International Journal of Basic and Applied Sciences, 4 (2) (2015) 156-161
www.sciencepubco.com/index.php/IJBAS
©Science Publishing Corporation
doi: 10.14419/ijbas.v4i2.4182
Research Paper

Influence of minimal subgroups on the product of smooth groups

A. M. Elkholy *, M. H. Abd El-Latif

Mathematics Department, Faculty of Science, Beni Suef University, Beni-Suef 62511, Egypt
**Corresponding author E-mail: aelkholy9@yahoo.com*

Abstract

A maximal chain in a finite lattice L is called smooth if any two intervals of the same length are isomorphic. We say that a finite group G is totally smooth if all maximal chains in its subgroup lattice $L(G)$ are smooth. In this article, we study the product of finite groups which have a permutable subgroup of prime order under the assumption that the maximal subgroups are totally smooth.

Keywords: *Permutable subgroups; Smooth groups; Subgroup lattices.*

1. Introduction

Only finite groups wiil be considered in this paper. Notation is standard and is taken mainly from Doerk and Hawkes [2]. In addition, for a fixed group G, the maximal length of the subgroup lattice $L(G)$ will be denoted by n, and $\pi(G)$ will denote the set of all distinct primes dividing $|G|$.
A subgroup H of a group G is permutable in G if it permutes with every subgroup of G. A subgroup H of a group G is said to be permutable in a subgroup K of G if it permutes with every subgroup of K. This concept was introduced by Asaad and Shaalan [3]. A group G is called smooth if G has a maximal chain of subgroups in which any two intervals of the same length are isomorphic. Finite smooth groups have been studied by Schmidt [4, 5]. A group G is said to be totally smooth if every maximal chain of subgroups is smooth. Finite totally smooth groups have been studied in [1].
A lattice L is said to be complemented if every element of L has a complement in L. Recall that a P-group is either an elementary abelian group of order p^n for a prime p, or a semidirect product of an elementary abelian normal subgroup P of order p^{n-1} for a prime p and a cyclic q-group inducing a power automorphism group on P, where p and q are different primes (see [6; p. 49]).
The purpose of this article is studying the product of finite totally smooth groups which have a permutable subgroup of prime order under the assumption that the maximal subgroups are totally smooth. Clearly, the structure of groups with $n \leq 2$ is well known. So we assume that $n \geq 3$.

2. Main results

The following Lemma will be used in the sequel:

Lemma 2.1 *A group G is totally smooth if and only if one of the following holds:*
(i) G is cyclic of prime power order.
(ii) G is a P-group.
(iii) G is cyclic of square free order (See [1]; Theorem 1).
In this article, we will deal only with groups whose order is divided at least by two primes. So we assume firstly that $|\pi(G)| = 2$.

Theorem 2.2 *Let $G = HK$ be the product of its proper subgroups H and K with $n \geq 3$ and $|\pi(G)| = 2$. Assume that all maximal subgroups of G are totally smooth. Let N be a minimal normal subgroup of H. If N is permutable in K, then one of the following holds:*
(i) G is a nonabelian P-group.
(ii) $n = 3$ and $|G| = p^2q$, where p and q are distinct primes in $\pi(G)$.
(iii) $G = PQ$, where P is a cyclic Sylow p-subgroup of order p^2 and Q is an elementary abelian normal subgroup of G of order $q^e(e > 1)$.

Proof. *Since all maximal subgroups of G are totally smooth, it follows by Lemma 2.1, that H and K are cyclic of prime power orders, P-groups, or cyclic of square free order. Let P be a Sylow p-subgroup of G and Q be a Sylow q-subgroup of G. We have the following cases:*

