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Research paper

Discontinuous galerkin method of the first order parabolic differential equation

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Abstract

The objective of this paper is to propose a modest numerical error analysis by applying DG finite element method for the parabolic differential equation. The DG method is an imperative numerical method with much mass compensation and more flexible meshing than other numerical methods. The DG method starts by discretizing the domain into a set of non-overlapping elements. This study gives a general introduction and discuss about the discontinuous Galerkin Method of first order parabolic problem. The parabolic problem satisfies the condition of the existence and uniqueness of DG solution. The main goal of this study is to theoretically explore the convergence of the solution of the above methods and show the validity of the results.

Keywords: *Use about five key words or phrases in alphabetical order, Separated by Semicolon*

1. Introduction

This study provides a theoretical concept to approximate the error of the solutions of a parabolic differential equation. This is focused on the weak formulation of discontinuous Galerkin (DG) method of the first order parabolic problem. Finite element methods (FEM) have been proven valuable in the numerical approximation of solutions to self-adjoint or ‘‘nearly’’ self-adjoint parabolic partial differential equation (PDE) problems and related indefinite PDE systems or to their parabolic counterparts. Possible reasons for the success of FEM are their applicability in very general computational geometries of interest and the availability of tools for their rigorous error analysis. The error analysis is usually based on the variational interpretation of the FEM as a minimization problem over finite-dimensional sets. In 1971, Reed and Hill [1], proposed a new class of FEM, namely the discontinuous Galerkin finite element method for the numerical solution of the nuclear transport PDE problem, which contains a linear first-order hyperbolic PDE. DG methods were first proposed and analyzed in the early 1970s as a technique to numerically solve partial differential equations. The first review of this method was displayed in 1974 by Lesaint and Raviart [2]. By using stronger stability of DG system, this methods was later scrutinized by Johnson, Navert, and J.Pitk¨aranta [3], and Johnson and J.Pitk¨aranta [4]. The origin of the DG method for parabolic problems cannot be traced back to a single publication as features such as jump penalization in the modern nous were advanced progressively. The comprehensive book by Beatrice Riviere [5], discussed the Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations; covered theory, implementation and other information. Jan S. Hesthaven and Tim Warburton [6], has been described Nodal Discontinuous Galerkin Methods; Algorithms, Analysis and also described their applications. P. E. Lewis and J.P. Ward [7], provided a new class of Finite Element Method with its Principles and Application. D.N. Arnoldis [8], presented an interior penalty finite element method with discontinuous elements. R. Becker, P. Hansbo, and M.G. Larson.[9], discussed about energy norm as well as the posteriori error estimation for discontinuous Galerkin methods. B. Cockburn [10], established the Discontinuous Galerkin methods for convection-dominated problems which was existed vast information about discontinuous Galerkin(DG) FEM. B. Cockburn, G.E. Karniadakis, and, C.-W. Shu (eds.) [11], characterized some key element of discontinuous Galerkin(DG) method with its application. Hailiang Liu, Jue Yan [12, 13], initiated another new technique of Discontinuous Galerkin named the Direct Discontinuous Galerkin (DDG) method for Diffusion with Interface Corrections. B. Cockburn,G. E. Karniadakis and C.-W. Shu (eds.) [14] characterized the DG method in the theoretical and computational perception. I. Babu˘ska [15] provided particular evidence to apply the Lagrangian multipliers on the finite element method. S. Brenner and L. Scottfor [16] established the essential mathematical analysis and theory of finite element method. B. Cockburn, G. Kanschat, and D. Schötzau [17] signified a novel extent of the DG method by using penalty parameter which is known as the local discontinuous Galerkin (LDG) method.

2. Formulation of the problem

Let Ω be a polygonal domain in $\mathbb{R}^d$, $d = 2$ or 3. The side of the boundary $\partial\Omega$ of the domain is Γ. Let n be the unit normal vector to the boundary exterior to Ω. For f given in $L^2(\Omega)$, g given in $H^{\frac{1}{2}}(\Gamma)$.
Consider the parabolic problem,

$$\nabla . u + cu = f \text{ in } \Omega \tag{1}$$

With the boundary condition:

$$u = g \text{ on } \Gamma \tag{2}$$

Now, let consider a weight function v. Multiplying (1) by v and integrating over the domain Ω, to have,

$$(\nabla . u + cu)v = fv$$

$$\Rightarrow (\nabla . u)v + cuv = fv$$

$$\Rightarrow \int_\Omega ((\nabla . u)v + cuv)\, dx = \int_\Omega fvdx$$

$$\Rightarrow \int_\Omega (\nabla . u)v\, dx + \int_\Omega cuv\, dx = \int_\Omega fv\, dx$$

$$\Rightarrow \sum \int_\Omega (\nabla . u)v\, dx + \sum \int_\Omega cuv\, dx = \sum \int_\Omega fv\, dx \tag{3}$$

