



# Bäcklund transformation, auto-Bäcklund transformation and exact solutions for KdV-type equations

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## Abstract

In plasma physics, fluid dynamics and nonlinear optics, Korteweg-de Vries (KdV)-type equations are used to describe certain phenomena (ion acoustic wave in plasma, quantum hydrodynamic model, wave motion on the surface of shallow water and the unidirectional propagation of long wave of small amplitude and exists in many physical branches). In this paper, KdV-type equations are investigated. We are used Bäcklund Transformation to obtain new exact solutions for the (KdV)-type equations. The method of characteristics is used and the Bäcklund transformation are employed to generate new solutions from the old ones. By the homogenous balance method, we derive an auto-Bäcklund Transformation (ABT) for the KdV equation. Thus, families of solution for KdV-type equations are obtained.

**Keywords:** KdV-type equations, Bäcklund transformation, auto-Bäcklund transformation and exact solutions.

## 1. Introduction

Nonlinear evolution equations (NLEEs) are important mathematical models to describe physical phenomena. They are also an important field in the contemporary study of nonlinear physics, especially in soliton theory. The research on the explicit solution and integrability is helpful in clarifying the movement of matter under nonlinear interaction and plays an important role in scientifically explaining the physical phenomena see for example, fluid mechanics, plasma physics, quantum hydrodynamic model, optical fibers, solid state physics, chemical kinematic, chemical physics and geochemistry. In this paper we will consider the following KdV-type equations.

### The KdV equation

$$q_t + 6q_x + 6qq_x + q_{xxx} = 0. \quad (1)$$

### The MKdV equation in $q_x$

$$q_{tx} + (q_x)^2 q_{xx} + q_{xxx} = 0. \quad (2)$$

The investigation of the exact solutions of nonlinear partial differential equations (NPDEs) plays an important role in the study of nonlinear physical phenomena. In the past decades, there has been significant progression in the development of methods such as the inverse scattering method [1-2], Hirota's bilinear method [3], Painlevé

expansions [4-7], truncated Painlevé [8], homogeneous balance method [9-10], the linearized transformation method [11], tanh function method [12-15] and several ansatz method [16-17]. The Bäcklund transformation (BT) technique is one of the direct methods for generating a new solution of NLEEs from a known solution of that equation (see, for example, [18-20]). Previously, Konno and Wadati [21], for example, had derived some BTs for the NLEEs of the AKNS class [2,21]. These BTs explicitly express the new solutions in terms of the known solutions of the NLEEs and the corresponding wave functions, which are solutions of the associated AKNS system. The AKNS system is a linear eigenvalue problem in the form of a system of first-order partial differential equations. Therefore, the problem of obtaining new solutions by BTs is equivalent to obtaining the wave function. The main aim of this paper is to derive BT, ABT and construct solutions for the KdV-type equations. [1].

## 2. The AKNS system and the BT for the NLEEs

It is well known that many integrable NLEEs can be derived from the following AKNS system eigenvalue problem, defined in the form

$$\psi_x = U\psi, \psi_t = V\psi \quad (3)$$

$$\psi = (\psi_1, \psi_2)^T, U = \begin{pmatrix} \eta & q \\ r & -\eta \end{pmatrix}, V = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \quad (4)$$

where  $\eta$  is a parameter, independent of  $x$  and  $t$ ,  $q$  and  $r$  are function of  $x$  and  $t$ , while  $A, B$  and  $C$  are scalar function of  $q(x, t)$ . The integrability condition  $\psi_{xt} = \psi_{tx}$  yields at once the evolution equations in the general form

$$-A_x + qC - rB = 0, \quad q_t - B_x - 2qA + 2\eta B = 0, \quad r_t - C_x - 2\eta C + 2rA = 0. \quad (5)$$

By suitably choosing  $A, B, C$  and  $r$  in (5), we obtain various NLEEs which  $q$  must satisfy. Konno and Wadati [21] introduced the relation

$$\Gamma = \frac{\psi_1}{\psi_2} \quad (6)$$

and for any of the NLEEs derived a BT with the following form:

$$q' = q + F(\Gamma, \eta), \quad (7)$$

where  $q'$  is the new solution generated from the old  $q$ .

### 2.1. The KdV equation

$$r = -1$$

$$A = -4\eta^3 - 2\eta q - q_x - 6\eta, \quad B = -4\eta^2 q - 6q - 2q^2 - 2\eta q_x - q_{xx}, \quad C = 4\eta^2 + 2q + 6. \quad (8)$$