case 1 . *H is cyclic. It follows by Lemma 2.1 that either $|H| = p^\alpha$ with $\alpha \geq 1$ or $|H| = pq$ with $p \neq q$.*
Assume that $|H| = p^\alpha$ with $\alpha \geq 1$. Hence $|N| = p$.
Suppose, further, that K is cyclic of prime power order. Since $|\pi(G)| = 2, |K| = q\beta$ with $q \neq p$. It follows that H is complemented in G. Let K_1 be a proper subgroup of K. If $|H| = p$, then $H = N$ is permutable in K and HK_1 is a proper subgroup of G. By hypothesis, HK_1 is totally smooth and hence Lemma 2.1 shows that HK_1 is a nonabelian P-group or cyclic of square free order. Since H and K are cyclic, it follows that $|HK_1| = pq$. Hence K would be of order q^2 as K_1 is any proper subgroup of K. Then $|G| = pq^2$ and (ii) holds. So assume that $|H| > p$. Hence H has a permutable subgroup N in K. By hypothesis, NK would be a proper subgroup of G and by using Lemma 2.1, it is cyclic of order pq or a nonabelian P-group. Since K is cyclic, $|K| = q$.
Let p be the largest prime in $\pi(G)$. Since K is cyclic, we get HG. If H_1 is a maximal subgroup of H, it follows that H_1G and hence $H_1K < G$. Since H_1 is cyclic, it follows by hypothesis and Lemma 2.1 that H_1K would be of order pq. Then $|G| = p^2q$ and we are done. So let q be the largest prime in $\pi(G)$. Then KG and hence $|G| = p^2q$.
Now suppose that K is cyclic of order pq, Hence $|Q| = q$. If $n = 3$, $|G| = p^2q$ and we are done. So let $n \geq 4$. It follows that $|H| \geq p^2$. As H is cyclic and P is totally smooth, P would be cyclic. If $p > q$, PG which implies that every subgroup of P is normal in G. Then there exists a proper subgroup L of G with $|L| = p^2q$ which is not totally smooth, a contradiction. Thus $p < q$. Since P is cyclic, QG. Once again, $|G| = p^2q$ and $n = 3$, a contradiction.
Final, suppose that K is a P-group. It follows that K is elementary abelian or a nonabelian P-group. Assume that K is elementary abelian of order q^β with $\beta > 1$.
If $|H| = p$, H is permutable in K and hence HK_1 is a maximal subgroup of G where K_1 is a maximal subgroup of K. Since HK_1 is totally smooth, we have by Lemma 2.1, that HK_1 is cyclic of order pq or a nonabelian P-group. If $|HK_1| = pq$, then $n = 3$ and G would be of order pq^2. Otherwise, HK_1 is a nonabelian P-group with $p < q$. Hence KG. Since K_1 is any maximal subgroup of K and HK_1 is a nonabelian P-group, H does not centralize any subgroup of K. Then G is a nonabelian P-group of order $pq^\beta(\beta > 1)$ and (i) holds. So assume that $|H| > p$. Hence H has a permutable subgroup N in K. By Lemma 2.1, NK would be cyclic of order pq or a nonabelian P-group. If NK would be of order pq, $|K| = p$ which contradicts that $\beta > 1$. Thus NK would be a nonabelian P-group with $p < q$. Since H is cyclic, KG. If H_1 is a maximal subgroup of H, H_1K is totally smooth proper subgroup of G. Since $\beta > 1$, H_1K would be a nonabelian P-group which implies that $|H_1| = p$ and hence $|H| = p^2$. If K has a normal subgroup in G, H would be of order p, we get a contradiction since $|H| > p$. Thus K is a minimal normal subgroup of G and (iii) holds. Now assume that K is a nonabelian P-group.
If $n = 3$, $|G| = p^2q$ and we are done. So assume that $n \geq 4$. Obviously, $Q < K$. If $p > q$, PG. We get by Lemma 2.1, P is cyclic or elementary abelian. If P is cyclic, $n = 3$ which contradicts that $n \geq 4$. Thus P is elementary abelian. Then H would be of order p. Since K is a nonabelian P-group, there exists a proper subgroup L of K with $|L| = p$ and LK. As P is elementary abelian, LP and hence LG. Since $n \geq 4$, we get $p^2 \mid\mid\mid |G/L|$. As G/L is totally smooth, G/L would be a nonabelian P-group. Then Q does not centralize any subgroup of P and every subgroup of P is normal in G. Therefore, Q induces a nontrivial power automorphism on P and hence G is a nonabelian P-group. So let $p < q$.
Assume that $|H| > p$. Then P would be cyclic and hence QG. Clearly, $L_1Q < G$ where L_1 is a maximal subgroup of P. Since $q > p$, $|L| = p$ and hence $|P| = p^2$. If Q has a proper subgroup Q_1 such that Q_1G, we get $Q_1P < G$. By hypothesis, Q_1P is totally smooth. By applying Lemma 2.1, $|P| = p$, a contradiction. Thus Q is a minimal

normal subgroup of G and (iii) holds. So assume that $|H| = p$. It follows that $HQ < G$ as $G = HK$ is a product of its proper subgroups H and K. If Q has a proper subgroup Q_1, we get Q_1HQ. Since Q_1K, it follows that Q_1G. Clearly, Q_1H is a totally smooth proper subgroup of G. By Lemma 2.1, H would be of order p, a contradiction. Thus Q would be of order q and hence $n = 3$ which contradicts that $n \geq 4$.