For the proposed DG method, the following DG norm is introduced

$$\|u\|_{DG}^2 = Ch\, \|u\|_{H^s(\Omega)}^2 + Ch_1\, \|u\|_{H^s(\Omega)}^2$$

Using the divergence theorem on every element integral (as v is now elementwise discontinuous), using the anti-clockwise orientation, to get:

$$-\sum \int_\Omega ((\nabla . u). \nabla v)\, dx + \sum \int_{\partial\Omega} ((\nabla . u). n)v\, ds + \sum \int_\Omega cuv\, dx = \int_\Omega fv\, dx$$

Where, n = The outward normal to each element edge. Then, introduce a bilinear form $B(u, v)$ as :

$$B(u, v) = -\int_\Omega ((\nabla . u). \nabla v)\, dx + \int_{\partial\Omega} ((\nabla . u). n)v\, ds + \int_\Omega cuv\, dx$$

Therefore, the DG finite element method is defined as

$$B(u, v) = \int_\Omega fv\, dx \tag{4}$$

3. Stability analysis

The following theorem is introduced for the stability analysis of this method.

Theorem: *Assume that there exist positive constants A and B such that the solution u satisfies the following bounds.*

$$B(u, u) \leq A\, \|u\|_{L^2(\Omega)}^2 + B\, \|\nabla . u\|_{L^2(\Omega)}^2$$

Proof: Let, define the bilinear form of the problem:

$$B(u, v) = -\int_\Omega \big((\nabla . u). \nabla v\big) dx + \int_{\partial\Omega} \big((\nabla . u). n\big) v\, ds + \int_\Omega cuv\, dx \tag{5}$$

Putting $v = u$, to obtain,

$$B(u, u) = -\int_\Omega \big((\nabla . u). \nabla u\big) dx + \int_{\partial\Omega} \big((\nabla . u). n\big) u\, ds + \int_\Omega cu^2\, dx \tag{6}$$

Using Cauchy's inequality on $((\nabla . u) \cdot n)u$, $\|((\nabla . u) \cdot n)u\| \leq \|(\nabla . u) \cdot n\| \cdot \|u\|$
and,

$$B(u, u) \leq -\int_\Omega \big((\nabla . u). \nabla v\big) dx + \int_{\partial\Omega} \big((\nabla . u). n\big) v\, ds + \int_\Omega cuv\, dx \tag{7}$$

Using trace inequality on 2^{nd} term of (3.7),

$$\forall u \in \mathbb{P}_k (\Omega), \forall e \subset \partial\Omega,$$

$$\|(\nabla . u) \cdot n\|_{L^2(e)} \leq \tilde{C}_t |e|^{\frac{1}{2}} |\Omega|^{-\frac{1}{2}} \|\nabla . u\|_{L^2(\Omega)}$$

Also,

$\forall\, u \in \mathbb{P}_k(\Omega), \forall e \subset \partial\Omega,\ \|u\|_{L^2(e)} \le C_t h_\Omega^{-\frac{1}{2}} \|u\|_{L^2(\Omega)}$

It is known that, $\|v\|_{L^2(\Omega)} = \left(\int_\Omega v^2\right)^{\frac{1}{2}}$

$\therefore \int_\Omega ((\nabla.u).\nabla u)\, dx = \|\nabla.u\|^2_{L^2(\Omega)}$ And, $\int_\Omega cu^2 dx = c\,\|u\|^2_{L^2(\Omega)}$

From the equation (7),

$$B(u,u) = -\|\nabla.u\|^2_{L^2(\Omega)} + \left(\tilde{C}_t|e|^{\frac{1}{2}}|\Omega|^{-\frac{1}{2}}\|\nabla.u\|_{L^2(\Omega)}\right)\cdot\left(C_t h_\Omega^{-\frac{1}{2}}\|u\|_{L^2(\Omega)}\right) + c\|u\|^2_{L^2(\Omega)} \le -\|\nabla.u\|^2_{L^2(\Omega)} + \frac{\epsilon}{2}\left(\tilde{C}_t|e|^{\frac{1}{2}}|\Omega|^{-\frac{1}{2}}\|\nabla.u\|_{L^2(\Omega)}\right)^2 + \frac{1}{2\epsilon}\left(C_t h_\Omega^{-\frac{1}{2}}\|u\|_{L^2(\Omega)}\right)^2 + c\,\|u\|^2_{L^2(\Omega)}$$

