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Research paper

Closure condition of a function by the Newton-Raphson method

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Abstract

The work was developed with the purpose of briefly presenting the Newton-Raphson method for the calculation of real roots of equations. The objective was to present an experiment that would consolidate with the study in simulation format to apply the concepts developed through the root isolation procedures, definition of the initial value by the appropriate interval, as well as in the stopping criteria, by the ge-ometric observation of the tangent line of the graphs and the iteration processes, approximating the values of the reality of the zeros of the function. We also discuss the possible flaws of the simulation system presented, given the rules of the method used.

Keywords: *Method; Newton-Raphson; Roots; Iteration; Convergence.*

1. Introduction

Calculations for roots of polynomials up to degree 2 are easily determined, without seeking sophisticated methods or approximations. Increasing the degree to greater than 2, we will need to look for other methods besides Bhaskara, as it will need some criteria to start and close the calculation.

This work presents the Newton-Raphson method, because in a faster and more objective way, it answers some needs for the questions of roots of high-degree equations, also naturally having its limitations.

We will approach an introductory idea, going through the main procedures for the isolation of the root and perception of the appropriate interval, as well as defining the criteria of stop, convergence, geometrically observing the graphs or by the iterated shape, through the errors and their approximations.

Finally, we will present in an experimental way, in simulation format, an analysis of one of the equations available by the system to discuss the behavior and possible failures according to the criteria of stop and real value of the roots of the equation.

2. Newton-Raphson method

The calculation of functions, equations, expressions, depending on the system analyzed, is used to extract various information. In the case of this work involving polynomials, the discussion of how to solve such questions goes back to ancient Babylon, where the scientists of the time were able to solve the exact roots, which we also known as the "zeros" of quadratic equations. The more time passes, the more studies advance, and scientists can reveal more problems that were previously open.

We know that to solve the quadratic equations, we can use the Bhaskara method to find the roots, and for cases of polynomials with degree 3 and 4, we have as examples the Ferrari and Cardano method [1].

On the other hand, Abel, in his work, shows that there is no way to determine exactly the roots of polynomials with a degree greater than 4. Within this reality, the search for the determination of roots of this nature led to what we now know as the Newton-Raphson method, which is also known only as Newton's method [2].

There are other methods for resolution, such as Bisection, Secants, False Position, Fixed Point, among others, having advantages and limitations, being direct or iterative and within this reality, other methods are emerging to solve the limitations and optimize the iteration time.

The case of the Fixed Point, for example, needs an iteration function, being suitable for certain convergence criteria. The method of this work aims at a fixed iteration function that meets these criteria [3].

The combination of this name by the method was precisely because they created simultaneously. Isaac Newton (1642-1727) wrote the method in 1671 in his book "Method of Fluxions" published in 1736, exemplifying the result by finding the root with values between 2 and 3 of the equation $x^3 - 2x - 5 = 0$ [4].

In the case of Joseph Raphson (1648-1715) he discovered the same technique, but in a simpler way, publishing before the publication of Newton's book, in his book "Analysis aequationum universalis" in 1690 [5].

The Newton-Raphson method (NRM) has a quadratic convergence in most cases, being linear in cases of root with multiplicity and within this limitation, the mathematicians Rabinowitz and Ralston modified the function of the iteration of the NRM, developing the method of multiple roots to recover the convergence of the quadratic form of this multiplicity of roots, although it also has limitations due to the need for two derivatives, because depending on the function it becomes complicated to solve [6].
If we consider a function $f: I \to \mathbb{R}$ with a continuous derivative and not zero in the entire interval I and take the initial value x_0 belonging to that interval, we then define the method in the form x_0 , where $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ it x_1 also belongs to the interval I, until we find the limit of values for the root of the function f. In general, we have for this method the form

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (1)$$

Aiming for $f(x_n) = 0$. This method can be interpreted geometrically, through a given initial value, taking the tangent line in x_0 to the graph of f, with x_1 the point of intersection of this line being with the axis of the abscissas [7].