To derive the new solution  $q'$  from the known solution  $q$  we use the BT

$$q' = q - 2 \frac{\partial \Gamma(x, t)}{\partial x} \quad (9)$$

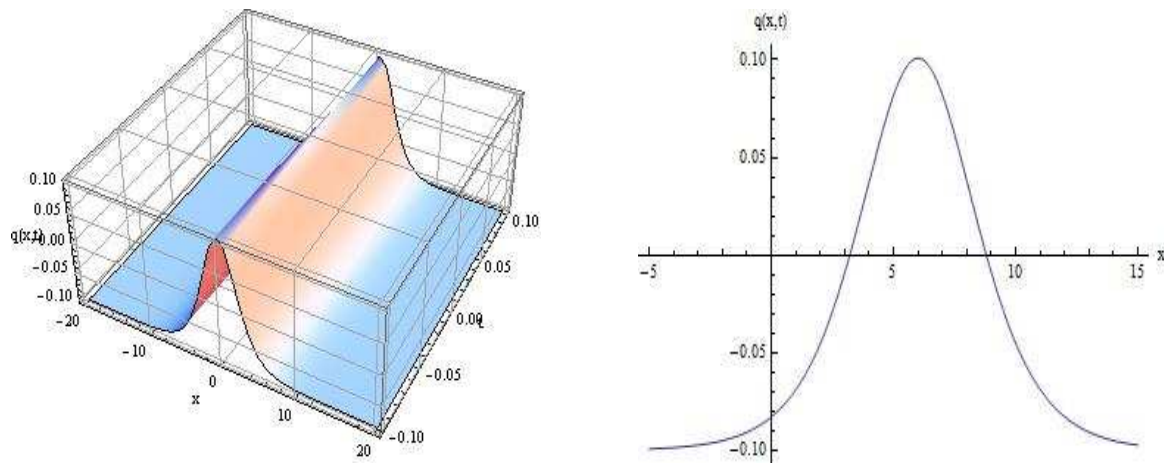
### 2.2. For the MKdV equation

$$r = -q = \frac{u_x}{2}$$

$$A = -4\eta^3 - \frac{1}{2}\eta(u_x)^2, \quad B = \frac{1}{2}u_{xxx} + \eta u_{xx} + 2\eta^2 u_x + \frac{1}{4}u_x^3, \quad C = -\frac{1}{2}u_{xxx} + \eta u_{xx} - 2\eta^2 u_x - \frac{1}{4}u_x^3. \quad (10)$$

To derive the new solution  $q'$  from the known solution  $q$  we use the BT

$$q' = q + 4t \tan^{-1}(\Gamma). \quad (11)$$



**Figure 1:** (a) (b)  
The solution solution of (15) for different values  $\eta = .2$ ,  $c = 5$  and  $q_0 = -0.1$ .

**Proposition.** The MKdV equation (2) is invariant under the BT

$$\frac{\partial u'}{\partial x} + \frac{\partial u}{\partial x} = 4\eta \sin\left(\frac{u' - u}{2}\right) \tag{12a}$$

$$\frac{\partial u'}{\partial t} - \frac{\partial u}{\partial t} = 4A \sin\left(\frac{u' - u}{2}\right) + 2(B - C) + 2(B + C) \cos\left(\frac{u' - u}{2}\right) \tag{12b}$$

where  $u$  is a solution of the equation,  $u'$  also satisfy the same equation and the BT for the class of the AKNS system can be constructed from the inverse scattering formalism (Proof, see Appendix A).

### 3. The known solution is a constant

#### 3.1. The KdV equation

To construct the solution of equation (1), we start with the zero solutions  $q(0) = q_0$ . By substituting this solution in the two matrices  $U$  and  $V$  we obtain:

$$U = \begin{pmatrix} \eta & q_0 \\ -1 & -\eta \end{pmatrix}, V = \begin{pmatrix} -4\eta^3 - 2\eta q_0 - 6\eta & -4\eta^2 q_0 - 6q_0 - 2q_0^2 \\ 4\eta^3 + 2q_0 + 6 & 4\eta^3 + 2\eta q_0 + 6\eta \end{pmatrix}. \tag{13}$$

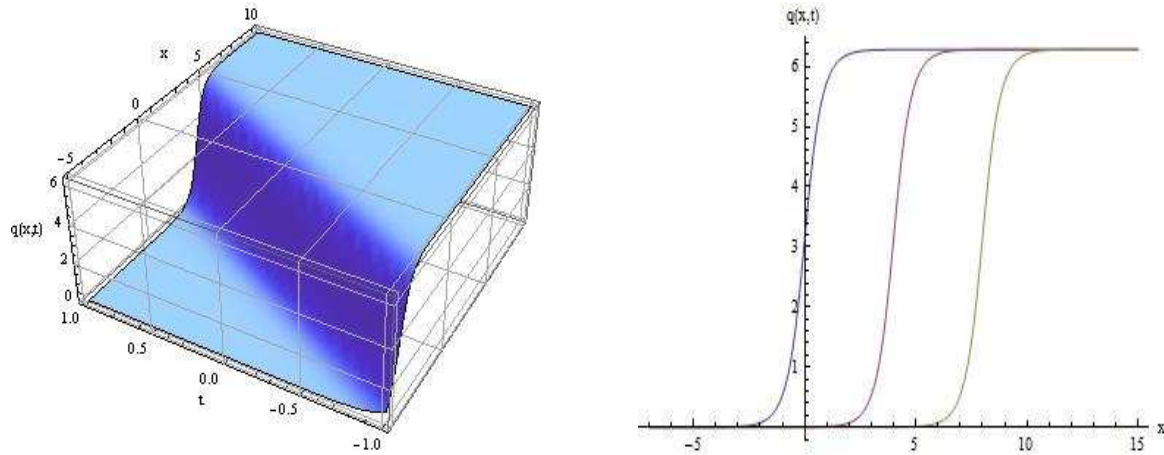
From the AKNS system, we get the solution

$$\psi = \psi_0 e^{U\rho} = \psi_0 \left[ I + U\rho + \frac{U^2 \rho^2}{2!} + \frac{U^3 \rho^3}{3!} + \frac{U^4 \rho^4}{4!} + \dots \right] = \psi_0 \begin{pmatrix} \cosh \alpha\rho + \frac{\eta}{\alpha} \sinh \alpha\rho & \frac{q_0}{\alpha} \sinh \alpha\rho \\ \sinh \rho & \cosh \alpha\rho - \frac{\eta}{\alpha} \sinh \alpha\rho \end{pmatrix}, \tag{14}$$

where  $\alpha^2 = \eta^2 - q_0$ . Then the solution of KdV equation corresponding to the known solution  $q(0) = q_0$  another  $\psi_0 = (0, 1)^T$ ,

$$q' = \frac{q_0 (-\eta^2 - 3\alpha^2 + (\alpha^2 + \eta^2) \cosh 2\alpha\rho - 2\eta\alpha \sinh 2\alpha\rho)}{2(\alpha \cosh \alpha\rho - \eta \sinh \alpha\rho)^2}, \tag{15}$$

where  $\rho = x - (4\eta^2 + 2q_0 + 6)t$ , and  $\alpha^2 = \eta^2 - q_0$ . We have represented this solution for a set of parameter values in Fig. 1.