Thus H is cyclic of order pq. Then every minimal subgroup of H is permutable in K. It is clear that if $n = 3$, we are done. So assume that $n \geq 4$.

Suppose that K is of prime power order and let $N < H$ with $(|N|, |K|) = 1$. So we can assume that $|K| = p^\beta$ and hence $|N| = q$. Clearly, $NK \leq G$. Since $n \geq 4$, $|K| \geq p^2$. If $G = NK$, we get a proper subgroup U of G containing H with $p^2 \mid |U|$ which is not smooth since H is cyclic of order pq, a contradiction. Thus $NK < G$. By hypothesis and Lemma 2.1, NK would be nonabelian P-group of order $p^e q$ $(e \geq 2)$. Hence P would be elementary abelian and $|Q| = q$. Similar, there exists a non-totally smooth subgroup V of G containing H with $p^2 \mid |V|$ which contradicts our hypothesis. Thus K is a nonabelian P-group of order $p^\alpha q$ or cyclic of order pq.

Suppose that K is cyclic of order pq. Since $G = HK$ is a product of its proper subgroups H and K, it follows that $n = 3$ which contradicts that $n \geq 4$. Thus K is a nonabelian P-group of order $p^\alpha q$ $(p > q)$. Since $G = HK$, we get a normal subgroup N of H with NK. By hypothesis, $NK \leq G$. If $NK = G$ and since H is cyclic, we get $n = 3$ which contradicts our assumption that $n \geq 4$. Thus $NK < G$. If $q^2 \mid |NK|$, then $[NK/1]$ is not smooth since K is a nonabelian P-group which contradicts our hypothesis. Thus N would be of order p. It follows that N would be normal in NK as K is a nonabelian P-group. Once again, there exists a subgroup of G containing H which is not smooth, a contradiction.

case 2. *H is a P-group.*

It follows that H is elementary abelian or a nonabelian P-group. Suppose, first, that H is elementary abelian of order p^α with $\alpha > 1$. Then H has a permutable subgroup N in K. It is clear that if $n = 3$, then $|G| = p^2q$ and we are done. So let $n \geq 4$.

Assume that K is cyclic of order q^β, $\beta \geq 1$. Since $n \geq 4$, we get by hypothesis that NK is a totally smooth proper subgroup of G. It follows by lemma 2.1 that NK is cyclic of order pq or a nonabelian P-group. If NK is cyclic, NNK and $|K| = q$. Hence NG. Since $n \geq 4$, we get $|H| > p^2$. By hypothesis and lemma 2.1, G/N would be a nonabelian P-group with $p > q$. Since $|H| > p^2$, it follows that H has a permutable subgroup N_1 in K with N_1G. Then there is a subgroup of G containing NN_1 and K which is not smooth, a contradiction. Thus NK is a nonabelian P-group for any minimal normal subgroup N of H, $|K| = q$, and every subgroup of H is normal in G. Since K does not centralize any subgroup of H, it follows that G is a nonabelian P-group and we are done.

Let K be cyclic of order pq. Since $|H| > p$ and $n \geq 4$, there exists a permutable subgroup N of H with NK. It follows that $[NK/1]$ is not smooth, a contradiction. Thus K is a P-group. If K is elementary abelian group of order q^β with $\beta > 1$. Since H has a permutable subgroup N in K, it follows that NK is a proper subgroup of G. Our hypothesis and lemma 2.1 show that NK would be a nonabelian P-group with $q > p$ as $\beta > 1$. Since N is any minimal normal subgroup of H, there exists a subgroup Q_1 of Q of order q which is normal in G. Then HQ_1 is a proper subgroup of G. Since $|H| > p$, it follows by lemma 2.1 that $[HQ1/1]$ is not smooth; a contradiction as $p < q$. Thus K is a nonabelian P-group.

Suppose first that $|K| = p^\beta q$, $(p > q)$. It follows that p is the largest prime dividing $|G|$ and hence Q would be of order q. Then PG and P is elementary abelian. Then G has a normal subgroup P_1 of order p with $P_1 < K$. By hypothesis and lemma 2.1, G/P_1 would be a nonabelian P-group. Since P_1 is any minimal subgroup of P, G would be a nonabelian P-group.

