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Research Paper

A class of new exact solutions of the equations governing the steady plane flows of incompressible fluid of variable viscosity

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Abstract

The objective of this paper is to indicate a class of new exact solutions of the equations governing the steady plane flows of incompressible fluid of variable viscosity. The class consists of the stream function characterized by equation (2). Exact solutions are determined for $g(r)=const.$ and $g(r)\neq const.$ When $f(r)$ is arbitrary we can construct an infinite set of streamlines and the velocity components, viscosity function, generalized energy function L and temperature distribution T. Therefore, an infinite set of solutions to flow equations. When $f(r)$ is not arbitrary, there are two values of $f(r)$ and therefore, two exact solutions to flow equations. The streamlines are presented through Fig.(1–56) for some chosen from of $f(r)$.

Keywords: *A Class of Exact Solutions; Exact Solutions to the Flow Equations of Incompressible; Variable Viscosity; Navier-Stokes Equations; New Exact Solutions Variable Viscosity.*

1. Introduction

Due to complex mathematical structure of the fluid flow equations, it is extremely difficult to achieve exact solutions. However, researchers have developed methods/techniques through which some exact solutions were determinable. The readers interested in these methods/techniques may refer to [1]-[23] and the references therein.
The aim of this paper is to indicate a class of new exact solutions of the equations describing the steady plane flows of incompressible fluid of variable viscosity. The aim is achieved by transforming flows equations into Martin system – (ϕ,ψ). In Martin system, the coordinate lines $\psi=\text{constant}$ represents streamlines and coordinate lines $\phi=\text{constant}$ are left arbitrary. We take $\phi=r(x,y)$ to achieve our aim. When $f(r)$ is arbitrary an infinite set of velocity components implying an infinite set of solutions to flow equation. When $f(r)$ is not arbitrary there are only two values of $f(r)$ indicating a set of two solutions.
The streamlines of the class of flows under consideration are characterized by

$$\frac{\theta-f(r)}{g(r)}=\text{const} \tag{1}$$

where $f(r)$ and $g(r)\neq 0$ are continuously differentiable functions and r, θ the polar coordinates. The equation, with loss of generality, implies

$$\theta=f(r)+g(r)\nu(\psi) \tag{2}$$

where $\nu(\psi)$ is unknown function such that $\nu'(\psi)\neq 0$.

The paper is organized as follow: In section (2), we give basic flow equations and transform them into Martin system. In section (3), we take $\phi = r(x,y)$ and transform the equations of section (2) in to a new system of equations. The solutions to new system of equations are determined. In section (4) we discuss the solutions of section (3). In section (5) we present conclusions.

2. Basic flow equations

The basic non-dimensional equations of motion governing a steady plane flow of an incompressible fluid of variable viscosity, in the absence of external force with no heat addition are

$$u_x + v_y = 0 \tag{3}$$

$$u\ u_x + v\ u_y = -p_x + \frac{1}{\mathrm{R_e}}\,[(2\mu u_x)_x + \{\mu(u_y + v_x)\}_y] \tag{4}$$

$$u\ v_x + v\ v_y = -p_y + \frac{1}{\mathrm{R_e}}\,[(2\mu v_y)_y + \{\mu(u_y + v_x)\}_x] \tag{5}$$

$$u\ T_x + v\ T_y = \frac{T_{xx} + T_{yy}}{\mathrm{R_e\,P_r}} + \frac{\mathrm{E_c}}{\mathrm{R_e}}\,[2\mu(u_x^{\,2} + v_y^{\,2}) + \mu(u_y + v_x)^2] \tag{6}$$

where p is the pressure, μ is the viscosity, T is the temperature, u and v are velocity components.
On introducing the vorticity function w, the total energy function L, the function A and B defined by

$$w = v_x - u_y \tag{7}$$

$$L = p + \frac{1}{2}\,(u^2 + v^2) - \frac{2\mu u_x}{\mathrm{R_e}} \tag{8}$$

$$A = \mu\,(u_y + v_x)\,, \quad B = 4\mu\,u_x \tag{9}$$

The system of equations (3 – 6) can be rewritten as

$$u_x + v_y = 0 \tag{10}$$

$$-v\ w = -L_x + \frac{A_y}{\mathrm{R_e}} \tag{11}$$

$$u\ w = -L_y - \frac{B_y}{\mathrm{R_e}} + \frac{A_x}{\mathrm{R_e}} \tag{12}$$

$$u\ T_x + v\ T_y = \frac{T_{xx} + T_{yy}}{\mathrm{R_e\,P_r}} + \frac{\mathrm{E_c}}{4\mu\mathrm{R_e}}\,(B^2 + 4A^2) \tag{13}$$

Now following Martin we introduce the curvilinear coordinate (ϕ,ψ) in the physical plane in which the coordinate lines $\psi = \text{constant}$ are the streamlines and the coordinate lines $\phi = \text{constant}$ left arbitrary.
In transforming the flow equations into curvilinear coordinates (ϕ,ψ), Martin considered the transformation defined by

$$x = x(\phi,\psi) \text{ and } y = y(\phi,\psi) \tag{14}$$

The transformation in (14) defines a system of curvilinear coordinates (ϕ,ψ) in the physical plane (x,y) such that the Jacobian, $J = \dfrac{\partial(x,y)}{\partial(\phi,\psi)}$ of the transformation is non-zero and finite. The first fundamental form ds^2 in (ϕ,ψ) system is given by

$$ds^2 = E(\phi,\psi)\ d\phi^2 + 2\ F(\phi,\psi)\ d\phi\ d\psi + G(\phi,\psi)\ d\psi^2 \tag{15}$$

where

$$E(\phi,\psi) = x_\phi^2 + y_\phi^2 ,\ F(\phi,\psi) = x_\phi x_\psi + y_\phi y_\psi ,G(\phi,\psi) = x_\psi^2 + y_\psi^2 \tag{16}$$

Differentiating equation (14) with respect to x and y, and solving the resulting equations for ψ_x , ψ_y , ϕ_x , ϕ_y yield:

$$x_\phi = J\ \psi_y ,\qquad x_\psi = -J\ \phi_y ,\qquad y_\phi = -J\ \psi_x ,\qquad y_\psi = J\ \phi_x \tag{17}$$

wherein

$$J = \pm\sqrt{EG - F^2} = \pm (x_\phi\, y_\psi - y_\varphi\, x_\psi) = \pm W \tag{18}$$

If α is the angle between the tangent at the point $P(x,y)$ to the coordinate line ψ =constant and the x-axis, then

$$\tan\alpha = \frac{y_\phi}{x_\phi} \tag{19}$$

Equation(17), on utilizing equation (19), gives

$$x_\phi = \sqrt{E}\ Cos\ \alpha ,\quad x_\psi = \frac{1}{\sqrt{E}}[F\ Cos\ \alpha - J\ Sin\ \alpha]$$

$$y_\phi = \sqrt{E}\ Sin\ \alpha ,\quad y_\psi = \frac{1}{\sqrt{E}}[F\ Sin\ \alpha + J\ Cos\ \alpha] \tag{20}$$

The integrablity conditions

$$x_{\psi\phi} = x_{\phi\psi} \qquad y_{\psi\phi} = y_{\phi\psi} \tag{21}$$

For x and y , yield

$$\alpha_\phi = \frac{J\,\Gamma_{11}^2}{E} ,\ \alpha_\psi = \frac{J\,\Gamma_{12}^2}{E} \tag{22}$$

wherein

$$\Gamma_{11}^2 = \frac{1}{2W^2}[-FE_\phi + 2E\,F_\phi - E\,E_\psi] ,\ \Gamma_{12}^2 = \frac{1}{2W^2}[E\,G_\phi - F\,E_\psi] \tag{23}$$

Equation (19), applying the integrability condition $\alpha_{\phi\psi} = \alpha_{\psi\phi}$ for $\alpha(\phi,\psi)$, yields

$$K = \frac{1}{W}\left[\left(\frac{W\ \Gamma_{11}^2}{E}\right)_\psi - \left(\frac{W\ \Gamma_{12}^2}{E}\right)_\phi\right] \tag{24}$$

where K is called the Gaussian curvature and equation (24) is called Gaussian equation. This equation represents a necessary condition that $E(\phi,\psi)$, $F(\phi,\psi)$ and $G(\phi,\psi)$ are coefficients of the first fundamental form in equation (15). The system of equations (10 – 13), on utilizing equations (20 – 24), is transformed into a new system of equations given in the following theorem.

Theorem I:
If the streamlines ψ =constant and the curves ϕ =constant left arbitrary, generate a curvilinear net in the physical plane, the equations (10 – 13), are transformed in to the following system of equations

$$q = \frac{\sqrt{E}}{W} \tag{25}$$

$$-R_e w\, J\, E = R_e\, J\, E\, L_\psi + A_\phi\left((F^2-J^2)\cos 2\alpha - 2FJ\sin 2\alpha\right) + EA_\psi\left(J\sin 2\alpha - F\cos 2\alpha)\right) - B_\phi\left(\frac{1}{2}(F^2-J^2)\sin 2\alpha + FJ\cos 2\alpha\right)$$
$$+ E\, B_\psi\left(\frac{1}{2}F\sin 2\alpha + J\cos^2\alpha\right) \tag{26}$$

$$0 = -R_e J\, L_\phi + E\, A_\psi\cos 2\alpha - A_\phi\left[F\cos 2\alpha - J\sin 2\alpha\right] + B_\phi\left(\frac{1}{2}F\sin 2\alpha - J\sin^2\alpha\right) - \frac{E\, B_\psi}{2}\sin 2\alpha \tag{27}$$

$$w = \frac{1}{W}\left[\left(\frac{F}{W}\right)_\phi - \left(\frac{E}{W}\right)_\psi\right] \tag{28}$$

$$\frac{1}{J\, \mathrm{R_e}\, \mathrm{P_r}}\left[\left(\frac{GT_\phi - FT_\psi}{J}\right)_\phi + \left(\frac{ET_\psi - FT_\phi}{J}\right)_\psi\right] = -\frac{\mathrm{E_c}}{4\mu \mathrm{R_e}}\left[B^2 + 4A^2\right] + \frac{T_\phi}{J} \tag{29}$$

$$K = \frac{1}{W}\left[\left(\frac{W\,\Gamma_{11}^2}{E}\right)_\psi - \left(\frac{W\,\Gamma_{12}^2}{E}\right)_\phi\right] \tag{30}$$