**Figure 2:** (a) (b)

The kink solution of (19) for different values  $\eta = 1$  and  $c = 1$ . (a) in the 3D and (b) in the plane at  $t = 0, 1, 2$ .

### 3.2. The MKdV equation

To construct the solution of equation (2), we start with the zero solutions  $q(0) = 0$ . By substituting this trivial solution in the two matrices U and V we obtain:

$$U = \begin{pmatrix} \eta & 0 \\ 0 & -\eta \end{pmatrix}, V = \begin{pmatrix} -4\eta^3 & 0 \\ 0 & 4\eta^3 \end{pmatrix}$$

From the AKNS system (3)-(4)

$$d\psi = U\psi dx + V\psi dt = (Udx + Vdt) \psi \tag{16}$$

and thus we get

$$d\psi = U\psi d\rho, d\rho = dx - (4\eta^2) dt, \quad \rho = x - (4\eta^2 t) \tag{17}$$

Then the solution of (23) is

$$\psi = \psi_0 \begin{pmatrix} e^{\eta\rho} & 0 \\ 0 & e^{-\eta\rho} \end{pmatrix} \tag{18}$$

Choosing  $\psi_0 = (1, 1)^T$ , we obtain the new solution of the MKdV equation corresponding to the known solution  $q(0) = 0$

$$q' = 4\tan^{-1} \left( e^{2\eta(x-4\eta^2 t)} \right). \tag{19}$$

We have represented this solution for a set of parameter values in Fig. 2.

### 4. The known solution $q = q(x, t)$ is a simple function

In this case the system (3)-(4) cannot be solved for the vector  $\Psi$  as a whole, but it can be solved in the component  $\psi_1$  and  $\psi_2$  separately. From (3)-(4), after inserting the known solution  $q(x, t)$  of the KdV equation into the corresponding matrices U and V we will have the following system of partial differential equations for unknowns  $\psi_1$  and  $\psi_2$  :

$$\psi_{1x} = \eta\psi_1 + q\psi_2 \tag{20a}$$

$$\psi_{2x} = r\psi_1 - \eta\psi_2 \tag{20b}$$

$$\psi_{1t} = A\psi_1 + B\psi_2 \tag{20c}$$

$$\psi_{2t} = C\psi_1 - A\psi_2 \tag{20d}$$

These equations are compatible under the conditions of the assumed values of the matrices  $U$  and  $V$  connected with the KdV equation under consideration. Solving for  $\psi_1$  from (20b) yields

$$\psi_1 = \frac{1}{r} (\psi_{2x} - \eta\psi_2) \quad (21)$$

Substituting this  $\psi_1$  into (20d) we have

$$C\psi_{2x} - r\psi_{2t} = \frac{1}{2} (C_x - r_t) \psi_2 \quad (22)$$

This is linear first-order partial differential equation with  $\psi_2$  as the unknown function; it can be solved by the method of characteristics or another method. After  $\psi_2$  has been obtained from (22), and substituting it into (21), we will obtain  $\psi_1$ . To be more specific, we apply the last technique in the following example:

#### 4.1. The KdV equation

Let

$$q(x, t) = \frac{x - 6t}{3(2t + 1)}, \quad t \neq -\frac{1}{2}. \quad (23)$$

By direct calculation one check that equation (23) is a solution of the KdV equation (1). Inserting equation (23) into equation (22), together with equations (3) and (4), gives

$$\left( 4\eta^2 + \frac{2(x - 6t)}{3(2t + 1)} + 6 \right) \psi_{2x} + \psi_{2t} = \frac{\psi_2}{3(2t + 1)}. \quad (24)$$

equation (24) has the following system ODEs as its characteristic equations:

$$\frac{dx}{dt} = (4\eta^2 + 6) + \frac{2(x - 6t)}{3(2t + 1)}, \quad (25)$$

$$\frac{d\psi_2}{\psi_2} = \frac{dt}{3(2t + 1)}. \quad (26)$$

Solving these two equations gives the general solution of the unknown  $\psi_2$  in equation (24), which reads:

$$\psi_2 = (2t + 1)^{1/6} F(\xi) \quad (27a)$$

$$\xi = (x + 3)(2t + 1)^{-1/3} - 3(\eta^2 + 1)(2t + 1)^{2/3}, \quad (27b)$$

where  $F$  is an arbitrary function. Substituting equations (23) and (27) into equation (21) gives the general solution of  $\psi_1$ , which reads:

$$\psi_1 = -(2t + 1)^{-1/6} F'(\xi) - \eta(2t + 1)^{1/6} F(\xi). \quad (28)$$

To determine the function  $F(\xi)$ , we substitute from equations (23), (27) and (28) into equation (20a), then  $F(\xi)$  must satisfy the following Airy equation [22]