Now consider $|K| = q^\beta p$. Hence $Q < K$ and p is the smallest prime dividing $|G|$. Since $|H| > p$, H has a permutable subgroup N of H. Among all such minimal normal subgroups N of H, choose N such that NK. Hence NQ is a proper subgroup of G which is totally smooth. Applying lemma 2.1, NQ is cyclic of order pq or a nonabelian P-group $(q > p)$. Then QNQ for each $N < H$ as H is elementary abelian. Hence QG.

If $|Q| = q$ and since $n \geq 4$, then G has a proper subgroup U containing Q such that$p^2 \mid |U|$ which is not totally smooth. Since $p < q$, we get a contradiction. Thus $|Q| > q$ and hence NQ is a nonabelian P-group. Let Q_1 be a maximal subgroup of Q. Clearly, Q_1NQ for each $N < H$ and so Q_1G. Similar, we get a contradiction since $p^2||G/Q_1|$ and $[G/Q_1]$ is not totally smooth. Thus assume that H is a nonabelian P-group of order $p^\alpha q$ with $\alpha \geq 1$. It follows that H has a normal subgroup N of order p.

If K would be cyclic of order q^β, we get $n = 3$ and $|G| = pq^2$ since $G = HK$. Thus assume that K is cyclic of order p^β. Clearly, $|Q| = q$ and so PG. If P is cyclic, $|G| = p^2q$ and we are done. So suppose that $n \geq 4$ and P is elementary abelian. Hence $|K| = p$. Since G/N is totally smooth, G/N would be nonabelian P-group and hence Q does not centralize any subgroup of P. Therefore, G is a nonabelian P-group. So assume that K is cyclic of order pq. Since $G = HK$ is the product of its proper subgroups H and K, there exists a minimal normal subgroup N of H such that NK. Hence $NK < G$ and by lemma 2.1 we get $[NK/1]$ is not totally smooth, a contradiction. Thus

$n = 3$ and (ii) holds.

ocmFinal, consider K is a P-group. Hence K is elementary abelian or a nonabelian P-group of order $p^\beta q$. Assume that K is elementary abelian of order q^β with $\beta > 1$. As H is a nonabelian P-group, H has a normal subgroup N of order p and by hypothesis NK is a subgroup of G. Since $G = HK$ is a product of its proper subgroup H and K, NK would be a proper subgroup of G which is totally smooth. Since $p > q$, it follows by lemma 2.1 that $|K| = q$ which contradicts our assumption that $\beta > 1$. Thus K is elementary abelian of order p^β. Similar, we get $|Q| = q$ and P is elementary abelian normal Sylow p-subgroup of G. Therefore, G/N is a nonabelian P-group and Q does not centralize any p-subgroup of P. Hence Q induces a nontrivial power automorphism on P. Then G is a nonabelian P-group and we are done. To complete the proof, K would be a nonabelian P-group of order $p^\beta q$. Hence NNK and so NG. Similar, G/N is a nonabelian P-group and consequently Q would be of order q. Once again, G would be a nonabelian P-group. This completes our proof.

Now we are in a position to prove the case when $|\pi(G)| \geq 3$.

Theorem 2.3 *Let $G = HK$ be the product of its proper subgroups H and K with $n \geq 3$ and $|\pi(G)| \geq 3$. Assume that all maximal subgroups of G are totally smooth. If every minimal normal subgroup of H is permutable in K, then one of the following holds:*
(i) G is cyclic of square free order.
(ii) $n = 3$ and $|G| = pqr$, where p, q, and r are distinct primes in $\pi(G)$.

Proof.