So,

$$B(u,u) \le -\|\nabla.u\|^2_{L^2(\Omega)}\{1 - \frac{\epsilon}{2}\left(\tilde{C}_t|e|^{\frac{1}{2}}|\Omega|^{-\frac{1}{2}}\right)^2\} + \|u\|^2_{L^2(\Omega)}\{\frac{1}{2\epsilon}\left(C_t h_\Omega^{-\frac{1}{2}}\right)^2 + c\}$$

If, consider, $A = \frac{1}{2\epsilon}\left(C_t h_\Omega^{-\frac{1}{2}}\right)^2 + c$ and $B = -(1 - \frac{\epsilon}{2}\left(\tilde{C}_t|e|^{\frac{1}{2}}|\Omega|^{-\frac{1}{2}}\right)^2)$
Then from the above equation, it is found that,

$$\Rightarrow B(u,u) \le A\,\|u\|^2_{L^2(\Omega)} + B\|\nabla.u\|^2_{L^2(\Omega)} \tag{8}$$

Hence, this completes the proof.

4. Consistency of the solution

Let consider,

$$\Rightarrow -\int_\Omega ((\nabla.u).\nabla v)dx + \int_{\partial\Omega}((\nabla.u).n)v\,ds + \int_\Omega cuv\,dx = \int_\Omega fv\,dx$$

$$\Rightarrow -\int_\Omega ((\nabla.u)\cdot\nabla v - cuv)dx + \int_{\partial\Omega}((\nabla.u)\cdot n)\,v\,ds = \int_\Omega fv\,dx$$

$$\Rightarrow -\int_\Omega ((\nabla.u)\cdot\nabla v - cuv)dx + \int_{\partial\Omega}((\nabla.u)\cdot n_E)v\,ds = \int_\Omega fv\,dx \tag{9}$$

Define, n_E is the outward normal to E. Sum over all elements, switch to the normal vectors n_e,

$$\sum_{E\in\mathcal{E}_h}\int_{\partial\Omega}((\nabla.u)\cdot n_E)\,v\,ds - \sum_{e\in\partial\Omega}\int_e((\nabla.u)\cdot n_e)\,v\,ds = \sum_{e\in\Gamma_h}\int_e[(\nabla.u)\cdot n_e)\,v]\,ds$$

$$\Rightarrow \sum_{E\in\mathcal{E}_h}\int_{\partial\Omega}((\nabla.u)\cdot n_E)\,v\,ds = \sum_{e\in\partial\Omega}\int_e((\nabla.u)\cdot n_e)\,v\,ds + \sum_{e\in\Gamma_h}\int_e[(\nabla.u)\cdot n_e)\,v]\,ds$$

By regularity of the solution u,

$(\nabla.u)\cdot n_e = \{(\nabla.u)\cdot n_e\}$

Substituting all of these values in the equation (3.9), to get,

$$-\sum_{E\in\varepsilon_h}\int_\Omega((\nabla.u)\cdot\nabla v - cuv)dx + \sum_{e\in\partial\Omega}\int_e((\nabla.u)\cdot n_e)\,v\,ds + \sum_{e\in\Gamma_h}\int_e\{(\nabla.u)\cdot n_e)\}[v]\,ds = \int_\Omega fv\,dx$$

Subtract, $\epsilon\sum_{e\in\Gamma}\int_e(\nabla v\cdot n_e)\,u\,ds$ from both sides and use the Dirichlet boundary condition $u = g$ on Γ,

$$-\sum_{E\in\varepsilon_h}\int_\Omega((\nabla.u)\cdot\nabla v - cuv)dx + \sum_{e\in\partial\Omega}\int_e((\nabla.u)\cdot n_e)\,v\,ds + \sum_{e\in\Gamma_h}\int_e\{(\nabla.u)\cdot n_e)\}[v]\,ds - \epsilon\sum_{e\in\Gamma}\int_e(\nabla v\cdot n_e)\,u\,ds = \int_\Omega fv\,dx - \epsilon\sum_{e\in\Gamma}\int_e(\nabla v\cdot n_e)\,g\,ds$$

Finally, it can be said that the jumps $[u] = [(\nabla.u)\cdot n_e]$ are zero that is on the interior edges.
Then, clearly have,

$$-\sum_{E\in\varepsilon_h}\int_\Omega((\nabla.u)\cdot\nabla v - cuv)dx = \int_\Omega fv\,dx$$

$\Rightarrow -\sum_{E\in\varepsilon_h}\int_\Omega \left((\nabla.u)\cdot \nabla v\right)dx + \int_\Omega\ cuv\ = \int_\Omega\ fv\,dx$

which immediately yields in the distributional sense, for all $E\ \in\ \mathcal{E}_h$,

$\nabla.u\ +\ cu\ =\ f$

This proves that the given parabolic problem is consistent.