$$\frac{d^2 F(\xi)}{d\xi^2} + \frac{\xi}{3} F(\xi) = 0. \quad (29)$$

Therefore, we obtain the function  $F(\xi)$  as follows:

$$F(\xi) = c_1 Ai(\xi) + c_2 Bi(\xi), \quad (30)$$

where  $Ai(\xi)$  and  $Bi(\xi)$  are two Airy functions [22], while  $c_1$  and  $c_2$  are two arbitrary constants. After  $F(\xi)$  has been determined, equations (27), (28) and (6) leads to

$$\Gamma = -(2t + 1)^{-1/3} \frac{F'(\xi)}{F(\xi)} - \eta, \quad (31)$$

then substituting this  $\Gamma$  and equation (23) into the BT (7) yields the new solution  $q'$  of the KdV equation (1) corresponding to the known solution (23)

$$q'(x, t) = \frac{x - 6t}{3(2t + 1)} + 2(2t + 1)^{-2/3} \frac{d}{d\xi} \left( \frac{F'(\xi)}{F(\xi)} \right). \quad (32)$$

## 5. The known solution is a traveling wave

In this we suppose that the components  $q$  and  $r$  of the all matrix  $U$  are function of  $\rho$

$$q = q(\rho), \quad r = r(\rho), \quad \text{where } \rho = x - Kt. \quad (33)$$

then the components  $A$ ,  $B$  and  $C$  of the matrix  $V$  determined by equations (5) are also functions of  $\rho$ :

$$A = A(\rho), \quad B = B(\rho), \quad \text{and } C = C(\rho). \quad (34)$$

We require the quantity

$$\beta = (A + K\eta)^2 + (B + Kq)(C + Kr), \quad (35)$$

to be constant with respect to  $\rho$  (or  $x$  and  $t$ ). Under these assumptions, the following result holds, which is crucial in the subsequent exact solution. Solving system (22) by applying the method of characteristics as section 4, the PDE (22) possesses the following characteristic equations:

$$\frac{dt}{-r} = \frac{dx}{C} = \frac{2d\psi_2}{(C_x - r_t)\psi_2}. \quad (36)$$

Using equations (33) and (34) substituting in equation (36) gives

$$\frac{dt}{-r} = \frac{d\rho}{(C + Kr)} = \frac{2d\psi_2}{(C + Kr)'_{\rho}\psi_2}. \quad (37)$$

These equations yield the following system of ODEs

$$\frac{d(\ln \psi_2)}{d\rho} = \frac{(C + Kr)'_{\rho}}{2(C + Kr)} \quad (38)$$

$$\frac{d\rho}{dt} = -\frac{(C + Kr)}{r} \quad (39)$$

Integrating (39) we get

$$-t + k_1 = \int \frac{rd\rho}{(C + Kr)} \quad (40)$$

where  $k_1$  is an integration constant. Integrating equation (38) leads to

$$\psi_2 = k_2 (C + Kr)^{1/2}, \quad (41)$$

where  $k_2$  is another integration constant.

Denoting

$$\sigma(\rho) = \int \frac{rd\rho}{(C + Kr)}. \quad (42)$$

Substituting from equation (42) into equation (40), we have

$$\sigma(\rho) + t = k_1. \quad (43)$$

From equations (41) and (43), we obtain the general solution of equation (22):

$$\psi_2 = (C + Kr)^{1/2} F(\xi), \quad (44)$$

where

$$\xi = \sigma(\rho) + t \quad (45)$$

and  $F(\xi)$  is a differential function of  $\xi$ . Substituting equation (44) into equation (21) gives the general solution for  $\psi_1$ :

$$\psi_1 = (C + Kr)^{-1/2} (F'_\xi + (A + K\eta) F). \quad (46)$$

To determine the function  $F(\xi)$ , we substitute equations (44) and (46) into equation (20a), and we find that  $F(\xi)$  must satisfy the following second order ODE:

$$F''_{\xi\xi} - \beta F = 0, \quad (47)$$

where  $\beta$  is a constant defined in equation (35). According to the sign of  $\beta$ , equation (47) will have the following three different solutions:

$$F(\xi) = c_1\xi + c_2 \quad \text{when } \beta = 0, \quad (48a)$$

$$F(\xi) = c_1 \sinh \omega(\xi + c_2) \quad \text{when } \beta > 0 \quad \omega^2 = \beta, \quad (48b)$$

$$F(\xi) = c_1 \sin \omega(\xi + c_2) \quad \text{when } \beta < 0 \quad \omega^2 = -\beta, \quad (48c)$$

where  $c_1$  and  $c_2$  are integration constants. Substituting these solutions into equations (46) and (44), respectively, we obtain the corresponding different solution of system (3)-(4):

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} (C + Kr)^{-1/2} [(A + K\eta)(c_1\xi + c_2) + c_2] \\ (C + Kr)^{1/2} (c_1\xi + c_2) \end{pmatrix}, \quad \text{when } \beta = 0, \quad (49a)$$

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} c_1 (C + Kr)^{-1/2} [(A + K\eta) \sinh \omega (c_2 + \xi) + \omega \cosh \omega (\xi + c_2)] \\ c_1 (C + Kr)^{1/2} \sinh \omega (\xi + c_2) \end{pmatrix}, \quad \text{when } \beta > 0, \quad (49b)$$

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} c_1 (C + Kr)^{-1/2} [(A + K\eta) \sin \omega (c_2 + \xi) + \omega \cos \omega (\xi + c_2)] \\ c_1 (C + Kr)^{1/2} \sin \omega (\xi + c_2) \end{pmatrix}, \quad \text{when } \beta < 0. \quad (49c)$$

These results (49a)-(49c) are valid for any NLEE contained in the AKNS system (3)-(4), provided they satisfy assumption (33) and (34).