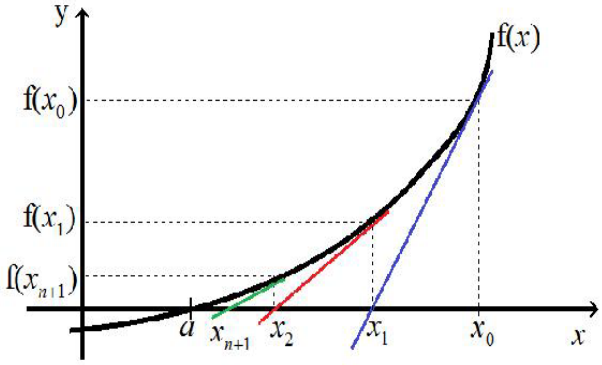

Fig. 1: Curve *f(x)* and Tangent Lines [8].

The polynomial of degree n must be an integer and this function in a generalized form is expressed.

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \quad (2)$$

With $a_0 \neq 0$ and $a_0, a_1, \ldots, a_n$ the real numbers that are the coefficients [9].
To analyze the roots of the polynomial or also known as zeros of functions, we must adopt some procedures. First, we must find the interval that contains the root, which is the condition of isolation of the root, and the second procedure is the application of the method in an iterated way to find the desired root, which is the condition of the refinement of the interval [3].
According to Campos Filho, we can use the method of graphical analysis of the function or tabulation for root isolation [10]. In the graphical analysis, we visualize through the sketch the points of intersection with the x-axis, as well as rewriting the function, $f(x) = f_1(x) - f_2(x)$ with the roots located at the points of intersection between the graphs $f_1(x)$ and $f_2(x)$ [11].
In the case of tabulation, we use Bolzano's Theorem or Annulment, where we consider the function *f* continuous in the interval $[a, b]$ and if there is the condition f(a) and f(b) with contrary signs, we're certainly going to have a c in $[a, b]$, such that $f(c) = 0$, Soon we'll have $f(a). f(b) < 0$. If we derive f and keep the sign of $[a, b]$, So we're going to have a single zero of *f* on that interval, with $f'(x) \neq 0$ [12].
We must have one ξ that applying to f, we will have $f(\xi) = 0$. Geometrically when we analyze the master equation of the NRM, equation (1), we see that x_{n+1} it is geometrically the abscissa of the point where the tangent to the graph of *f* at the point x_n meets the axis of the horizontal. Within this idea of NRM is that the closer the tangent approaches the curve, then its intersection with the horizontal axis approaches the point ξ , to objectify in $f(\xi) = 0$, as shown in figure 1.
But within this geometric analysis, there is a flaw in the NRM, because if we have a function $f(x) = e^x$, which has no root, the sequence x_n of approximations will not converge. For the case of inflection, i.e., the $f''(x) = 0$ method fails to converge.
We cannot have a case of zero slope, in which $f'(x) = 0$ where it will cause a null division for NRM, not reaching the horizontal axis [13].

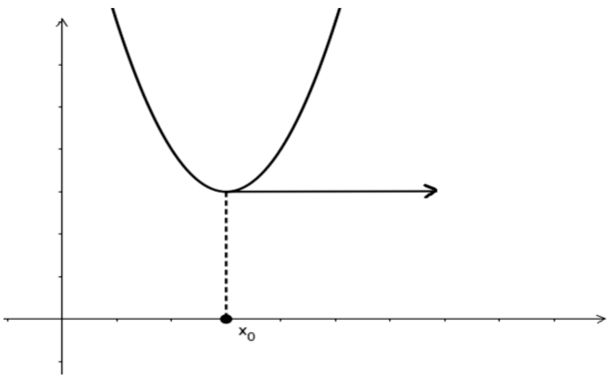

Fig. 2: Case of a Null Slope [13].

Doing an example of convergence analysis, when we estimate the root of the function $f(x) = x - x^3$ in the interval $[-1/2, 1/2]$, looking at figure 3, we can immediately state that there is a single root of *f* in this interval.

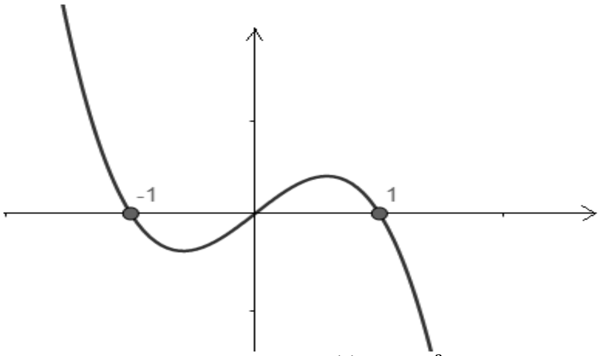

Fig. 3: Function Graph $f(x) = x - x^3$ [13].