5. Error analysis

The following theorem is proposed for the error estimate of the problem governed by the equation (1).

Theorem: *Let u_h be the DG finite element solution and u be the exact solution of (1) arising from (4).Then there exists a constant C such that*

$\|u\ - u_h\|^2_{L^2(\Omega)}\ \le\ Ch\,\|u\|^2_{DG}$

Proof: Bilinear form of the problem be,

$$B(u,v) = -\int_\Omega \left((\nabla.u).\nabla v\right)dx + \int_{\partial\Omega}\left((\nabla.u).n\right)v\,ds + \int_\Omega\ cuv\,dx \tag{10}$$

Let, the exact solution of the problem be u and the approximation solution be u_h .
Putting $u\ = u_h$ in (10),

$$B(u_h,v) = -\int_\Omega \left((\nabla.u_h).\nabla v\right)dx + \int_{\partial\Omega}\left((\nabla.u_h).n\right)v\,ds + \int_\Omega\ cu_h v\,dx \tag{11}$$

Now,

$B(u,v) -\ B(u_h,v) = -\int_\Omega \left((\nabla.u - \nabla.u_h)\cdot\nabla v\right)dx + \int_{\partial\Omega}\left(\left((\nabla.u)\cdot\ n\right) - \left((\nabla.u_h)\cdot\ n\right)\right)v\,ds + \int_\Omega\ c(u-u_h)v\,dx$

Let, define, $=\ u\ - u_h$. Then,

$B(u,u\ - u_h) -\ B(u_h,v) = -\int_\Omega \left((\nabla.u - \nabla.u_h)\cdot\ \nabla(u\ - u_h)\right)dx + \int_{\partial\Omega}\left(\left((\nabla.u)\cdot\ n\right) - \left((\nabla.u_h)\cdot\ n\right)\right)\ (u\ - u_h)ds +$
$\int_\Omega\ c(u-u_h)\ (u\ - u_h)dx$

$$\Rightarrow B(u,u\ - u_h) -\ B(u_h,v) = -\int_\Omega \left((\nabla.u - \nabla.u_h)\cdot\ \nabla(u\ - u_h)\right)dx + \int_{\partial\Omega}\left(\left((\nabla.u)\cdot\ n\right) - \left((\nabla.u_h)\cdot\ n\right)\right)(u\ - u_h)ds + \int_\Omega\ c(u-u_h)^2dx \tag{12}$$

The first term of right-hand side of the equation (12) can be written as,

$\int_\Omega \left((\nabla.u - \nabla.u_h)\cdot\ \nabla(u\ - u_h)\right) = \int_\Omega \left((\nabla.(u-u_h)\cdot\ \nabla(u\ - u_h)\right)$

$= \int_\Omega\ \nabla.(u\ - u_h)^2$

It is known that, $\int_\Omega\ \nabla.(u\ - u_h)^2 = \|\nabla.(u-u_h)\|^2_{L^2(\Omega)}$
From the 2^{nd} term of right-hand side of the equation (12),

$\int_{\partial\Omega}\left(\left((\nabla.u)\cdot\ n\right) - \left((\nabla.u_h)\cdot\ n\right)\right)(u\ - u_h)$

Using Cauchy-Schwarz's inequalit,

$|(((\nabla.u)\ \cdot\ n) - ((\nabla.u_h)\ \cdot\ n))(u\ - u_h)|\ \le\ \|((\nabla.u)\ \cdot\ n) - ((\nabla.u_h)\ \cdot\ n)\|_{L^2(\Omega)}\ \cdot\ \|u\ - u_h\|_{L^2(\Omega)}$

Using Trace inequality,

$\|((\nabla.u)\ \cdot\ n) - ((\nabla.u_h)\ \cdot\ n)\|_{L^2(\Omega)} \le C_t h_\Omega^{-\frac{1}{2}}\,\|\nabla.(u-u_h)\|_{L^2(\Omega)}$