Now, applying the results obtained here and the known traveling wave solution of the KdV-type equations obtained in section 3. allows us to construct a new class of solutions of the corresponding KdV-type equations by means of the BTs. The constant  $\beta$  defined by (35) is zero and, therefore, the corresponding solution of the AKNS system (3)-(4) is (49a). By substituting equation (49a) into equation (6) we get the common expression of  $\Gamma$  for the KdV-type equations.

$$\Gamma = (C + Kr)^{-1} \left[ (A + K\eta) + \left( \frac{1}{c + \xi} \right) \right], \quad c = c_1/c_2. \quad (50)$$

### 5.1. The KdV equation

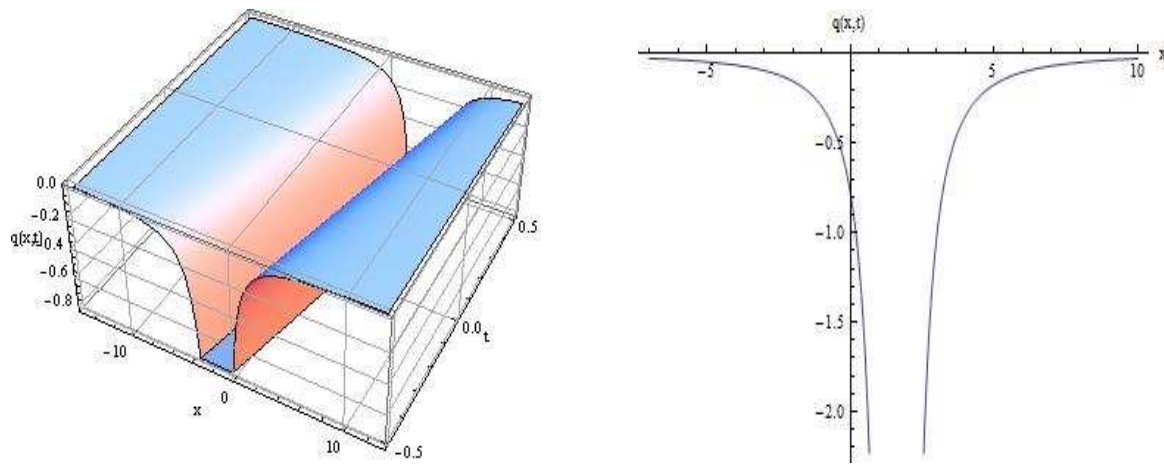
$$\text{Let } q = \frac{q_0 (-\eta^2 - 3\alpha^2 + (\alpha^2 + \eta^2) \cosh 2\alpha\rho - 2\eta\alpha \sinh 2\alpha\rho)}{2(\alpha \cosh \alpha\rho - \eta \sinh \alpha\rho)^2}, \quad (51)$$

where  $\rho = x - (4\eta^2 + 2q_0 + 6)t$ , and  $\alpha^2 = \eta^2 - q_0$ . Clearly (51) satisfies the KdV equation (1) as can be seen by direct substitution. Substituting in equation (35) we obtain that the constant  $\beta$  is equal zero. Substituting in two relations (43) and (45)

$$\xi = \int \frac{rd\rho}{(C + Kr)} + t = \frac{1}{16\alpha^2 q_0} \left( -2\eta \cosh 2\alpha\rho + \frac{(2\eta^2 - q_0)}{\alpha} \sinh 2\alpha\rho - 2q_0\rho + 16\alpha^2 q_0 t \right). \quad (52)$$

Substituting in relation (50), then

$$\Gamma = \frac{-2\eta^2 \sinh \alpha\rho + 2\eta\alpha \cosh \alpha\rho + q_0 \sinh \alpha\rho}{\eta \sinh \alpha\rho - \alpha \cosh \alpha\rho} - \frac{1}{4\alpha^2 q_0} \left( \frac{1}{c + \xi} \right) (\eta \sinh \alpha\rho - \alpha \cosh \alpha\rho)^2, \quad (53)$$



**Figure 3:** (a) (b)  
The singular solution solution of (54) for different values  $\eta = .2$ ,  $c = 5$  and  $q_0 = -0.1$ .

Then substituting from relations (53) and (51) into (7), we obtain

$$q' = \frac{q_0 (-\eta^2 - 3\alpha^2 + (\alpha^2 + \eta^2) \cosh 2\alpha\rho - 2\eta\alpha \sinh 2\alpha\rho)}{2 (\alpha \cosh \alpha\rho - \eta \sinh \alpha\rho)^2} - 2 \left( \frac{-2\eta^2 \sinh \alpha\rho + 2\eta\alpha \cosh \alpha\rho + q_0 \sinh \alpha\rho}{\eta \sinh \alpha\rho - \alpha \cosh \alpha\rho} - \frac{1}{4\alpha^2 q_0} \left( \frac{1}{c + \xi} \right) (\eta \sinh \alpha\rho - \alpha \cosh \alpha\rho)^2 \right)_x \tag{54}$$

We have represented this solution for a set of parameter values in Fig. 3.