wherein ϕ and ψ are considered as independent variables. This is a system of six equations in seven unknowns E, F, G, W, L, T and q. In equations (26 – 29) the functions $A(\phi,\psi)$ and $B(\phi,\psi)$ are given by

$$A(\phi,\psi) = \mu\Big[-\frac{(F\cos\alpha - J\sin\alpha)}{4E^2J^5}\left\{E_\phi(2EJ^3\cos\alpha + F\sqrt{E}\sin\alpha) - 4E^2J^2J_\phi\cos\alpha - 2E\sqrt{E}\,F_\phi\sin\alpha + E\sqrt{E}\,E_\psi\sin\alpha\right\}$$
$$+\frac{\cos\alpha}{2J^3}\left\{E_\psi(F\sin\alpha + J\cos\alpha) - 2EJ_\psi\cos\alpha - E\,G_\phi\sin\alpha\right\} + \frac{(F\sin\alpha + J\cos\alpha)}{2EJ^3}\left\{(J\,E_\phi - 2EJ_\phi)\sin\alpha\right.$$
$$\left. + \cos\alpha\left[-FE_\phi + 2E\,F_\phi - E\,E_\psi\right]\right\} - \frac{\sin\alpha}{2J^3}\left\{(E_\psi(J\sin\alpha - F\cos\alpha) - 2EJ_\psi\sin\alpha + EG_\phi\cos\alpha\right\}\Big] \tag{31}$$

$$B(\phi,\psi) = \frac{4\mu}{EJ^3}\left[E_\phi(F\sin\alpha + J\cos\alpha)^2 - 2E(F\sin\alpha + J\cos\alpha)(F_\phi\sin\alpha + J_\phi\cos\alpha) + E^2(J_\psi\sin 2\alpha + G_\phi\sin^2\alpha)\right] \tag{32}$$

wherein

$$Cos\,\alpha = \frac{1}{\sqrt{E}}, \quad Sin\,\alpha = \frac{\sqrt{E-1}}{\sqrt{E}} \quad Cos\,2\alpha = \frac{2-E}{E}, \quad Sin\,2\alpha = \frac{2\sqrt{E-1}}{E} \tag{33}$$

3. Exact solutions

Since our objective is to determine a class of exact solutions to flow equations for which the streamlines are characterized by equation (1) and to achieve it we set

$$\phi = r(x,y) \tag{34}$$

where

$$x = r\,Cos\,\theta, \qquad y = r\,Sin\,\theta \tag{35}$$

Utilizing equations (34) and (35) in equations (25-33), we get

$$q = \frac{\sqrt{E}}{W} \tag{36}$$

$$-R_e w = R_e L_\psi - JA_r + \sqrt{E-1}A_\psi + B_\psi \tag{37}$$

$$0 = -R_e L_r + \frac{A_\psi(2-E)}{J} + A_r\sqrt{E-1} - \frac{\sqrt{E-1}B_\psi}{J} \tag{38}$$

$$J T_{rr} - 2\sqrt{E-1}\; T_{\nu r}\,\nu' + \frac{E}{J} T_{\nu\nu}(\nu')^2 + \left(J_r - \frac{E_\psi}{2\sqrt{E-1}} - R_e P_r\right) T_r \;\; + \left(\frac{E_\psi}{J} - \frac{E_r}{2\sqrt{E-1}} - \frac{E J_\psi}{J^2} + \frac{E}{J}\left(\frac{\nu''}{\nu'}\right)\right) T_\nu \nu'$$
$$= -\frac{J E_c P_r}{4\mu}\left(B^2 + 4A^2\right) \tag{39}$$

$$w = \left[\frac{f'(r)}{r\,g(r)} + \frac{f''(r)}{g(r)} - \frac{2f'(r)g'(r)}{g^2(r)}\right]\left(\frac{1}{\nu'(\psi)}\right) + \left[\frac{g'(r)}{r\,g(r)} + \frac{g''(r)}{g(r)} - \frac{2\{g'(r)\}^2}{g^2(r)}\right]\left(\frac{\nu(\psi)}{\nu'(\psi)}\right)$$

$$+\left[\frac{1}{r^2 g^2(r)} + \frac{\{f'(r)\}^2}{g^2(r)}\right]\left(\frac{\nu''(\psi)}{\{\nu'(\psi)\}^3}\right) + \frac{2f'(r)g'(r)\nu(\psi)\nu''(\psi)}{g^2(r)\{\nu'(\psi)\}^3} + \frac{\{g'(r)\}^2\nu^2(\psi)\,\nu''(\psi)}{g^2(r)\{\nu'(\psi)\}^3} \tag{40}$$

where

$$A(r,\psi) = \frac{\mu}{J}\left[\;\frac{-2J_r\sqrt{E-1}}{J} + \frac{E_r}{2\sqrt{E-1}} + \frac{-(2-E)J_\psi}{J^2}\;\right] \tag{41}$$

$$B(r,\psi) = 4\mu\;\frac{1}{J^3}\left[-J\;J_r + \sqrt{E-1}\;J_\psi\right] \tag{42}$$

$$E = 1 + r^2\;\left[f'(r) + g'(r)\;\nu(\psi)\right]^2 \tag{43}$$

$$F = J\sqrt{E-1} \tag{44}$$

$$G = r^2\;g^2(r)\;\nu'(\psi)^2 \tag{45}$$

$$W = J = r\;g(r)\;\nu'(\psi) \tag{46}$$

$$\sqrt{E-1} = r f'(r) + r\,g'(r)\;\nu(\psi) \tag{47}$$

In order to determine the solution of the flow equations we require an equation which the functions f, g, ν and the viscosity μ must satisfy for the class of flows under consideration and this is obtained by using the integrability condition $L_{r\psi} = L_{\psi r}$. The integrability condition, utilizing equations (43) and (46) yields

$$r\,g\,\nu' A_{rr} - 2r(f' + g'\nu)A_{r\psi} - \frac{\left[1 - r^2(f' + g'\nu)^2\right]}{r\,g\,\nu'}A_{\psi\psi} + g\,\nu' A_r - A_\psi\left((f' + g'\nu) + r(f'' + g''\nu)\right) - \left\{B_r - \frac{(f' + g'\nu)B_\psi}{g\,\nu'}\right\}_\psi = R_e w_r \tag{48}$$

Once a solution of this equation is determined, the function L and temperature distribution T are determined from equations (37-38) and (39), respectively.
To determine the exact solutions of equation (48), the presence of term $(f' + g'\nu)$ in equation (48) and equation (2) suggests to consider the following two cases.

Case I $\quad g$ = Constant (49)

Case II $\quad g \neq$ Constant (50)

Case I: For the sake of simplicity we take $g = 1$. On substituting $g = 1$ in equations (36-42), in the absence of body force become

$$q = \frac{\sqrt{1 + (rf')^2}}{r\,\nu'(\psi)}\,, \tag{51}$$

$$w = \left[\frac{f'(r)}{r} + f''(r)\right]\left(\frac{1}{\nu'(\psi)}\right) + \left[\frac{1}{r^2} + \{f'(r)\}^2\right]\left(\frac{\nu''(\psi)}{\{\nu'(\psi)\}^3}\right), \tag{52}$$

$$-\mathrm{R_e} w = \mathrm{R_e}\;\nu' L_\nu - r\,\nu'(\psi)A_r + r f'\nu' A_\nu + \nu' B_\nu\,, \tag{53}$$

$$0 = -\mathrm{R_e}\ L_r + r f' A_r + \frac{\{1-(rf')^2\}A_\nu}{r} - f' B_\nu \tag{54}$$

$$(r\nu'(\psi))^3 T_{rr} - 2(r\nu'(\psi))^3 f' T_{r\nu} + r(\nu'(\psi))^3 \{1+(rf')^2\} T_{\nu\nu} + (r\nu'(\psi))^2 \{\nu'(\psi) - R_e P_r\} T_r$$
$$-r^2(\nu'(\psi))^3 (rf')' T_\nu = -\frac{E_c P_r J^3}{4\mu}[B^2 + 4A^2] \tag{55}$$

$$-r\nu'(\psi)\ A_{rr} + 2rf'\ \nu'(\psi)\ A_{\nu r} + \nu'(\psi)\left\{\frac{\{1-(rf')^2\}}{r}\right\}A_{\nu\nu} - \nu'(\psi) f' B_{\nu\nu}$$
$$+\nu'(\psi)\ B_{\nu r} - \nu'(\psi)\ A_r + \nu'(\psi)\ (rf')'\ A_\nu = -w_r\ \mathrm{R_e} \tag{56}$$

where

$$A(r,\psi) = \frac{\mu}{r\nu'(\psi)}\left[rf'' - f' - \frac{(1-(rf')^2)\ \nu''(\psi)}{r(\nu'(\psi))^2}\right], \tag{57}$$

$$B(r,\psi) = \frac{4r\mu}{(r\nu'(\psi))^3}\left[-\nu'(\psi)^2 + rf'\ \nu''(\psi)\right], \tag{58}$$

The above system of equations indicates that its solutions strongly depend on the function $\nu(\psi)$ and its derivatives. Since $\nu'(\psi) \neq 0$, therefore we consider the following cases

Case I (a) $\nu''(\psi) = 0$

Case I(b) $\nu''(\psi) \neq 0$

Case I(a): $\nu'' = 0$ gives

$$\nu(\psi) = a\psi + b \tag{59}$$

where $a \neq 0$ and b are real constants.
Inserting equation (59) in equations (51-58) we get

$$q = \frac{\sqrt{1+(rf')^2}}{ar}, \tag{60}$$

$$\mathrm{R_e}\ a\ L_\nu = a\ r\ A_r - r\ f'\ aA_\nu - aB_\nu - \frac{\mathrm{R_e}(rf'' + f')}{ar} \tag{61}$$

$$\mathrm{Re} L_r = r\ f'\ A_r + \frac{(1-r^2 f'^2)}{r}\ A_\nu - f'\ B_\nu \tag{62}$$

$$a\ r\ T_{rr} - 2\ a\ r\ f'\ T_{\nu r} + \frac{a(1+r^2 f'^2)}{r} T_{\nu\nu} - a\ (rf'' + f')\ T_\nu + (a - R_e P_r)\ T_r = \frac{-a\, r\, \mathrm{E_c}\, \mathrm{P_r}(B^2 + 4A^2)}{4\mu} \tag{63}$$

$$-r\ a\ A_{rr} + 2\ a\ rf'\ A_{\nu r} + \ -a\ A_r + aA_\nu\ (rf')' - af' B_{\nu\nu} + aB_{\nu r} = -w_r\ \mathrm{R_e} \tag{64}$$

$$w = \frac{1}{ar}(rf'' + f') \tag{65}$$

where

$$A = \frac{\mu}{ar}\ (r f'' - f') \tag{66}$$

$$B = \frac{-4\mu}{a\ r^2} \tag{67}$$

It is obvious from equation (64) that it is difficult to obtain exact solutions. However, we see that on eliminating μ from equations (66) and (67) the function A can be eliminated from equation (64). On eliminating μ, we get

$$A = X(r)\, B \tag{68}$$

where

$$X(r) = \left(\frac{-1}{4}\right)(r^2 f'' - r f') \tag{69}$$

provided $(r^2 f'' - r f') \neq 0$.