As all maximal subgroups of G are totally smooth, we get by Lemma 2.1, that H and K are cyclic of prime power orders, P-groups, or cyclic of square free order. Let N be a minimal normal subgroup of H. We have the following cases:

case 1. *H is cyclic.*
It follows that either $|H| = p^\alpha$ with $\alpha \geq 1$ or H is of order $p_1p_2...p_m$ where $p_i \neq p_j$ $(i \neq j)$ and $i, j = 1, 2, ...m$.
Let $|H| = p^\alpha$ with $\alpha \geq 1$. Since $|\pi(G)| \geq 3$, $|\pi(K)| \geq 2$. So we can assume that either K is cyclic of square free order or a nonabelian P-group of order $q^\beta r$, $(q > r)$.
Suppose first that K is a nonabelian P-group of order $q^\beta r$, $(q > r)$ and let $K = QR$ where Q is a Sylow q-subgroup of K and R is a Sylow r-subgroup of K. If $|H| > p$, we get by hypothesis that $NK < G$ where N is a normal subgroup of H of order p. Furthermore, $|\pi(NK)| = 3$. Since K is a nonabelian P-group, we have by lemma 2.1 that $[NK/1]$ is not totally smooth which contradicts our hypothesis. Thus H would be of order p. Since all maximal subgroups of G are supersolvable, it follows that G is solvable and so G has a Sylow basis. Hence $HQ < G$.
If $|Q| = q$, then $|G| = pqr$ and (ii) holds. So let $|Q| > q$. Hence by lemma 2.1, HQ would be a nonabelian P-group $(q > p)$. Then there exists a proper subgroup Q_1 of Q which is normal in HQ. Since K is a nonabelian P-group, Q_1K. Therefore, Q_1G. Clearly, $HR < G$. Then $Q_1HR < G$. Since R does not centralize Q_1 and $|\pi(Q_1HR)| = 3$, we get $[Q_1HR/1]$ is not totally smooth which contradicts our hypothesis. Thus K is cyclic of square free order. Let P_i be a Sylow p_i-subgroup of G with $p \neq p_i$. By the solvability of G and since $|H| = p^\alpha$, HP_i is a subgroup of G. By hypothesis and lemma 2.1, H would be of order p as H is cyclic. Hence the Sylow subgroups of G are of prime orders. If G is abelian, then G is cyclic. Otherwise, $|\pi(G)| = 3$ and G would be of order pqr.
Now assume that H is a cyclic of order $p_1p_2...p_m$ with $(m \geq 2)$. Then there exists a minimal normal subgroup N of H with NK such that $NK \leq G$. Since $|\pi(G)| \geq 3$, K would be a P-group or cyclic of square free order or cyclic of prime power order.
Suppose first that K is cyclic of order p^α. It follows that NK is a totally smooth proper subgroup of G. By lemma 2.1, NK would be cyclic of square free order or a nonabelian P-group. As K is cyclic, $|K| = p$. If $n = 3$, then G is cyclic or (ii) holds. Thus assume that $n \geq 4$. Since G is solvable, $PP_j < G$ where P is a Sylow p-subgroup of G and P_j is a Sylow p_j-subgroup of G $(p \neq p_j)$. We argue that $|P| = p$.
Suppose for a contradiction that $|P| > p$. We have by lemma 2.1 that PP_j is a nonabelian P-group $(p > p_j)$. Then there exists a proper subgroup L of P which is normal in PP_j and hence it is normal in G as P_j is any Sylow p_j-subgroup of G $(p \neq p_j)$. Since $|\pi(G)| \geq 3$, it follows that $LP_iP_j < G$ where $p_j \neq p_i \neq p$. We get by hypothesis and lemma 2.1 that LP_iP_j is cyclic, a contradiction since P_j does not centralize L. Thus $|K| = |P| = p$ and so the Sylow subgroups of G are of prime orders. As $n \geq 4$, $|\pi(G)| \geq 4$. Hence PP_1P_2 is a proper subgroup of G and by lemma 2.1, it would be cyclic. Thus P_iG, $i = 1, 2$. By applying lemma 2.1, G/P_i would be cyclic as $|\pi(G/P_i)| \geq 3$. Then $G \leq P_1 \cap P_2 = 1$ hence G is abelian. Therefore G is cyclic of square free order.
Now Let K be a cyclic of square free order such that $|\pi(K)| \geq 2$. Since $G = HK$ is a product of its proper subgroups H and K, it follows that there exists a minimal subgroup N of H with NK. Hence $NK \leq G$. Obviously, if $p^2 \mid |G|$ for some prime $p \in \pi(G)$ and since G is solvable, we get a normal subgroup L of order p. Similar, LP_1P_j is a cyclic

subgroup of G. Then by lemma 2.1, $[PP_j/1]$ is not smooth, a contradiction. Thus the Sylow subgroups of G are of prime orders.