$\le C_t h_\Omega^{-\frac{1}{2}}.Ch_\Omega^{\min(k+1,s)-1}\ |u|_{H^s(\Omega)} =\ C_t\,.Ch_\Omega^{-\frac{1}{2}+\min(k+1,s)-1}\ |u|_{H^s(\Omega)}$

Let, $C_t\cdot\ C\ =\ \tilde{C}$. Then,

$$\|((\nabla.u)\cdot n)-((\nabla.u_h)\cdot n)\|_{L^2(\Omega)} \leq \tilde{C}h_\Omega^{\min(k+1,s)-\frac{3}{2}}|u|_{H^s(\Omega)}$$

Now, substituting these values in (12),

$$B(u-u_h, u-u_h) \leq -\|\nabla.(u-u_h)\|^2_{L^2(\Omega)}$$

$$+\left\{\tilde{C}h_\Omega^{\min(k+1,s)-\frac{3}{2}}|u|_{H^s(\Omega)}.\|u-u_h\|_{L^2(\Omega)}\right\}+C\|u-u_h\|^2_{L^2(\Omega)} \tag{13}$$

Now,

$$\|\nabla.(u-u_h)\|^2_{L^2(\Omega)} \leq \left\{Ch^{\min(k+1,s)-1}\|u\|_{H^s(\Omega)}\right\}^2$$

$$= C^2h^{\min(2k+2,2s)-2}\|u\|^2_{H^s(\Omega)}$$

$$= C\,h^{\min(2k+2,2s)-2}\|u\|^2_{H^s(\Omega)}$$

[By considering the constant C^2 as C]
And

$$\|u-u_h\|_{L^2(\Omega)} \leq Ch_\Omega^{\min(k+1,s)-2}\|u\|_{H^s(\Omega)}$$

Again,

$$\|u-u_h\|^2_{L^2(\Omega)} \leq \{Ch_\Omega^{\min(k+1,s)-2}\|u\|_{L^2(\Omega)}\}^2$$

$$= C^2h_\Omega^{\min(2k+2,2s)-4}\|u\|^2_{L^2(\Omega)}$$

$$= C\,h_\Omega^{\min(2k+2,2s)-4}\|u\|^2_{L^2(\Omega)}$$

[By considering the constant C^2 as C]
The last term of right-hand side of the equation (12) can be written as,

$$\int_\Omega c(u-u_h)^2dx = c\|u-u_h\|^2_{L^2(\Omega)}$$

Substituting all of these values in (12),

$$B(u-u_h, u-u_h) \leq -Ch_\Omega^{\min(2k+2,2s)-2}\|u\|^2_{H^s(\Omega)}+\left\{\tilde{C}h_\Omega^{\min(k+1,s)-\frac{3}{2}}\|u\|_{H^s(\Omega)}.Ch_\Omega^{\min(k+1,s)-2}\|u\|_{H^s(\Omega)}\right\}$$

$$+cC\,h_\Omega^{\min(2k+2,2s)-4}\|u\|^2_{L^2(\Omega)}$$

$$\leq \left\{-Ch_\Omega^{\min(2k+2,2s)-2}+cC\,h_\Omega^{\min(2k+2,2s)-4}\right\}\|u\|^2_{L^2(\Omega)}+\left\{C\tilde{C}h_\Omega^{\min(k+1,s)-\frac{3}{2}}.h_\Omega^{\min(k+1,s)-2}\right\}\|u\|^2_{H^s(\Omega)}$$

$$\leq Ch\|u\|^2_{H^s(\Omega)}+Ch_1\|u\|^2_{H^s(\Omega)} \tag{14}$$

The constant terms is considered as,

$-C+cC = C$ and $C\tilde{C} = C$.

So, it can be written as :

$$\|u-u_h\|^2_{L^2(\Omega)} \leq Ch\|u\|^2_{H^s(\Omega)}+Ch_1\|u\|^2_{H^s(\Omega)}$$

By applying proposed DG norm, then

$$\|u-u_h\|^2_{L^2(\Omega)} \leq Ch\|u\|^2_{DG}$$

This completes the proof.

6. Conclusion

The paper investigated the error of the numerical solution by applying the Discontinuous Galerkin finite element method for the first order parabolic differential equation. It is considered discontinuous Galerkin finite element approximations of a model scalar linear parabolic equation. It is a different and straightforward approach to seek error analysis from all other finite element schemes which are given in the literature. The technique used in this paper can also be extended to obtain the $L^2(\Omega)$ error estimate of the time dependent and higher order problems with the optimal order of convergence.

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