Inserting equation (68) in equation (64) we get

$$a r X\, B_{rr} - a\,(1+2M X)\, B_{\nu r} + \frac{a\{M - X(1-M^2)\}}{r} B_{\nu\nu} + a B_r \{2rX' + X\} - a\,(2MX' + M'X)\, B_\nu + a B (r X')' = R_e \left(\frac{M'}{a r}\right)' \tag{70}$$

where

$$M(r) = r f'(r) \tag{71}$$

The form of equation (70) suggests to seek a solution of the form

$$B = S(r) + K(r)\nu(\psi) \tag{72}$$

where $S(r)$ and $K(r)$ are to be determined.
Substituting equation (72) in equation (70) we get

$$\nu(\psi)\,[r X K''(r) + (X + 2r X')K'(r) + (X' + r X'')K(r)] + [a r X S''(r) + a(X + 2rX')S'(r) + a(rX')' S(r)] = R_e \left(\frac{M'}{a r}\right)' + a\,(1+2M X)\, K'(r) + a\,(2MX' + M'X)\, K(r) \tag{73}$$

As r and ψ are independent variables therefore equation (73) provides

$$r X K''(r) + (X + 2r X')K'(r) + (X' + r X'')K(r) = 0 \tag{74}$$

and

$$a r X S''(r) + a(X + 2rX')S'(r) + a(rX')' S(r) = Z_1(r) \tag{75}$$

where

$$Z_1(r) = R_e \left(\frac{M'}{a r}\right)' + a\,(1+2M X)\, K'(r) + a\,(2MX' + M'X)\, K(r) \tag{76}$$

Equation (74) can be rewritten as

$$[r(X K)']' = 0 \tag{77}$$

Equation (77) yields

$$K(r) = \frac{k_1 \ln r}{X(r)} + \frac{k_2}{X(r)} \tag{78}$$

where k_1 and k_2 are constants.
Also equation (75) can be rewritten as

$$[r(X S)']' = Z_1(r) \tag{79}$$

Whose solution is

$$S(r) = \frac{1}{X} \int \left[\frac{1}{r} \int Z_1(r) dr \right] dr + \frac{k_3 \ln r + k_4}{X} \tag{80}$$

where k_3 and k_4 are constants.
On substituting equation (78) and equation (80) in equation (72) we get

$$B = \frac{1}{X} \int \left[\frac{1}{r} \int Z_1(r) dr \right] dr + \frac{k_3 \ln r + k_4}{X} + \left(\frac{k_1 \ln r + k_2}{X} \right) \nu(\psi) \tag{81}$$

The expression for viscosity employing equation (66) or equation (67) is

$$\mu(r,\psi) = \frac{-a r^2}{4} \left[\frac{1}{X} \int \left[\frac{1}{r} \int Z_1(r) dr \right] dr + \frac{k_3 \ln r + k_4}{X} + \left(\frac{k_1 \ln r + k_2}{X} \right) \nu(\psi) \right] \tag{82}$$

The expression of function L is determined by finding the solution of equation (61) and (62). The solution, employing equations (81) and (68), is

$$a R_e L = -R_e \left(\frac{M'}{a r} \right) \nu + a r (S X)' \nu - a K (M X + 1) \nu + a r (K X)' \left(\frac{\nu^2}{2} \right)$$
$$+ a \int M (S X)' dr + \int \left[\frac{K}{r} \left\{ a X (1 - M^2) - a M \right\} \right] dr + p_1 \tag{83}$$

The temperature distribution T is determined from equation (64). Equation (64) utilizing equations (68), (81) and (82) becomes

$$a\ r\ T_{rr} - 2\ a^2\ r\ f'\ T_{\nu r} + \frac{a(1 + r^2 f'^2)}{r} T_{\nu\nu} - a\ (r f'' + f')\ T_\nu + (a - R_e P_r)\ T_r = \frac{E_c P_r (1 + 4X^2)}{r} \{ S(r) + K(r) \nu \} \tag{84}$$

Right-hand side of equation (84) suggests to seek a solution of the form

$$T = T_1(r) + T_2(r) \nu(\psi) \tag{85}$$

On using equation (85) into equation (84), we get

$$T_1'' + \frac{(a - R_e P_r)}{a r} T_1' = Z_2(r) \tag{86}$$

$$T_2'' + \frac{(a - R_e P_r)}{a r} T_2' = Z_3(r) \tag{87}$$

where

$$Z_2(r) = \frac{E_c P_r (1 + 4X^2)}{a r^2} S(r) + 2\ a^2\ r\ f'\ T_2' + a\ (r f'' + f')\ T_2 \tag{88}$$

$$Z_3(r) = \frac{E_c P_r (1 + 4X^2)}{a r^2} K(r) \tag{89}$$

It is obvious from equation (86) and (87) that their solutions depend on the value of $(a-R_e P_r)$. The solution of equations (86) and (87) when $(a-R_e P_r)\neq 0$ is

$$T_1(r) = \int r^{\frac{-(a-R_e P_r)}{a}}\left[\int\left\{r^{\frac{(a-R_e P_r)}{a}} Z_2(r)\right\}dr\right]dr + H_3 \int r^{\frac{-(a-R_e P_r)}{a}} dr + H_4 \tag{90}$$

$$T_2(r) = \int r^{\frac{-(a-R_e P_r)}{a}}\left[\int\left\{r^{\frac{(a-R_e P_r)}{a}} Z_3(r)\right\}dr\right]dr + H_1 \int r^{\frac{-(a-R_e P_r)}{a}} dr + H_2 \tag{91}$$

Now when $(a-R_e P_r)=0$ the solution of equations (86) and (89) is

$$T_1(r) = \int\int Z_2(r) dr dr + H_5\, r + H_6 \tag{92}$$

$$T_2(r) = \int\int Z_3(r) dr dr + H_7\, r + H_8 \tag{93}$$

where H_i, $i=1,2,...8$ are constant.
On eliminating μ from (66) and (67), the equation (69) indicates that it is valid only for $(rf''-f')\neq 0$. When $(rf''-f')=0$, the function $A=0$ and we get

$$f(r)=\frac{1}{2}c_1 r^2 + c_2 \tag{94}$$

Equation (64) on substituting $A=0$

$$c_1\, r\, B_{\nu\nu} - B_{r\nu} = 0 \tag{95}$$

The solution of equation (95) is

$$B = \frac{1}{2}\, c_1 b_1 r^2 + b_1 \nu(\psi) + \int I(r) dr + c_3 \tag{96}$$

In above $c_1 \neq 0$, c_2, c_3 and b_1 are constants. The function $I(r)$ is an unknown function. Utilizing equation (96) in equation (67) we get

$$\mu = \frac{-a r^2}{4}\left[c_3 + b_1\nu(\psi) + \frac{1}{2}\, c_1 b_1 r^2 + \int I(r) dr\right] \tag{97}$$

The solution of equations (61) and (62), for $f(r)=\frac{1}{2}c_1 r^2 + c_2$ is

$$\operatorname{Re}\, L = \left(-b_1 - \frac{2c_1 \operatorname{Re}}{a^2}\right)\nu - \frac{1}{2}\, c_1 b_1 r^2 + M_0 \tag{98}$$

where M_0 is a real constant

The energy equation (63) employing equations (96) and (97), becomes

$$a\, r^2\, T_{rr} - 2\, c_1\, a\, r^3\, T_{\nu r} + a r(1+c_1^2 r^4)\, T_{\nu\nu} - 2c_1\, a\, r^2\, T_\nu + r\,(a-R_e P_r)\, T_r = \mathrm{E_c P_r}\left[c_3 + b_1\nu(\psi) + \frac{1}{2}\, c_1 b_1 r^2 + \int I(r) dr\right] \tag{99}$$

Equation (99) suggests to seek a solution of the form

$$T = A_1 \nu^3(\psi) + R_2(r)\nu^2(\psi) + R_1(r)\nu(\psi) + R_0(r) \tag{100}$$

Equation (99), employing equation (100), provides

$$a\, r^2\, R_2'' + r\,(a-R_e P_r)\, R_2' = 6 a c_1 A_1 r^2 \tag{101}$$

$$a\ r^2\ R_1'' + r\ (a - R_e\,P_r)\ R_1' = 4ac_1 r^3 R_2' + 4ac_1 r^2 R_2 - 6aA_1 r(1+c_1^2 r^4) + b_1 E_c\,P_r \tag{102}$$

$$a\ r^2\ R_0'' + r\ (a - R_e\,P_r)\ R_0' = 2ac_1 r^3 R_1' + 2ac_1 r^2 R_1 - 2a\ r(1+c_1^2 r^4)R_2 + E_c\,P_r\left(c_3 + \frac{1}{2}c_1 b_1 r^2 + \int I(r)dr\right) \tag{103}$$