Assume first that $G = NK$ and let K_1 be a maximal subgroup of K. Hence $NK_1 < G$. By hypothesis and lemma 2.1, NK_1 is a nonabelian P-group or cyclic. If $|\pi(G)| = 3$, we are done since NK_1 is of square free order. So let $|\pi(G)| \geq 4$. It follows that NK_1 would be cyclic and hence every Sylow subgroup of K centralizes N as K_1 is any maximal subgroup of K. Then G is abelian and hence it is cyclic of square free order. Now assume that $NK < G$. Since the Sylow subgroups of G are of prime orders and $|\pi(K)| \geq 2$, NK would be cyclic of square free order and $|\pi(G)| \geq 4$. Once again, G is cyclic.

Consider K is a P-group. Hence it is elementary abelian or a nonabelian P-group. Suppose first that K is a nonabelian P-group of order $p^{\alpha}q$, $p > q$. If $NK < G$, we have a contradiction since K is a nonabelian P-group and $|\pi(NK)| = 3$. Thus $NK = G$ and hence $|\pi(G)| = 3$.

Suppose, for a contradiction, that $n \geq 4$. Since N is of prime order and $(|N|, |K|) = 1$, it follows that $|G|$ would be divided by p^2 where p is the largest prime in $\pi(K)$. Let K_1 be a maximal nonabelian P-subgroup of K. It follows that NK_1 is a proper subgroup of G. As $|\pi(NK_1)| = 3$, we have by lemma 2.1 that NK_1 would be cyclic. Since K_1 is a nonabelian P-group, we get a contradiction. Thus $n = 3$ and we are done. So assume that K is elementary abelian of order p^{β}, $\beta > 1$. It follows NK is a totally smooth proper subgroup of G where N is a minimal subgroup of H. Since $|K| > p$, we get by lemma 2.1 that NK would be a nonabelian P-group where p is the largest prime in $\pi(NK)$. Let $L < K$. Then LNK and hence LG as N is any minimal subgroup of H. Let N_1 and N_2 be minimal subgroups of H such that $N_1 \neq N_2$. Clearly, $LN_1N_2 < G$. Since $|\pi(LN_1N_2)| = 3$, LN_1N_2 would be cyclic which contradicts that N_i does not centralize L $(i = 1, 2)$. Thus $|K| = p$, a contradiction as $\beta > 1$.

case 2. *H is a P-group. Hence H is elementary abelian or a nonabelian P-group.*

Suppose, first, that H is elementary abelian of order p^{α} with $\alpha > 1$. Therefore H has a permutable subgroup N in K. Hence K is a nonabelian P- group or cyclic of square free order. Suppose that K is a nonabelian P-group. We get $|\pi(G)| = 3$ and $(|N|, |K|) = 1$. Then NK would be cyclic which contradicts our choice of K. Thus K is cyclic of square free order. Let P be a Sylow p-subgroup of G. As P is totally smooth and H is elementary abelian p-subgroup, it follows that P would be elementary abelian. If p would be dividing $|K|$, we get G has a normal subgroup N of order p. Hence $|\pi(G/N)| \geq 3$ which implies that G/N would be cyclic of square free order since G/N is totally smooth. Then $|P| = p^2$. Let Q be a Sylow q-subgroup of G with $q \neq p$. Since G is solvable, PQ is a subgroup of G. Since Q centralizes N, $[PQ/1]$ is not smooth which contradicts our hypothesis. Thus $p|K|$. It follows that $NK < G$. Similar, we get a contradiction as $|H| > p$. Thus H is a nonabelian P-group of order $p\alpha p_1$, $p > p_1$. Hence $|N| = p$.

Consider, first, that K is a nonabelian P-group. Then $|\pi(G)| \leq 4$. Let P_i be a Sylow p_i-subgroups of G with $p_i \neq p$. Since G is solvable, we have PP_i is a totally smooth proper subgroup of G which is a nonabelian P-group or cyclic. If $|P| > p$, PP_i would be a nonabelian P-group and NPP_i. Then NG as P_i is any Sylow p_i-subgroups of G. Hence $NP_1P_2 < G$, for $i = 1, 2$. Since P_i does not centralize N, it follows by lemma 2.1 that $[NP_1P_2/1]$ is not smooth which contradicts our hypothesis. Thus $|P| = p$.

If $|\pi(G)| = 4$, we get by hypothesis and lemma 2.1 that PK is a cyclic subgroup of G and hence PPK. Since PH, we get PG. Once again $[PP_1P_2/1]$ is not smooth, a contradiction. Therefore $|\pi(G)| = 3$ and p would be the largest prime in $\pi(G)$ which implies that $n = 3$ and (ii) holds.