The first derivative of this function, we have $f'(x) = 1 - 3x^2$ and by the NRM, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n - x_n^3}{1 - 3x_n^2} = -\frac{2x_n^3}{1 - 3x_n^2}$$

With initial approximation $x_0 = 0{,}2$, where Table 1, shows the example of iterations for this case.

Table 1: Iterations for $x_0 = 0{,}2$ [13]

Iterations n	x_n
0	0,20000
1	-0,01818
2	0,00001
3	0,00000
4	0,00000
5	0,00000

The method is iterative, seeking to find a necessary approximation for the root from a given initial value, within a defined range, with isolated root and refinement across iterations. The iterations are terminated when the precision is sufficient [3].
Within this iteration process, a discussion is also necessary to establish an initial condition for the development of the method, which is error. The discussion about the error is the fundamental factor for the closure of the iterations, which is nothing more than the stop criterion. In general, there are rounding and truncation errors, which are errors caused by the limitations of machinery systems, as it limits the number of significant digits used in the representation of a number [11].
According to Chapra and Canale [6], in the case of absolute and relative errors, they deal with the proximity between the value obtained and the real value. The cause of the general error is the use of finite approximations to represent real numbers. The absolute error consists of the modulus of the difference between the actual value ξ and the approximate value $\bar{x}$ found in the form

$$E_a = |\xi - \bar{x}| \tag{3}$$

Equation (3) is sensitive to be used precisely because it requires a capacity of an analytical method for real value ξ. For this reason, an estimate is usually adopted for this absolute error given by the modulus of the difference between the current and the previous approximation in the form

$$E_a = |x_{n+1} - x_n| \tag{4}$$

According to Franco [14], the relative error given in the form

$$E_r = |x_{n+1} - x_n|/|\bar{x}| \tag{5}$$

This discussion about the error is important for the stopping criteria, where when we consider the successive approximations x_{k+1} and x_k and ε the established precision, we have $|f(x_k)| \leq \varepsilon$ $|x_{k+1} - x_k| \leq \varepsilon$ $\frac{|x_{k+1} - x_k|}{|x_{k+1}|} \leq \varepsilon$, and the limit number of iterations [14].
According to Machado and Alves, in a more detailed discussion of convergence within an interval, $[a, b]$ it goes from the first derivative to the second of the product of functions. If we have the form, $f(a).f(b) > 0$ either there will be an even number of real roots, with their multiplicities, or there will be no real root in the interval (a, b).
For the case of, $f(a).f(b) < 0$ there will be an odd number of real roots, with their multiplicities in the interval (a, b). If we have the form, the $f'(a).f'(b) > 0$ function will either be only increasing or only decreasing never alternating in the interval (a, b).
If we have the form, the $f'(a).f'(b) < 0$ function will alternate between increasing and decreasing in the interval (a, b). If we have the form, the $f''(a).f''(b) > 0$concavity of the function will not invert into the interval (a, b). If we have the form, the $f''(a).f''(b) < 0$ concavity of the function will invert into the interval (a, b).
After this careful analysis, it is evident that to converge to a given root in the interval (a, b), the following relation must be followed

$$f(a).f(b) < 0, f'(a).f'(b) > 0 \text{ e } f''(a).f''(b) < 0 \tag{6}$$

3. Simulation demonstration and analysis of results

The simulational application was developed to complement the study analyzed on the Newton-Raphson method. One of the demonstrations of Fritz "Wolfram Demonstrations Project" was used to follow in an "experimental" way the treatment of functions for this type of method [15].

The general objective is to observe the behavior and apply the method as previously studied through the available functions.

The simulation presents the operator with four functions, with six iterations and initial value ranging from 0.01 to 6.11, unless the operator needs to further increase the pre-established initial value range. From then on, we used the procedures adopted as root isolation and subsequent iteration to analyze the graphs.

The equation analyzed was only $f(x) = x^3 - 3x^2 + x - 1$ We perform the first derivative by obtaining $f'(x) = 3x^2 - 6x + 1$ and the derivative by obtaining the second obtaining $f''(x) = 6x - 6$. We performed the graph on the symbolab website of the analyzed function and presented it as shown in figure 4. Looking at figure 4, we can see that the interval of the desired root is between 2 and 3. The simulator does not present the error condition, but limits the number of iterations, that is, up to 6.