When $(a - R_e P_r) \neq 0$ the equations (101-103) give

$$R_2(r) = \frac{3aA_1c_1r^2}{(2a - R_e\,P_r)} + \frac{a\,n_1}{R_e\,P_r}r^{\frac{R_e P_r}{a}} + n_2\,,\ (2a - R_e\,P_r) \neq 0 \tag{104}$$

$$R_1(r) = -\frac{6aA_1c_1^2r^5}{5(5a - R_e\,P_r)} - \frac{6aA_1r}{(a - R_e\,P_r)} + \frac{9a^2A_1c_1^2r^4}{(4a - R_e\,P_r)(2a - R_e\,P_r)} + \frac{2ac_1r^2n_2}{(2a - R_e\,P_r)} + \frac{a\,n_1}{R_e\,P_r}r^{\frac{R_e P_r}{a}} + n_4 - \frac{b_1E_c\ \ln r}{R_e}$$
$$+\frac{a\,n_1 r^{\frac{R_e P_r}{a}}\left[2a\ (c_1r^2n_1 + n_3) + R_e\,P_r(2c_1r^2n_1 + n_3)\right]}{R_e\,P_r(2a - R_e\,P_r)}, \tag{105}$$

where $(2a - R_e\,P_r) \neq 0$, $(4a - R_e\,P_r) \neq 0$, $(5a - R_e\,P_r) \neq 0$

$$R_0(r) = n_6 + \int M_1(r)dr \tag{106}$$

$$M_1(r) = n_5\,r^{-\left\{\frac{(a-R_e P_r)}{a}\right\}} + r^{-\left\{\frac{(a-R_e P_r)}{a}\right\}}\int\left\{E_c\,P_r(c_3 + \frac{1}{2}c_1b_1r^2 + \int I(r)dr)\right\}r^{\left\{\frac{(a-R_e P_r)}{a}\right\}}dr$$
$$+ r^{-\left\{\frac{(a-R_e P_r)}{a}\right\}}\int\left\{2ac_1r^2R_1' + 2ac_1r^2R_1 - 2ar(1+c_1^2r^4)R_2\right\}r^{\left\{\frac{(a-R_e P_r)}{a}\right\}}dr \tag{107}$$

and n_1, n_2, n_3, n_4, n_5, n_6 are non-zero constants.
When $(a - R_e P_r) = 0$, equations (101–103) yield

$$R_2(r) = 3c_1A_1r^2 + n_7\,r + n_8 \tag{108}$$

$$R_1(r) = \frac{-6A_1c_1^2}{20}r^5 + 3c_1^2A_1r^4 + \frac{4}{3}c_1n_7r^3 + 2c_1n_8r^2 - 6A_1r\ln r + (6A_1 + n_9)r - \frac{b_1E_cP_r}{a}\ln r + n_{10} \tag{109}$$

$$R_0(r) = \iint M_2(r)dr\,dr + n_{11}\,r + n_{12} \tag{110}$$

where

$$M_2(r) = 2c_1\,r\,R_1'(r) + 2c_1\,R_1(r) - \frac{2}{r}(1+c_1^2\,r^4)R_2(r) + \frac{E_cP_r}{a\,r^2}(c_3 + \frac{c_1b_1}{2}r^2 + \int I(r)dr) \tag{111}$$

and n_7, n_8 n_9, n_{10} n_{11}, n_{12} are non-zero constants.
Case I(b):
In this case $\nu'' \neq 0$, the equations (57) and (58) can be rewritten as

$$A = \frac{\mu}{r^2\nu'}\,[\,rM' - 2M - \left(1 - M^2\right)\left(\frac{\nu''}{\nu'^2}\right)] \tag{112}$$

$$B = \frac{4\mu}{r^2\nu'}\left[-1 + M\left(\frac{\nu''}{\nu'^2}\right)\right] \tag{113}$$

where

$$M = rf' \tag{114}$$

Equations (112) and (113) indicate that we can eliminate μ from these equations if we set

$$\frac{\nu''}{\nu'^2}=1 \tag{115}$$

Equation (115) gives

$$\nu=\ln\left[\frac{-1}{(c_4\psi+c_5)}\right] \tag{116}$$

where $c_4\neq 0$ and c_5 are constants.

On eliminating μ from equations (112) and (113), employing (115), we get

$$B=Y(r)\,A \tag{117}$$

where

$$Y(r)=\frac{4(-1+M)}{rM'-2M-(1-M^2)},\quad M\neq 1 \tag{118}$$

Equation (56) employing equations (115) and (117), become

$$rA_{rr}-(2M+Y)\,A_{\nu r}+\left(\frac{MY-(1-M^2)}{r}\right)A_{\nu\nu}+A_r-(M'+Y')\,A_\nu=R_e\left(\frac{e^{-2\nu}}{c_4^2}\right)\left[\frac{M'}{r}+\frac{(1+M^2)}{r^2}\right]' \tag{119}$$

Equation (119) suggests to seek solution of the form

$$A=C(r,\nu)+e^{-2\nu}D(r) \tag{120}$$

where the functions $C(r,\nu)$ and $D(r)$ are to be determined. Substituting equation (120) in equation (119) we get

$$\begin{aligned} &r\,C_{rr}-(2M+Y)\,C_{r\nu}+\left(\frac{MY-(1-M^2)}{r}\right)C_{\nu\nu}+C_r-(M'+Y')\,C_\nu\\ &\quad+e^{-2\nu}\left[r\,D''+2(2M+Y)\,D'+4\left(\frac{MY-(1-M^2)}{r}\right)D+D'+2(M'+Y')\,D\right]\\ &\quad\quad=R_e\left(\frac{e^{-2\nu}}{c_4^2}\right)\left[\frac{M'}{r}+\frac{(1+M^2)}{r^2}\right]' \end{aligned} \tag{121}$$

Which on equating the coefficients of $e^{-2\nu}$ gives

$$r\,C_{rr}-(2M+Y)\,C_{r\nu}+\left(\frac{MY-(1-M^2)}{r}\right)C_{\nu\nu}+C_r-(M'+Y')\,C_\nu=0 \tag{122}$$

$$r^2\,D''+r\,(4M+2Y+1)\,D'+\left[4MY-4(1-M^2)+2r(M'+Y')\right]D=\left(\frac{R_e}{c_4^2}\right)r\left[\frac{M'}{r}+\frac{(1+M^2)}{r^2}\right]' \tag{123}$$

The equation (123) can be reduced to Cauchy equation if we set

$$(4M+2Y+1)=m_1 \tag{124}$$

and

$$4MY-4(1-M^2)+2r(M'+Y')=m_2 \tag{125}$$

The solution of the system of equations (119), (124) and (125) is

$$M=-1,\; m_1=-11,\; m_2=16 \tag{126}$$

Equation (123), utilizing equation (126), becomes

$$r^2\ D''-11\,r\ \ D'+16\,D=\left(\frac{R_e}{c_4^2}\right)\left[\frac{-4}{r^2}\right] \tag{127}$$

The solution of equation (127) is

$$D(r)=D_1\,r^{(6+2\sqrt{5})}+D_2\,r^{(6-2\sqrt{5})}+\left(\frac{-R_e}{11c_4^2}\right)\left(\frac{1}{r^2}\right) \tag{128}$$

where D_1 and D_2 are constants.
Equation (122), utilizing equation (126) becomes

$$r^2\ C_{rr}+6\,r\ \ C_{r\nu}+4\,C_{\nu\nu}+\ r\ \ C_r=0 \tag{129}$$

Equation (129) indicates to seek a solution of the form

$$C=C_1(r)+S_1(\nu)+C_2(r)S_2(\nu) \tag{130}$$

Substituting equation (130) in equation (129)

$$\{\,r\ \ (rC_1')'\,\}+4\,S_1''+\{4\,C_2\ \ S_2''+6\,r\ \ C_2'\ S_2'+r\ \ (rC_2')'\ S_2\}=0 \tag{131}$$

Differentiating equation (131) with respect to “ r ” we get

$$\left[r\,(rC_1')'\right]'\ +4\,C_2'\ S_2''+6\,(r\,C_2')'\ S_2'+\left[r\,(rC_2')'\right]'\ S_2=0 \tag{132}$$

Differentiating equation (132) with respect to “ ν ” we obtain

$$4\,C_2'\ S_2'''+6\,J\ \ S_2''+[r\,J]'\ S_2'=0 \tag{133}$$

where

$$J(r)=(r\,C_2'(r))' \tag{134}$$

Equation (133) can be written as

$$4\,C_2'\left(\frac{Z''}{Z}\right)+6\,J\ \left(\frac{Z'}{Z}\right)+[r\,J]'=0 \tag{135}$$

where

$$Z(\nu)=S_2'(\nu) \tag{136}$$

Differentiating equation (134) with respect to “ ν ” and arranging the terms we obtain

$$\frac{\left(\frac{Z''}{Z}\right)'}{\left(\frac{Z'}{Z}\right)'}=-\frac{3}{2}\ \frac{J(r)}{C_2'(r)}=\ d_1 \tag{137}$$

where d_1 is a non-zero arbitrary constant. Equation (138) provides

$$J(r)=\ -\frac{2}{3}\ d_1\ C_2'(r) \tag{138}$$

$$Z'' = d_1 Z' + d_2 Z \tag{139}$$

where d_2 is a constant

Equation (138) on utilizing equation (134) and integrating once gives

$$r C_2'(r) + \frac{2}{3} d_1 C_2(r) = d_3 \tag{140}$$

where d_3 is a constant. The solution of equation (140) is

$$C_2(r) = \frac{3d_3}{2d_1} + d_4 r^{-2d_1/3} \tag{141}$$

where d_4 is constant.

On substituting equation (136) in equation (139) and integrating once, we obtain

$$S_2''(\nu) - d_1 S_2'(\nu) - d_2 S_2(\nu) = d_5 \tag{142}$$

where d_5 is a constant. Now inserting equation (141) and equation (142) in equation (134), we obtain

$$C_2' \{ S_2''' - d_1 S_2'' + \frac{d_1^2}{9} S_2' \} = 0 \tag{143}$$

As $C_2'(r) \neq 0$, equation (143) gives

$$S_2''' - d_1 S_2'' + \frac{d_1^2}{9} S_2' = 0 \tag{144}$$

which can be rewritten as

$$\left(S_2'' - d_1 S_2' \right)' + \frac{d_1^2}{9} S_2' = 0 \tag{145}$$

Employing equation (141) in equation (145), we find

$$d_2 = -\frac{d_1^2}{9} \tag{146}$$

Inserting equations (145) and (146) in equation (132), we get

$$r \left(r C_1' \right)' = -4 C_2 d_5 + d_6 \tag{147}$$

whose solution is

$$C_1(r) = -4 d_5 \int \left[\frac{1}{r} \int \frac{C_2(r)}{r} dr \right] dr + d_6 \int \frac{\ln r}{r} dr + d_7 \ln r + d_8 \tag{148}$$

where d_5 , d_6 , d_7 , and d_8 are constants.