Assume that K is elementary abelian of order p_2^{β}. Once again, if $n = 3$, we get $|G| = pp_1p_2$ and we are done. So let $n \geq 4$. By hypothesis, NK is a totally smooth subgroup of G. Suppose that p is the largest prime in $\pi(G)$. Then NNK and $|K| = p_2$. Hence NG. As G/N is totally smooth, it follows by lemma 2.1 that G/N is a nonabelian P-group or cyclic of square free order. If G/N is a nonabelian P-group, $|\pi(G)| = 3$ which implies that $n = 3$, a contradiction. Thus G/N is cyclic with $|\pi(G/N)| \geq 3$ as $n \geq 4$. Since P_1 does not centralize N, $[NP1K/1]$ is not smooth which contradicts our hypothesis. Therefore p_2 is the largest prime in $\pi(G)$ which implies that $|P| = p$ since PP_2 is a totally smooth subgroup of G. Suppose, for a contradiction, that $|K| > p_2$. Since G is solvable, we have that G has a Sylow basis and since $|K| > p_2$, it follows that K has a normal subgroup L in G. Hence $LH < G$. Since H is a nonabelian P-group, $[LH/1]$ is not smooth which contradicts our hypothesis. Therefore $|K| = p_2$ and hence $n = 3$, a contradiction as $n \geq 4$.

To complete the proof, assume that K is cyclic. Suppose first that K is a cyclic of square free order and let P_j be Sylow p_j-subgroups of G with $p_j \neq p$ $(j = 1, 2, ..., m)$. The solvability of G shows that, $PP_j < G$. We argue that $|P| = p$. If $|P| > p$, then PP_j is a nonabelian P-group and $|P_j| = p_j$ for each $j = 1, 2, ..., m$. Then NPP_j and hence NG. It follows that $NP_1P_2 < G$. Then by lemma 2.1, it is cyclic; a contradiction since P_1 does not centralize N. Thus $|P| = p$. Suppose, for a contradiction, that $n \geq 4$. Since H is a nonabelian P-group, G has a proper subgroup M containing H with $|\pi(M)| = 3$ which is not smooth. Thus $n = 3$, $|G| = pp_1p_2$ and we are done. Now assume K is cyclic of prime power order. Then $(|N|, |K|) = 1$. Since NK is a totally smooth proper subgroup of G, K

would be of prime order. If KG, we get $|P| = p$ where P is a Sylow p-subgroup of G. Then $n = 3$ and (ii) holds. Otherwise, p is the largest prime in $\pi(G)$. Once again if $N < P$, we get a contradiction. Thus $|P| = p$ and $n = 3$. This completes our proof.

3. Conclusion

In this paper, we proved the following result:

Theorem 3.1 *Let $G = HK$ be the product of its proper subgroups H and K with $n \geq 3$ and $|\pi(G)| \geq 2$. Assume that all maximal subgroups of G are totally smooth. If every minimal normal subgroup of H is permutable in K, then one of the following holds:*
(i) G is a nonabelian P-group.
(ii) G is cyclic of square free order.
(iii) $n = 3$ and $|G| = pqr$, where p and q are not necessary distinct primes in $\pi(G)$.
(iv) $G = PQ$, where P is a cyclic Sylow p-subgroup of order p^2 and Q is an elementary abelian normal subgroup of G of order $q^e (e > 1)$.

Clearly, the proof of Theorem 3.1 is included in both Theorem 2.2 and Theorem 2.3.

References

[1] A. M. Elkholy, "On totally smooth groups", *Int. J. Algebra*, Vol.1, No.2, (2007), pp.63-70.

[2] K. Doerk, T. Hawkes, Finite soluble groups. Walter de Gruyter, *Berlin-New York*, (1992).

[3] M. Asaad, Shaalan, "On the supersolvability of finite groups", *Arch. Math*, 53, (1989), pp.318-326.

[4] R. Schmidt, "Smooth groups", *Geometriae Dedicata*, 84, (2001), pp.183-206.

[5] R. Schmidt, "Smooth *p*-groups", *J. Algebra*, 234, (2000), pp.533-5390.

[6] R. Schmidt, Subgroup lattices of groups. Walter de Gruyter, Berlin-New York, *Berlin-New York*, (1994).