### 5.2. The MKdV equation

Let  $q = 4\tan^{-1}(e^{2\eta\rho})$ . (55)

Clearly (55) satisfies the KdV equation (2) as can be seen by direct substitution. Substituting in equation (35) we obtain that the constant  $\beta$  is equal zero. Substituting in two relations (43) and (45)

$$\xi = \int \frac{rd\rho}{(C + Kr)} + t = \frac{1}{32\eta^3} (32\eta^3 t + e^{-4\eta\rho} - 4\eta\rho) \tag{56}$$

Substituting in relation (56), then

$$\Gamma = -\frac{e^{2\eta\rho} (2 + e^{4\eta\rho} - 32\eta^3(t + c) + 4\eta\rho)}{1 + 4\eta e^{4\eta\rho} (8\eta^2(t + c) - \rho)} \tag{57}$$

Then substituting from relations (57) and (55) into (11), we obtain

$$q' = 4 \left( \tan^{-1} (e^{2\eta\rho}) + \tan^{-1} \left( \frac{e^{2\eta\rho} (2 + e^{4\eta\rho} - 32\eta^3(t + c) + 4\eta\rho)}{1 + 4\eta e^{4\eta\rho} (8\eta^2(t + c) - \rho)} \right) \right) \tag{58}$$

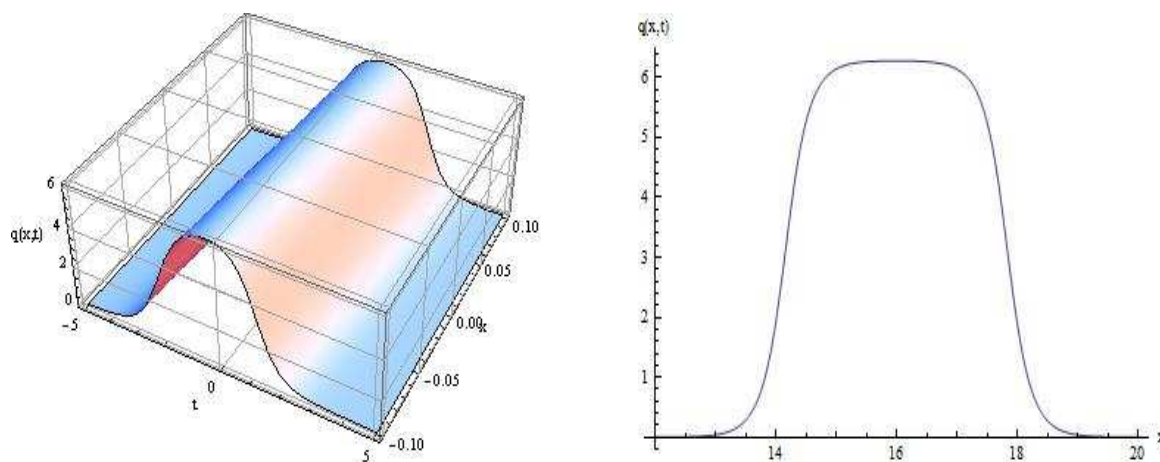
where  $\rho = x - 4\eta^2 t$ . We have represented this solution for a set of parameter values in Fig. 4.

## 6. Auto-Bäcklund transformation for KdV equation

Consider the KdV equation

$$q_t + 6q_x + 6qq_x + q_{xxx} = 0. \tag{59}$$





**Figure 4:** (a) (b)

The double kink of the solution (58) for different values  $\eta = 1$  and  $c = 1$ . (a)in the 3D and (b) in the plane at  $t = 1$ .

According the idea of improved HB [9-10], we seek for ABT [23-24] of equation (59), when balancing  $qq_x$  with  $q_{xxx}$  then gives  $N = 2$ . therefore, we may choose

$$q = \frac{\partial^2 f(\omega)}{\partial x^2} = f''(\omega)\omega_x^2 + f'(\omega)\omega_{xx} + u_0, \tag{60}$$

where  $u_0(x, t)$  is a solution of equation(59). Substituting equation (60) into equation (56), we obtain

$$[6f''f''' + f^{(5)}](\omega_x)^5 + [\omega_{xxt} + 6\omega_{xxx} + 6\omega_{xxx}u_0 + 6u_0\omega_{xx} + \omega_{xxxxx}]f' + [2\omega_x\omega_{xt} + \omega_t\omega_{xx} + 18\omega_x\omega_{xx} + 6\omega_x^2u_{0x} + 18\omega_x\omega_{xx}u_0 + 5\omega_x\omega_{xxxx} + 10\omega_{xx}\omega_{xxx}]f'' + [\omega_t\omega_x^2 + 6\omega_x^3 + 6\omega_x^3u_0 + 10\omega_x^2\omega_{xxx} + 15\omega_x\omega_{xx}^2]f''' + [10\omega_x^3\omega_{xx}]f^{(4)} + [6\omega_{xx}\omega_{xxx}](f')^2 + [6\omega_x^2\omega_{xxx} + 18\omega_x\omega_{xx}^2]f'f'' + [6\omega_x^3\omega_{xx}](f'')^2 + [6\omega_x^3\omega_{xx}]f'f''' + [u_{0t} + 6u_{0x} + 6u_0u_{0x} + u_{0xxx}] = 0, \tag{61}$$

where  $f^{(i)} = d^i f/d\omega^i$ , the subscripts denote the partial derivatives, setting the coefficient of  $\omega_x^5$  to zero yields an ordinary differential equation for  $f$

$$6f''f''' + f^{(5)} = 0, \tag{62}$$

which admits a solution

$$f = 2ln\omega. \tag{63}$$

According to (77), we get

$$(f'')^2 = -f^{(4)}, \quad f'f''' = -\frac{2}{3}f^{(4)} \quad f'f'' = -f''' \quad (f')^2 = -2f''. \tag{64}$$