Now the equations (142), (143) and (146) must satisfy equation (131), and therefore we get

$$S_1 = -\frac{d_3}{6} \int \left\{ \int [9 S_2'(\nu) - d_1 S_2(\nu)] d\nu \right\} d\nu - \frac{d_6}{8} \nu^2 + d_9 \nu + d_{11} \tag{149}$$

The solution of equation (145) is

$$S_2(\nu) = d_{12}\,Exp[\frac{(3+\sqrt{5})}{6}d_1\nu] + d_{13}\,Exp[\frac{(3-\sqrt{5})}{6}d_1\nu] + \frac{9d_{11}}{d_1^2} \tag{150}$$

which is the solution of equation (143). In above equations $d_9, .., d_{11}$, d_{12} and d_{13} are constants of integration.
On substituting the values $C_1(r)$, $C_2(r)$, $S_1(\nu)$, $S_2(\nu)$ and $D(r)$ in equation (120) we get

$$\begin{aligned} A = & -\frac{3d_3d_5}{d_1}(\ln r)^2 - \frac{9d_4d_5}{d_1^2}r^{-2d_1/3} + d_6\frac{(\ln r)^2}{2} + d_7\ln r + d_8 - \frac{d_3}{6}\int\{\int[9S_2'(\nu) - d_1S_2(\nu)]d\nu\}d\nu - \frac{d_6}{8}\nu^2 + d_9\nu + d_{10} \\ & + \{\frac{3d_3}{2d_1} + d_4 r^{-2d_1/3}\}\{d_{12}\,Exp[\frac{(3+\sqrt{5})}{6}d_1\nu] + d_{13}\,Exp[\frac{(3-\sqrt{5})}{6}d_1\nu] + \frac{9d_{11}}{d_1^2}\} \\ & + e^{-2\nu}\{D_1 r^{(6+2\sqrt{5})} + D_2 r^{(6-2\sqrt{5})} + \left(\frac{-R_e}{11c_4^2}\right)\left(\frac{1}{r^2}\right)\} \end{aligned} \tag{151}$$

The viscosity distribution from equation (57) or (58) is

$$\begin{aligned} \mu = & \left(\frac{-r^2c_4e^{\nu}}{2}\right)[-\frac{3d_3d_5}{d_1}(\ln r)^2 - \frac{9d_4d_5}{d_1^2}r^{-2d_1/3} + d_6\frac{(\ln r)^2}{2} + d_7\ln r + d_8 - \frac{d_3}{6}\int\{\int[9S_2'(\nu) - d_1S_2(\nu)]d\nu\}d\nu - \frac{d_6}{8}\nu^2 + d_9\nu + d_{10} \\ & + \{\frac{3d_3}{2d_1} + d_4 r^{-2d_1/3}\}\{d_{12}\,Exp[\frac{(3+\sqrt{5})}{6}d_1\nu] + d_{13}\,Exp[\frac{(3-\sqrt{5})}{6}d_1\nu] + \frac{9d_{11}}{d_1^2}\} \\ & + e^{-2\nu}\{D_1 r^{(6+2\sqrt{5})} + D_2 r^{(6-2\sqrt{5})} + \left(\frac{-R_e}{11c_4^2}\right)\left(\frac{1}{r^2}\right)\}] \end{aligned} \tag{152}$$

The solution of equations (53-54), utilizing equations (112-113), (116),(126) and (152) is

$$R_e L = -[C_1(r) + S_1(\nu) + C_2(r)\,S_2(\nu) + e^{-2\nu}D(r)] - 4S_1'\ln r - 4S_2'\int\frac{C_2}{r}dr + 8e^{-2\nu}\int\frac{D}{r}dr + (d_7 + 6d_9)\,\nu - \frac{3d_6}{4}\nu^2 + p_4 \tag{153}$$

provided

$$d_3 = 0\,, \quad d_5 = d_{11} \tag{154}$$

Now for this case the equation for T is

$$\begin{aligned} & r^2T_{rr} + 2r\,T_{\nu r} + \frac{2}{r}T_{\nu\nu}(\nu') + r\left(1 - \frac{R_eP_r}{c_4}e^{-\nu}\right)T_r \\ & = \frac{10E_cP_r}{c_4}[e^{-\nu}\{d_6\frac{(\ln r)^2}{2} + d_7\ln r + d_8\} + e^{-\nu}\{-\frac{d_6}{8}\nu^2 + d_9\nu + d_{10}\} \\ & + e^{-\nu}\{d_4 r^{-2d_1/3}\}\{d_{12}\,Exp[\frac{(3+\sqrt{5})}{6}d_1\nu] + d_{13}\,Exp[\frac{(3-\sqrt{5})}{6}d_1\nu]\} \\ & + e^{-3\nu}\left(\frac{-R_e}{11c_4^2}\right)\left(\frac{1}{r^2}\right) + e^{-3\nu}\{D_1 r^{(6+2\sqrt{5})} + D_2 r^{(6-2\sqrt{5})}\} \end{aligned} \tag{155}$$

It is obvious from the equation (155) that it is extremely difficult to determine the exact solution of equation (155). However, we obtained by setting

$$T = e^{-\nu}T_1(r) + S_3(\nu) + r^bS_4(\nu) \tag{156}$$

in the temperature equation (155). Inserting equation (156) in equation (155) and arranging the terms we get

$$\begin{aligned} & e^{-\nu}\left[r^2T_1''(r) - rT_1'(r) + 2T_1(r)\right] + r^b\left[b(b-1)S_4(\nu) + 2bS_4'(\nu) + 2S_4''(\nu) + br^b\left(1 - \frac{R_eP_r}{c_4}e^{-\nu}\right)S_4(\nu)\right] + 2S_3''(\nu) - \frac{R_eP_r}{c_4}e^{-2\nu}rT_1'(r) \\ & = \frac{10E_cP_r}{c_4}[e^{-\nu}\{d_6\frac{(\ln r)^2}{2} + d_7\ln r + d_8\} + e^{-\nu}\{-\frac{d_6}{8}\nu^2 + d_9\nu + d_{10}\} \\ & + e^{-\nu}\{d_4 r^{-2d_1/3}\}\{d_{12}\,Exp[\frac{(3+\sqrt{5})}{6}d_1\nu] + d_{13}\,Exp[\frac{(3-\sqrt{5})}{6}d_1\nu]\} \\ & + e^{-3\nu}\left(\frac{-R_e}{11c_4^2}\right)\left(\frac{1}{r^2}\right) + e^{-3\nu}\{D_1 r^{(6+2\sqrt{5})} + D_2 r^{(6-2\sqrt{5})}\} \end{aligned} \tag{157}$$

Comparing terms of right-hand side and left-hand side of equation (157) we get

$$T_1(r) = \frac{5 d_8 E_c P_r}{c_4} \tag{158}$$

$$S_3''(\nu) = \frac{5 E_c P_r}{c_4} (d_9 \nu + d_{10}) e^{-\nu} \tag{159}$$

and

$$S_4''(\nu) - 2S_4'(\nu) + \left(2 + \frac{R_e P_r}{c_4} e^{-\nu} \right) S_4(\nu) = \frac{5 E_c P_r}{c_4} d_4 \{ d_{12} e^{\frac{(1+\sqrt{5})}{2}\nu} + d_{13} e^{\frac{(1-\sqrt{5})}{2}\nu} \} + e^{-3\nu} \left(\frac{-R_e}{11 c_4^2} \right) \tag{160}$$

provided

$$D_1 = D_2 = d_6 = d_7 = 0 , d_1 = 3 , b = -2 , \tag{161}$$

Solution of equation (159) and (160) is

$$S_3(\nu) = \frac{5 E_c P_r}{c_4} \iint (d_9 \nu + d_{10}) e^{-\nu} d\nu + s_1 \nu + s_2 \tag{162}$$

$$\begin{aligned} S_4(\nu) = {} & \frac{C_1}{A1} e^{\nu} \text{Gamma}[1-2i] \text{BesselJ}[-2i, 2\sqrt{A1 e^{-\nu}}] + \frac{C_2}{A1} e^{\nu} \text{Gamma}[1+2i] \text{BesselJ}[2i, 2\sqrt{A1 e^{-\nu}}] \\ & + \frac{i}{2} e^{\nu} \text{Gamma}[1-2i] \text{Gamma}[1+2i] \\ & \Big\{ \text{BesselJ}[2i, 2\sqrt{A1 e^{-\nu}}] \int e^{-4\nu} \left\{ A4 + A3 e^{(7-\sqrt{5})\nu/2} + A2 e^{(7+\sqrt{5})\nu/2} \right\} \text{BesselJ}[-2i, 2\sqrt{A1 e^{-\nu}}] d\nu \\ & - \text{BesselJ}[-2i, 2\sqrt{A1 e^{-\nu}}] \int e^{-4\nu} \left\{ A4 + A3 e^{(7-\sqrt{5})\nu/2} + A2 e^{(7+\sqrt{5})\nu/2} \right\} \text{BesselJ}[2i, 2\sqrt{A1 e^{-\nu}}] d\nu \Big\} \end{aligned} \tag{163}$$

where

$$A1 = \frac{R_e P_r}{c_4} , \quad A2 = \frac{5 E_c P_r d_4 d_{12}}{c_4} , \quad A3 = \frac{5 E_c P_r d_4 d_{13}}{c_4} , \quad A4 = -\frac{R_e}{11 c_4^2} \tag{164}$$

The solution (163) of equation (160) is obtained using Mathematica.
We know that equation (118) does not hold for $M = 1$. To determine the solution we have to use equation (112) and (113) which on substituting $M = 1$, give

$$B = 0 \tag{165}$$

$$A = \frac{-2\mu}{r^2 \nu'} \tag{166}$$

Equation (56), utilizing equation (165) and $M = 1$ becomes

$$r^2 A_{rr} - 2 r A_{\nu r} + r A_r = R_e \left(\frac{e^{-2\nu}}{c_4^2} \right) \left(\frac{-4}{r^2} \right) \tag{167}$$

Following the previous case $M \neq 1$, we seek a solution of the form

$$A = N(r,\nu) + e^{-2\nu} D_1(r) \tag{168}$$

Inserting Equation (168) in equation (167) gives

$$\{ r^2 N_{rr} - 2 r N_{\nu r} + r N_r \} + e^{-2\nu} \{ r^2 D_1'' + 5r D_1' \} = R_e \left(\frac{e^{-2\nu}}{c_4^2} \right) \left(\frac{-4}{r^2} \right) \tag{169}$$

Which on comparing left hand side and right hand side give

$$r^2 D_1'' + 5r D_1' = \left(\frac{R_e}{c_4^2}\right)\left(\frac{-4}{r^2}\right) \tag{170}$$

$$r^2 N_{rr} - 2\ r\ N_{\nu r} + r\ N_r = 0 \tag{171}$$

The solution of equation of (171) is

$$D_1(r) = \frac{R_e}{c_4{}^2}\left(\frac{1}{r^2}\right) - \frac{4d_{14}}{r^4} + d_{15} \tag{172}$$

