementary particles could not be done in quantum mechanic. These solutions are of soliton-like solutions and constitute a model can be used to describe the complex internal configuration of elementary particles. In the near future, we will do similar work but with the 4-vector potential chosen in the form $A(\ A_0, 0,\ A_2,\ A_3)$.

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