By making use equation (64), equation (61) can be written as a finite series of  $f', f''$  and  $f'''$ , then we get

$$[\omega_{xxt} + 6\omega_{xxx} + 6\omega_{xxx}u_0 + 6u_0\omega_{xx} + \omega_{xxxxx}]f' + [2\omega_x\omega_{xt} + \omega_t\omega_{xx} + 18\omega_x\omega_{xx} + 6\omega_x^2u_{0x} + 18\omega_x\omega_{xx}u_0 + 5\omega_x\omega_{xxxx} - 2\omega_{xx}\omega_{xxx}]f'' + [\omega_t\omega_x^2 + 6\omega_x^3 + 6\omega_x^3u_0 + 4\omega_x^2\omega_{xxx} - 3\omega_x\omega_{xx}^2]f''' = q_t + 6q_x + 6qq_x + q_{xxx} = 0, \tag{65}$$

Decomposing the coefficient of  $f'''$  as

$$[\omega_t\omega_x + 6\omega_x^2 + 6\omega_x^2u_0 + 4\omega_x\omega_{xxx} - 3\omega_{xx}^2]\omega_x = A\omega_x. \tag{66}$$

Integrate the coefficient of  $f'$  with respect to  $x$  once and setting the constant of integration to zero. Denote

$$B = \omega_{xt} + 6\omega_{xx} + 6u_0\omega_{xx} + \omega_{xxxx}. \tag{67}$$

Rewriting the coefficient of  $f''$  we have

$$Coeff.of \ f'' = A_x + B\omega_x. \tag{68}$$

Substituting equations (66), (67) and (68) into equation (65) gives

$$A\omega_x f''' + (A_x + B\omega_x)f'' + B_x f' = 0, \tag{69}$$

that is

$$\frac{2A\omega_x}{\omega^3} - \frac{A_x + B\omega_x}{\omega^2} + \frac{B_x}{\omega} = \left( \frac{B\omega - A}{\omega^2} \right)_x = 0. \tag{70}$$

Thus we have a ABT as follows

$$q(x, t) = 2[\ln\omega(x, t)]_x + u_0(x, t). \tag{71}$$

We can write the relation (70) as

$$B\omega - A = (\omega_{xt} + 6\omega_{xx} + 6u_0\omega_{xx} + \omega_{xxxx})\omega - [\omega_t\omega_x + 6\omega_x^2 + 6\omega_x^2u_0 + 4\omega_x\omega_{xxx} - 3\omega_{xx}^2] = 0. \tag{72}$$

By equation (72)

$$(D_x D_t + 6(1 + u_0)D_x^2 + D_x^4) w \cdot w = 0, \tag{73}$$

where

$$D_x^m D_t^n a.b = (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'}) a(x, t) b(x', t') \Big|_{x'=x, t'=t}. \tag{74}$$

Now setting  $u_0(x, t) = 0$  and we use the auto-BT consisted of (63) and (7) to obtain two-soliton solution of equation (59). We supposed that the solution of equation (73) in the form

$$w(x, t) = 1 + \sum_{i=1}^{\infty} \varepsilon^i \phi_i, \tag{75}$$

where  $\phi_i(x, t), (i = 1, 2, 3, \dots)$  to be determined later,  $\varepsilon$  be a small parameter. Substituting (75) into (73) and collecting all terms with the same order of  $\varepsilon$  equal to zero then we obtain

$$\varepsilon^0 : \quad 0 = 0, \tag{76}$$

$$\varepsilon^1 : \quad (D_x D_t + 6D_x^2 + D_x^4) (\phi_1 \cdot 1 + 1 \cdot \phi_1) = 0, \tag{77}$$

$$\varepsilon^2 : \quad (D_x D_t + 6D_x^2 + D_x^4) (\phi_2 \cdot 1 + \phi_1 \cdot \phi_1 + 1 \cdot \phi_2) = 0, \tag{78}$$

$$\varepsilon^3 : \quad (D_x D_t + 6D_x^2 + D_x^4) (\phi_3 \cdot 1 + \phi_2 \cdot \phi_1 + \phi_1 \cdot \phi_2 + 1 \cdot \phi_3) = 0 \quad \dots \tag{79}$$

Note that

$$(D_x D_t + 6D_x^2 + D_x^4) (\phi_1 \cdot 1) = (D_x D_t + 6D_x^2 + D_x^4) (1 \cdot \phi_1) = \phi_{ixt} + 6\phi_{ixx} + \phi_{ixxxx}, \tag{80}$$

the linear equation  $\phi_{1xt} + 6\phi_{1xx} + \phi_{ixxxx}$  take an exponential solution

$$\phi(x, t) = \exp(\xi_1) + \exp(\xi_2) \quad \xi_i = \alpha_i x + \omega_i t + \xi_{i0}, \tag{81}$$

where  $\alpha_i, \omega_i,$  and  $\xi_{i0}, (i = 1, 2)$  are constants and satisfy

$$\alpha_i \omega_i + 6\alpha_i^2 + \alpha_i^4 = 0, \quad (i = 1, 2) \quad (82)$$