Left- hand side of equation (172) suggest to seek a solution of the form

$$N = C_3(r) + S_5(\nu) + C_4(r) S_6(\nu) \tag{173}$$

which on substituting in equation (171) gives

$$\{\ r^2 C_3'' + rC_3'\ \} + S_6\ \{\ r^2 C_4'' + rC_4'\ \} - 2rC_4' S_6' = 0 \tag{174}$$

Differentiating equation (175) with respect to ν and rearranging the terms of r and ν we get

$$rC_4''(r) + (1 - d_{16}) C_4'(r) = 0 \tag{175}$$

$$S_6''(\nu) - \frac{d_{16}}{2} S_6'(\nu) = 0 \tag{176}$$

where d_{14} is a separation constant. Solution of equations (175) and (176) are

$$C_4(r) = \frac{d_{17}}{d_{16}} r^{d_{16}} + d_{18} \tag{177}$$

$$S_6(\nu) = \frac{-2d_{19}}{d_{16}} + d_{20}\, e^{d_{16}\nu/2} \tag{178}$$

To obtain the solution of equation (175) we insert equation (176) and (177) in equation (174) and we get

$$r^2 C_3'' + rC_3' = 2d_{19}\ d_{17}\, r^{d_{16}} \tag{179}$$

Solution of equation (179) is

$$C_3(r) = \frac{2d_{17} d_{19}}{d_{16}^2} r^{d_{16}} + d_{21} \ln r + d_{22} \tag{180}$$

Substituting equation (177-178) and equation (180) in equation (168), we get

$$A = d_{22} - \frac{2d_{18} d_{19}}{d_{16}} + d_{21} \ln r + S_5(\nu) + d_{18} d_{20}\, e^{d_{16}\nu/2} + \frac{2d_{17} d_{19}}{d_{16}^2} r^{d_{16}} + \frac{d_{17}}{d_{16}} r^{d_{16}} \left[d_{20} e^{d_{16}\nu/2} - \frac{2d_{19}}{d_{16}} \right] + e^{-2\nu}\ \{\ \frac{R_e}{c_4{}^2}\left(\frac{1}{r^2}\right) - \frac{4d_{14}}{r^4} + d_{15}\ \} \tag{181}$$

The viscosity is obtained from equation (167) by substituting equation (114) and equation (181) which is

$$\mu(r,\nu) = \left(\frac{-r^2 e^{-\nu}}{2c_4}\right) [\ d_{22} - \frac{2d_{18} d_{19}}{d_{16}} + d_{21} \ln r + S_5(\nu) + d_{18} d_{20}\, e^{d_{16}\nu/2} + \frac{2d_{17} d_{19}}{d_{16}^2} r^{d_{16}} + \frac{d_{17}}{d_{16}} r^{d_{16}} \left[d_{20} e^{d_{16}\nu/2} - \frac{2d_{19}}{d_{16}} \right]$$
$$+ e^{-2\nu}\ \{\ \frac{R_e}{c_4{}^2}\left(\frac{1}{r^2}\right) - \frac{4d_{14}}{r^4} + d_{15}\ \}] \tag{182}$$

The solution of equations (61) and (62), utilizing equations (165) and equation (181), is

$$R_e\, L = R_e \left(\frac{1}{r^2 {c_4}^2} e^{-2\nu} \right) + r\,[C_3'(r)\;\nu + C_4'(r)\int S_6(\nu) d\nu\;\; + \left(\frac{e^{-2\nu}}{-2} \right)\; D_1'(r)\,] - A\; + 2d_{21} \ln r + 2d_{22}\;\; - \frac{4d_{18} d_{19}}{d_{16}} + p_3 \tag{183}$$

where p_3 is constant.

The equation (64) of temperature distribution, on substituting equations (165) and equation (181), becomes

$$\begin{aligned} & r^2 T_{rr}\;\; -2r\; T_{\nu r} + \frac{2}{r} T_{\nu\nu}\,(\nu') + r\left(1 - \frac{R_e\, P_r}{c_4} e^{-\nu}\right) T_r \\ &= \frac{2E_c\, P_r d_{21}}{c_4} \ln r\, e^{-\nu}\; + \frac{2E_c\, P_r d_{22}}{c_4} e^{-\nu} + \frac{2E_c\, P_r}{c_4}\; S_5(\nu)\; e^{-\nu}\;\; - \frac{4E_c\, P_r d_{19} d_{18}}{c_4 d_{16}} e^{-\nu}\;\; + \frac{2E_c\, P_r d_{18} d_{20}}{c_4} e^{\left(\frac{d_{16}}{2}-1\right)\nu}\;\; + \frac{2E_c\, P_r d_{17} d_{20}}{c_4 d_{16}} r^{d_{16}} e^{\left(\frac{d_{16}}{2}-1\right)\nu} \\ &\quad + \frac{2E_c\, P_r}{c_4}\; e^{-3\nu}\; \left\{ \frac{R_e}{{c_4}^2} \left(\frac{1}{r^2} \right) - \frac{4d_{14}}{r^4} + d_{15} \right\} \end{aligned} \tag{184}$$

It is obvious from equation (184) the general solution is extremely difficult to obtain, however the terms in equation (184) suggests to seek a solution of the form

$$T\; = e^{-\nu} T_2(r) + S_7(\nu) + r^b S_8(\nu) \tag{185}$$

On substituting equation (185) in equation (184) we get

$$\begin{aligned} & e^{-\nu}\left[r^2 T_2''(r) + 3r T_2'(r) + 2T_2(r) \right]\;\; + r^b \left[2S_8''(\nu) - 2bS_8'(\nu) + b^2 S_8(\nu) - \frac{R_e\, P_r}{c_4} e^{-\nu} b S_8(\nu) \right]\; + 2S_7''(\nu)\;\; - \frac{R_e\, P_r}{c_4} e^{-2\nu} r T_2'(r) \\ &= e^{-\nu} \left[\frac{2E_c\, P_r d_{21}}{c_4} \ln r \right]\; + e^{-\nu} \left[\frac{2E_c\, P_r}{c_4} \left(d_{22} - \frac{2d_{19} d_{18}}{d_{16}} \right) + \frac{2E_c\, P_r}{c_4} S_5(\nu) + \frac{2E_c\, P_r d_{18} d_{20}}{c_4} e^{\left(\frac{d_{16}}{2}-1\right)\nu} \right] \\ &\quad + \frac{2E_c\, P_r d_{17} d_{20}}{c_4 d_{16}} r^{d_{16}} e^{\left(\frac{d_{16}}{2}-1\right)\nu} + \frac{2E_c\, P_r}{c_4} e^{-3\nu} \left[\frac{R_e}{{c_4}^2} \left(\frac{1}{r^2} \right) \right]\qquad + \frac{2E_c\, P_r}{c_4} e^{-3\nu} \left[-\frac{4d_{14}}{r^4} + d_{15} \right] \end{aligned} \tag{186}$$

On comparing the right-hand side and left-hand side, we get

$$T_2(r) = \frac{E_c d_{17} d_{20}}{R_e} \left(\frac{r^{-2}}{-2} \right) \tag{187}$$

$$2S_7''(\nu) = \frac{2E_c\, P_r d_{22}}{c_4} e^{-\nu} + \frac{2E_c\, P_r}{c_4}\;\; e^{-\nu}\;\; S_5(\nu)\;\; - \frac{4E_c\, P_r d_{19} d_{18}}{c_4 d_{16}} e^{-\nu}\;\;\; + \frac{2E_c\, P_r d_{18} d_{20}}{c_4} e^{\left(\frac{d_{16}}{2}-1\right)\nu} \tag{188}$$

and

$$S_8''(\nu) + 2S_8'(\nu) + \left(2 + \frac{R_e\, P_r}{c_4} e^{-\nu} \right) S_8(\nu) = \;\frac{E_c d_{17} d_{20}}{2R_e} e^{-\nu} + \frac{E_c\, R_e\, P_r}{{c_4}^3} e^{-3\nu} \tag{189}$$

provided

$$d_{14} = 0\,, \qquad d_{15} = 0 \qquad d_{16} = -2 \qquad d_{21} = 0\,, \qquad b\; = -2 \tag{190}$$

Solution of equation (188) is

$$S_7(\nu) = \left[\frac{E_c\, P_r d_{22}}{c_4} - \frac{2E_c\, P_r d_{19} d_{18}}{c_4 d_{16}} \right] e^{-\nu} + \frac{E_c\, P_r}{c_4}\;\; \iint e^{-\nu} S_5(\nu) d\nu\;\;\; + \frac{E_c\, P_r d_{18} d_{20}}{2c_4} e^{-2\nu} + s_3\, \nu + s_4 \tag{191}$$

The solution of equation (189) is

$$S_8(\nu) = C_3 A5\, e^{-\nu}\,\text{BesselJ}[-2i,\, 2\sqrt{A5e^{-\nu}}\,]\,\text{Gamma}[1-2i]$$
$$+\, C_4 A5e^{-\nu}\,\text{BesselJ}[\,2i,\, 2\sqrt{A5e^{-\nu}}\,]\,\text{Gamma}[1+2i]$$
$$+\frac{i}{2}e^{-\nu}\,\text{Gamma}[1-2i]\text{Gamma}[1+2i]$$
$$\Big\{\text{BesselJ}[2i,\, 2\sqrt{A5e^{-\nu}}\,]\int e^{-\nu}(A7+A6e^{2\nu})\,\text{BesselJ}[-2i,\, 2\sqrt{A5e^{-\nu}}\,]d\nu$$
$$-\text{BesselJ}[-2i,\, 2\sqrt{A5e^{-\nu}}\,]\int e^{-\nu}(A7+A6e^{2\nu})\,\text{BesselJ}[2i,\, 2\sqrt{A5e^{-\nu}}\,]d\nu\Big\} \quad (192)$$

where C_3 and C_4 are constant and

$$A5 = \frac{R_e P_r}{c_4}, \qquad A6 = \frac{E_c\, d_{17}\, d_{20}}{2R_e}, \qquad A7 = \frac{E_c P_r R_e}{c_4^{\,3}} \quad (193)$$