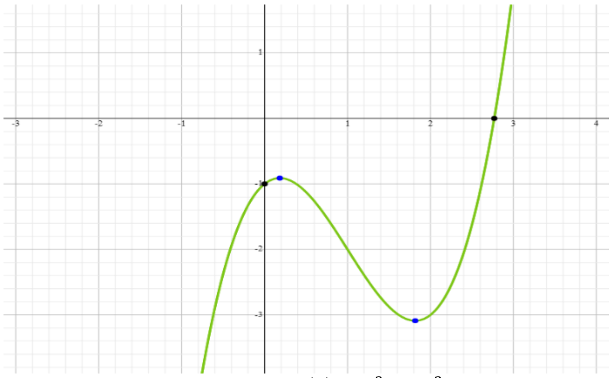

Fig. 4: Function Graph $f(x) = x^3 - 3x^2 + x - 1$.

According to the observation of the graph, the zero of the function touches only once on the horizontal axis, we can already have the number 2.85 as the initial value, by the system until we make the successive iterations and compare with the values of the simulator.

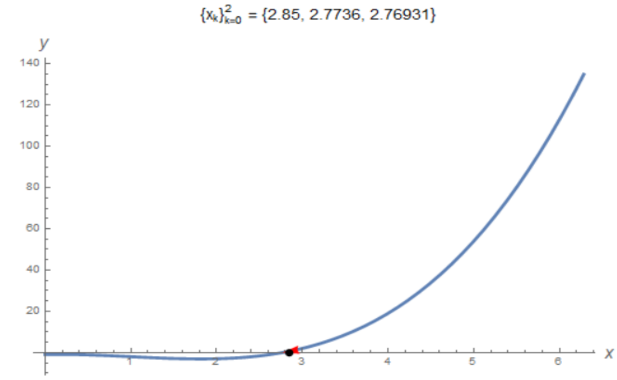

Fig. 5: Function Graph $f(x) = x^3 - 3x^2 + x - 1$ of Iteration 2, with an Initial Value of 2.85.

Looking at the calculations, they really match the simulator and iteration 2 is already enough for $x_2 = 2{,}76931$ because the value of zero is for a value of 0.000130. If we use the same initial value, but increase the iterations, the system closes with a root value for $x_k = 2{,}76929$.

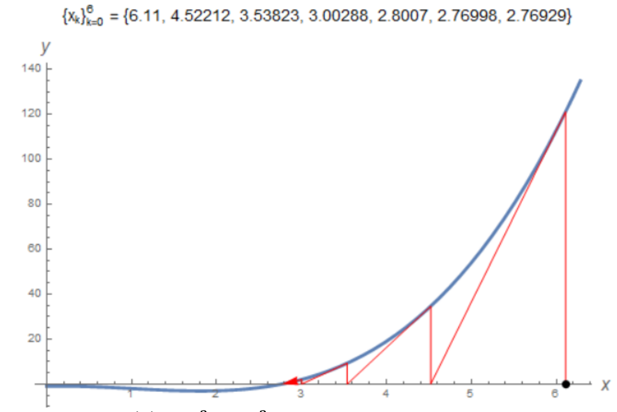

Fig. 6: Function Graph $f(x) = x^3 - 3x^2 + x - 1$ of Iteration 6, with an Initial Value of 6.11.

The red line indicates the tangent line procedure, where the values that touch the x-axis are approaching the value of the real root. In the same way, the system closes with $x_k = 2{,}76929$, which corresponds to a value of -0.000017, not matching the predicted value that should be zero.

For the purposes of more approximate calculation, the value should be $x_k = 2{,}76930$ as it would reach the value of 0.000056511557, that is, closer to reality.
For simulation purposes, the system had a good intention to develop the Newton-Raphson method, but it failed in the stopping criterion, not being a value consistent with reality.

4. Conclusion

The work sought to develop in a more practical way about the calculation of real roots through the Newton-Raphson method, without going into details of previous methods and their comparisons, as well as avoiding the deduction of the master equation of the method for objectivity purposes, focusing on the application of the Wolfram simulator, seeking to complement with the study of the method, in the comprehension of the two main procedures for the development of the calculations, considering the isolation of the root, evidencing the established interval, and configuring the stopping criteria, in order to approximate the values of the reality of the root of the function.

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