From (50), we get

$$2(\phi_{1xt} + 6\phi_{1xx} + \phi_{1xxx}) = -[D_x D_t + 6D_x^2 + D_x^4](\phi_1 \cdot \phi_1), \quad (83)$$

and

$$[D_x D_t + 6D_x^2 + D_x^4] e^{\xi_1} \cdot e^{\xi_2} = [(\omega_2 - \omega_1)(\alpha_2 - \alpha_1) + 6(\alpha_2^2 - \alpha_1^2) + (\alpha_2^4 - \alpha_1^4)] e^{\xi_1 + \xi_2}, \quad (84)$$

we have

$$\phi_2(x, t) = e^{\xi_1 + \xi_2 + A_{12}} \quad (85)$$

where

$$e^{A_{12}} = \frac{(\omega_2 - \omega_1)(\alpha_2 - \alpha_1) + 6(\alpha_2^2 - \alpha_1^2) + (\alpha_2^4 - \alpha_1^4)}{(\omega_2 + \omega_1)(\alpha_2 + \alpha_1) + 6(\alpha_2^2 + \alpha_1^2) + (\alpha_2^4 - \alpha_1^4)} \quad (86)$$

and

$$\phi_3(x, t) = 0 \quad \text{for } k \geq 3 \quad (87)$$

Therefore the two-soliton solution of equation (1) is obtained from

$$u(x, y, t) = 2 \left[ \ln \left( 1 + \left[ e^{\xi_1} + e^{\xi_2} + e^{\xi_1 + \xi_2 + A_{12}} \right] \right) \right]_x, \quad (88)$$

## 7. Conclusion

In this paper, the Bäcklund transformations and a series of new exact explicit solutions of the KdV-type equations have been established. We used the inverse scattering transform to derive the Bäcklund transformations for the MKdV equation see relation (A7) (in Appendix A). The solutions include (a) the solitary wave solutions, (b) singular traveling solutions, (c) single-soliton solutions, (d) Kink and double-kink solutions. In addition to some known solutions re-deriving in a systematic way, several entirely new and more general explicit exact solitary wave solutions can also be obtained. We derive the auto-Bäcklund transformation for KdV equation (1). There exist two types, the first one is the soliton, singular solutions (see Figs 3a, 3b) and the second type is the kink and double kink (see Figs 2, 4) for the MKdV equation. Finally the results of this paper are useful in the interpretation of some physical phenomena, especially in plasma physics, fluid and nonlinear optics.

## Appendix A

**Proof of Proposition.** Application of the integrability condition  $\Psi'_{xt} = \Psi'_{tx}$  in equations (12a), (12b) and using equation (10) shows that

$$\frac{\partial}{\partial t} \left( \frac{\partial u'}{\partial x} + \frac{\partial u}{\partial x} \right) = 4\eta A \sin(u' - u) + 4\eta(B - C) \cos\left(\frac{u' - u}{2}\right) + 2\eta(B + C) + 2\eta \cos(u' - u), \quad (A1)$$

and

$$\begin{aligned} \frac{\partial}{\partial x} \left( \frac{\partial u'}{\partial t} - \frac{\partial u}{\partial t} \right) &= 4A_x \sin\left(\frac{u' - u}{2}\right) + 2A \cos\left(\frac{u' - u}{2}\right) \left[ -2u_x + 4\eta \sin\left(\frac{u' - u}{2}\right) \right] \\ &+ 2(B_x - C_x) + 2(B_x + C_x) \cos\left(\frac{u' - u}{2}\right) - (B + C) \sin\left(\frac{u' - u}{2}\right) \left[ -2u_x + 4\eta \sin\left(\frac{u' - u}{2}\right) \right], \end{aligned} \quad (A2)$$

from the equations (A1) and (A2) we obtain  $u_{xt} = -\frac{3}{2}(u_x)^2 u_{xx} - u_{xxxx}$ . Similarly, application of the integrability requirement yields  $u'_{xt} = -\frac{3}{2}(u'_x)^2 u'_{xx} - u'_{xxxx}$ . Hence, the relations (12a) and (12b) leave (3) invariant; that is, they

constitute an ABT of the MKdV equation.  
Consider now the following scattering problem:

$$\frac{\partial \psi_1}{\partial x} = \eta \psi_1 - \frac{u_x}{2} \psi_2, \quad \frac{\partial \psi_2}{\partial x} = -\eta \psi_2 + \frac{u_x}{2} \psi_1, \quad (\text{A3})$$

by choosing the t-dependency of the eigenfunction  $\psi_1$  and  $\psi_2$  to be

$$\frac{\partial \psi_1}{\partial t} = A(x, t, \eta) \psi_1 + B(x, t, \eta) \psi_2, \quad \frac{\partial \psi_2}{\partial t} = C(x, t, \eta) \eta \psi_1 - A(x, t, \eta) \psi_2. \quad (\text{A4})$$

Defining  $\Gamma = \frac{\psi_1}{\psi_2}$  the scattering equation (A4) gives the Riccati forms

$$\frac{\partial \Gamma}{\partial x} = 2\eta \Gamma - \frac{u_x}{2} - \frac{u_x}{2} \Gamma^2, \quad \frac{\partial \Gamma}{\partial t} = 2A\Gamma + B - C\Gamma^2. \quad (\text{A5})$$

We Put

$$\left( \frac{u' - u}{4} \right) = \tan \Gamma. \quad (\text{A6})$$

Substituting from equation (A6) into equation (A5) given the relation (12), Thus. from equation (A6), the BT for the MKdV-type equation in the case of the class ZS/AKNS system take the form

$$u' = u + 4 \tan^{-1} \Gamma. \quad (\text{A7})$$

If we choose  $\Gamma = \frac{1}{\Gamma'}$  and  $u'$  as in (A7), then  $\Gamma'$  and  $u'$  satisfy equation (A5).

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