Case II:
For this case $g(r) \neq 1$ and basic flow equations are

$$q = \frac{\sqrt{1+(M+N\nu)^2}}{r\,g\,\nu'} \quad (194)$$

$$w = \left[\frac{f'(r)}{r\,g(r)} + \frac{f''(r)}{g(r)} - \frac{2f'(r)g'(r)}{g^2(r)}\right]\left(\frac{1}{\nu'(\psi)}\right) + \left[\frac{g'(r)}{r\,g(r)} + \frac{g''(r)}{g(r)} - \frac{2\{g'(r)\}^2}{g^2(r)}\right]\left(\frac{\nu(\psi)}{\nu'(\psi)}\right)$$
$$+\left[\frac{1}{r^2 g^2(r)} + \frac{\{f'(r)\}^2}{g^2(r)}\right]\left(\frac{\nu''(\psi)}{\{\nu'(\psi)\}^3}\right) + \frac{2f'(r)g'(r)\nu(\psi)\nu''(\psi)}{g^2(r)\{\nu'(\psi)\}^3} + \frac{\{g'(r)\}^2\nu^2(\psi)\,\nu''(\psi)}{g^2(r)\{\nu'(\psi)\}^3} \quad (195)$$

$$-R_e w = R_e\,\nu' L_\nu - (rg\nu')A_r + (M+N\nu)\nu' A_\nu + \nu' B_\nu \quad (196)$$

$$0 = -R_e L_r + \frac{A_\nu(2-E)}{(rg)} + A_r(M+N\nu) - \frac{(M+N\nu)B_\nu}{(rg)} \quad (197)$$

$$r\,g\,\nu' A_{rr} - 2(M+N\nu)\nu' A_{r\nu} - \frac{\left[1-(M+N\nu)^2\right]}{r\,g\,\nu'}(\nu'' A_\nu + \nu'^2 A_{\nu\nu})$$
$$+\, g\,\nu' A_r - \nu' A_\nu(M'+N'\nu) - \nu'\left\{B_r - \frac{(f'+g'\nu)B_\nu}{g}\right\}_\nu = R_e w_r \quad (198)$$

$$(rg\nu')T_{rr} - 2(M+N\nu)\,T_{\nu r}\,\nu' + \frac{\left[1+(M+N\nu)^2\right]}{(rg\nu')}T_{\nu\nu}(\nu')^2 + \left((rg\nu')_r - \frac{\left[1+(M+N\nu)^2\right]_\psi}{2(M+N\nu)} - R_e P_r\right)T_r$$
$$+\left[\frac{\left[2N(M+N\nu)\right]}{(rg)} - (M'+N'\nu) - \frac{\left[1+(M+N\nu)^2\right](rg\nu'')}{(rg\nu')^2} + \frac{\left[1+(M+N\nu)^2\right]}{(rg\nu')}\left(\frac{\nu''}{\nu'}\right)\right]T_\nu\nu'$$
$$= -\frac{(rg\nu')E_c P_r}{4\mu}\left(B^2 + 4A^2\right) \quad (199)$$

where

$$A(r,\psi) = \frac{\mu}{(rg\nu')}\left[\frac{-2(rg)_r(M+N\nu)}{(rg)} + (M'+N'\nu) - \frac{\left\{(1-(M+N\nu)^2\right\}\nu''}{rg\,\nu'^2}\right] \quad (200)$$

$$B(r,\psi) = \frac{4\mu}{(rg\nu')^3}\left[-\nu'^2(rg)\,(rg)_r + (M+N\nu)\,(rg\nu'')\right] \quad (201)$$

$$M(r) = r\,f'(r) \quad \text{and} \quad N(r) = r\,g'(r) \quad (202)$$

The above system of equations indicates that its solutions strongly depend on the function $\nu(\psi)$ and its derivatives. Since $\nu'(\psi) \neq 0$, following the Case I, we consider the following cases

Case II(a) $\nu''(\psi) = 0$ or $\nu'(\psi) = a$

Case II(b) $\nu''(\psi) \neq 0$

Case II(a):
For this case the equations(197-204) becomes

$$q = \frac{\sqrt{1+(M+N\nu)^2}}{a\,rg} \tag{203}$$

$$w = \left[\frac{f'(r)}{r\,g(r)} + \frac{f''(r)}{g(r)} - \frac{2f'(r)g'(r)}{g^2(r)}\right]\left(\frac{1}{a}\right) + \left[\frac{g'(r)}{r\,g(r)} + \frac{g''(r)}{g(r)} - \frac{2\{g'(r)\}^2}{g^2(r)}\right]\left(\frac{\nu(\psi)}{a}\right) \tag{204}$$

$$-R_e w = aR_e L_\nu - (a\,rg)A_r + a(M+N\nu)A_\nu + a B_\nu \tag{205}$$

$$0 = -R_e L_r + \frac{A_\nu(2-E)}{(rg)} + A_r(M+N\nu) - \frac{(M+N\nu)B_\nu}{(rg)} \tag{206}$$

$$a\,rg\,A_{rr} - 2a(M+N\nu)A_{r\nu} - \frac{a\left[1-(M+N\nu)^2\right]}{r\,g}A_{\nu\nu} + agA_r - a(M'+N'\nu)A_\nu - a\left\{B_r - \frac{(f'+g'\nu)B_\nu}{g}\right\}_\nu = R_e w_r \tag{207}$$

$$(a\,rg)T_{rr} - 2a(M+N\nu)T_{\nu r} + \frac{a\left[1+(M+N\nu)^2\right]}{r\,g}T_{\nu\nu} + \left((a\,rg)_r - \frac{a\left[1+(M+N\nu)^2\right]_\nu}{2(M+N\nu)} - R_e P_r\right)T_r$$
$$+ a\left[\frac{\left[2N(M+N\nu)\right]}{(rg)} - (M'+N'\nu)\right]\;]\,T_\nu = -\frac{(a\,rg)E_c P_r}{4\mu}\left(B^2+4A^2\right) \tag{208}$$

$$A(r,\psi) = \frac{\mu}{(a\,rg)}\left[\left\{\frac{-2(rg)_r M}{(rg)} + M'\right\} - \nu\left\{\frac{2(rg)_r N}{(rg)} - N'\right\}\right] \tag{209}$$

$$B(r,\psi) = \frac{-4a\mu}{(a\,rg)^2}(rg)_r \tag{210}$$

It is obvious from equation (207), it is extremely difficult to obtain its exact solutions. However, we see that by setting $A=0$ or $B=0$ we can reduce the equation (207) to simple whose solutions are determinable.

For

$$A = 0 \tag{211}$$

On substituting equations (202) and (209) in equation (211) and comparing the coefficient of ν, we get

$$\frac{2(rg)_r g'}{g} - (r\,g')' = 0 \tag{212}$$

$$\frac{-2(rg)_r f'}{g} + (rf')' = 0 \tag{213}$$

The solution of equation (212) is

$$g(r) = \frac{-1}{(C_0 r^2 + C_1)} \tag{214}$$

Utilizing equation (215) in equation (213) we get

$$r(C_0 r^2 + C_1)f'' + (3C_0 r^2 - C_1)f' = 0 \tag{215}$$

The equation (215) possesses trivial and non-trivial solutions. For trivial solution

$$f(r) = 0 \tag{216}$$

The equation (209), becomes

$$B_{r\nu} - \left(\frac{g'}{g}\right)\nu \; B_{\nu\nu} - \left(\frac{g'}{g}\right) B_{\nu} = \frac{-4R_e C_0 \nu}{a^2} g' \tag{217}$$

This suggests to seek a solution of the form

$$B = \nu^2 Q(r) \tag{218}$$

On substituting equation (218) in equation (217) we get

$$Q' - 2\left(\frac{g'}{g}\right) Q = \frac{-2R_e C_0}{a^2} g' \tag{219}$$

whose solution is

$$Q(r) = \frac{2R_e C_0}{a^2} g + C_2 g^2 \tag{220}$$

On substituting equation (220) in equation (218) we get

$$B = \nu^2 \left\{ \frac{2R_e C_0}{a^2} g + C_2 g^2 \right\} \tag{221}$$

Equation (204) gives the value of μ

$$\mu = \frac{-a(r\,g)^2}{4\,(rg)'} \left\{ \frac{2R_e C_0}{a^2} g + C_2 g^2 \right\} \nu^2 \tag{222}$$

Solution of equations (197) and (198) utilizing equation (212) and (213) is

$$a R_e L = \left\{ -\frac{R_e}{a}\left[\frac{N'}{r\,g} - \frac{2N^2}{(rg)^2}\right] - 2a\; Q(r) \right\} \left(\frac{\nu^2}{2}\right) + p_6 \tag{223}$$

The energy equation for this is

$$(a\,rg)T_{rr} - 2a(rg'\nu)\; T_{\nu r} + \frac{a\left[1+(rg'\nu)^2\right]}{r\,g} T_{\nu\nu} + \left((a\,rg)_r - \frac{a\left[1+(rg'\nu)^2\right]_\nu}{2(rg'\nu)} - R_e P_r\right) T_r$$
$$+ a\left[\frac{\left[2rg'(rg'\nu)\right]}{(rg)} - (rg')'\nu\right] T_\nu = E_c P_r a^2 (rg)(rg'+g)\nu^2 Q(r) \tag{224}$$

We seek a solution of equation (224) of the form

$$T(r,\nu) = \nu^2 R(r) + S(r) \tag{225}$$

Inserting equation (225) in (224) we find

$$a r^2 g^2 R'' + rg\left\{ag - R_e P_r - 4a\, r\, g'\right\} R' + 2a\left\{3(rg')^2 - rg\,(rg')'\right\} R = E_c P_r\, (rg'+g) \left\{ \frac{2R_e C_0}{a^2} g + C_2 g^2 \right\} \tag{226}$$

$$a\, r^2 g^2 S'' + rg\left\{ag - R_e P_r\right\} S' = -2a R(r) \tag{227}$$

On substituting the value of $g(r)$ from equation (214) in equation (226) we get

$$a r^2 (C_0 r^2 + C_1)^2 R'' + r (C_0 r^2 + C_1)\left\{a(9C_0 r^2 + C_1) + R_e P_r (C_0 r^2 + C_1)^2\right\} R' + 8a C_0 r^2 (2C_0 r^2 + C_1) R$$
$$= -E_c P_r (C_0 r^2 + C_1) \left\{ \frac{2 R_e C_0^2}{a^2} r^2 + \frac{2 R_e C_0 C_1}{a^2} - C_2 \right\} \quad (228)$$

where C_0, C_1 and C_2 are arbitrary constants. Equation (228) suggests that a solution can be determined by setting

$$C_1 = 0 \quad (229)$$

Equation (228) on substituting C_1 =0, becomes

$$a r^2 R'' + r\left\{9a + R_e P_r C_0 r^2\right\} R' - 16 a R = -E_c P_r \left\{ \frac{2 R_e C_0^2}{a^2} - \frac{C_2}{C_0 r^2} \right\} \quad (230)$$

The equation suggest to seek a solution of the form

$$R(r) = B_1 + B_2 r^{-2} \quad (231)$$

On substituting equation (231) in (230) we get

$$B_1 = \frac{E_c P_r R_e}{8a^2}\left(\frac{1}{a} + \frac{P_r C_2}{28}\right) \quad (232)$$

and

$$B_2 = \frac{-E_c P_r C_2}{28 a C_0} \quad (233)$$

Inserting equation (232) and (233) in equation (231) we get

$$R(r) = \frac{E_c P_r R_e}{8a^2}\left(\frac{1}{a} + \frac{P_r C_2}{28}\right) - \frac{E_c P_r C_2}{28 a C_0} r^{-2} \quad (234)$$

Employing equation (234) in equation (227) we get

$$S(r) = -2 C_0^2 \int \left[\frac{e^{\frac{-C_0 R_e P_r}{2a} r^2}}{r} \int r^3 e^{\frac{C_0 R_e P_r}{2a} r^2} R(r) dr \right] dr + C_3 \int \frac{e^{\frac{-C_0 R_e P_r}{2a} r^2}}{r} dr + C_4 \quad (235)$$

On substituting equation (234) and (235) in equation (225) we get the temperature distribution. The above solutions are when the function $A = 0$.
Now when the function B is zero equation (208) gives

$$g = \frac{c}{r} \quad (236)$$

where c is a non-zero constant.
Inserting equation (236) in equation (205) we get

$$a_1 c A_{rr} - 2a(M - g\nu) A_{r\nu} - \frac{a\left[1 - (M - g\nu)^2\right]}{c} A_{\nu\nu} + a g A_r + a A_\nu \left(-(M' - g'\nu) + \frac{2(-g)(M - g\nu)}{c}\right)$$
$$= \left(\frac{R_e}{ac}\right)\left[M' + \frac{2M}{r}\right]' + \frac{2 R_e \nu}{a r^3} \quad (237)$$

The right hand side suggests seeking a solution of the form

$$A(r,\psi) = R(r) + P(r)\,\nu \quad (238)$$

Equation (237) employing equation (238) become

$$acR'' - 2aMP' + agR' + aP\left(-M' - \frac{2gM}{c}\right) + \nu\left\{acP'' + 3agP' + aP\left(g' + \frac{2g^2}{c}\right)\right\} = \left(\frac{R_e}{ac}\right)\left[M' + \frac{2M}{r}\right]' + \frac{2R_e\nu}{ar^3} \tag{239}$$

Since ν and r independent variables, equation (239) yields

$$r^2P'' + 3rP' + P = -\frac{2R_e}{a^2cr} \tag{240}$$

and

$$(rR')' = \left(\frac{rR_e}{a^2c^2}\right)\left[M' + \frac{2M}{r}\right]' + \frac{2rM}{c}P' + \frac{r}{c}\left(M' + \frac{2M}{r}\right)P \tag{241}$$

The solution of equations (240) and (241) are

$$P(r) = \frac{s_1}{r} + \frac{s_2\ln r}{r} - \frac{R_e(\ln r)^2}{a^2cr} \tag{242}$$

and

$$R(r) = \frac{1}{r}\int\left\{\frac{1}{r}\int Z_1(r)dr\right\}dr + C_1\ln r + C_2 \tag{243}$$

where

$$Z_1(r) = \left(\frac{rR_e}{a^2c^2}\right)\left[M' + \frac{2M}{r}\right]' + \frac{2rM}{c}P' + \frac{r}{c}\left(M' + \frac{2M}{r}\right)P \tag{244}$$

On substituting equation (238) in equation (207) we get the value of μ which is

$$\mu = \frac{ac}{(M' - g'\nu)}\left(R(r) + P(r)\nu\right) \tag{245}$$

where the function $R(r)$ and $P(r)$ are given by equations (243) and (232) respectively.

The solution of equation (203) and (204) using equation (236), is

$$aR_eL = \left(\frac{-R_e}{ar^2} + acP'(r) + agP(r)\right)\left(\frac{\nu^2}{2}\right) + \left\{\left(\frac{-R_e}{ac}\right)\left[M'(r) + \frac{2M(r)}{r}\right] + acR'(r) - aM(r)P(r)\right\}\nu + \frac{a}{c}\int\left(1 - M(r)^2\right)P(r)dr + \int M(r)R'(r)dr + p_5 \tag{246}$$

Which gives the generalized energy function L for the case when $B = 0$.
The equation (206) for temperature distribution utilizing equation (236) becomes

$$acT_{rr} - 2a(M - g\nu)T_{\nu r} + \frac{a\left[1 + (M - g\nu)^2\right]}{c}T_{\nu\nu} + \left(\frac{ac}{r} - R_eP_r\right)T_r + \left(-\frac{2(M + N\nu)}{r} - (M' - g'\nu)\right)aT_\nu = -E_cP_r\left[RM' + (PM' - g'R)\nu - Pg'\nu^2\right] \tag{247}$$

The right-hand side of equation (247) suggests to seek solution of the form

$$T(r,\psi) = R_1(r) + R_2(r)\nu + R_3(r)\nu^2 \tag{248}$$

Inserting equation (248) in equation (247) we get

$$\nu^2\,[\,acR_3'' + 4agR_3' + \frac{2aR_3 g^2}{c} + 2aR_3\,(g' + \frac{2g}{r}) + \left(\frac{ac}{r} - R_e\,P_r\right)R_3'\,] + \nu\,[\,acR_2'' - 4aMR_3'\nu + 2ag\nu R_2' - \frac{4aR_3 Mg\nu}{c}$$
$$+\left(\frac{ac}{r} - R_e\,P_r\right)R_2'\nu + a\left\{-2R_3\left(M' + \frac{2M}{r}\right) + R_2\left(g' + \frac{2g}{r}\right)\right\}\nu\,]$$
$$+\,acR_1'' - 2aMR_2' + \frac{2aR_3(1+M^2)}{c} + \left(\frac{ac}{r} - R_e\,P_r\right)R_1' + \left(-a(M' + \frac{2M}{r})R_2\right)$$
$$= -E_c\,P_r\left[RM' + (PM' - g'R)\nu - P\,g'\nu^2\right] \tag{249}$$

Equation (249) on comparing the coefficients of ν^0, ν^1 and ν^2 we get

$$r^2 R_3'' + \left(5r - \frac{R_e\,P_r}{ac} r^2\right)R_3' + 4R_3 = Z_4(r) \tag{250}$$

$$r^2 R_2'' + \left(3r - \frac{R_e\,P_r\,r^2}{ac}\right)R_2' + R_2 = Z_3(r) \tag{251}$$

and

$$R_1'' + \left(\frac{1}{r} - \frac{R_e\,P_r}{ac}\right)R_1' = Z_2(r) \tag{252}$$

where

$$Z_2(r) = \frac{2MR_2'}{c} + \frac{1}{c}\left(M' + \frac{2M}{r}\right)R_2 - \left(\frac{2(1+M^2)}{c^2}\right)R_3 - \frac{E_c\,P_r\,M'R}{ac}R \tag{253}$$

$$Z_3(r) = 4ar^2 M\,R_3' + 2ar^2 R_3\left(M' + \frac{4M}{r}\right) - E_c\,P_r\left(r^2 P\,M' + c\,R\right) \tag{254}$$

$$Z_4(r) = \frac{-E_c\,P_r}{a}P(r) \tag{255}$$

The solution of equation (252) is

$$R_1(r) = \int\left\{\frac{e^{\left(\frac{R_e P_r}{ac}\right)r}}{r}\int r\,e^{-\left(\frac{R_e P_r}{ac}\right)r} Z_2(r)dr\right\}dr + C_5\int\left\{\frac{e^{\left(\frac{R_e P_r}{ac}\right)r}}{r}\right\}dr + C_6 \tag{256}$$

The solutions of equations (250) and (251) using Mathematica is

$$R_2(r) = C_3\left(-1 + \frac{1}{A1\,r}\right) + C_4\,\text{MeijerG}[\{\{\},\{1\}\},\{\{-1,-1\},\{\}\},-\text{A1 r}]$$
$$+\frac{1}{r}\,\{\,A1\,(-1+A1\,r)\;\int e^{-A1r}\,r\,\text{MeijerG}[\{\{\,\},\{1\}\},\{\{-1,-1\},\{\}\},-\text{A1 r}]\;Z_3(r)dr$$
$$+\,r\,\text{MeijerG}[\{\{\,\},\{1\}\},\{\{-1,-1\},\{\}\},-\text{A1 r}]\;\int A1(1-A1\,r)\,e^{-A1r}\,Z_3(r)dr\,\} \tag{257}$$

$$R_3(r) = C_1\left(\frac{1}{2} - \frac{2}{A1r} + \frac{1}{(A1r)^2}\right)$$
$$+\,C_2\,\text{MeijerG}[\{\{\},\{1\}\},\{\{-2,-2\},\{\}\},-\text{A1 r}]$$
$$+\frac{1}{r^2}\,\{\,A1^2\,(2 - 4A1\,r + A1^2 r^2)\;\int e^{-A1r}\,r^3\,\text{MeijerG}[\{\{\,\},\{1\}\},\{\{-2,-2\},\{\}\},-\text{A1 r}]\;Z_4(r)dr$$
$$+\,r^2\,\text{MeijerG}[\{\{\,\},\{1\}\},\{\{-2,-2\},\{\}\},-\text{A1 r}]\;\int A1^2\left(-2+4A1\,r - A1^2\,r^2\right)r\,e^{-A1r}\,Z_4(r)dr\,\} \tag{258}$$

where

$$A1 = \frac{R_e P_r}{a c} \tag{259}$$

Using $R_1(r)$, $R_2(r)$ and $R_3(r)$ in equation (248) we get temperature distribution.

Case II(b):
When $g(r) \neq 1$ and $\nu''(\psi) \neq 0$ the system of equations (195-200) is extremely difficult to solve in general. However the system simplifies into a very simple system if we assume

$$\nu(\psi) = e^{\psi} \tag{260}$$

On utilizing equation (260) in equations (198) and (199) we get

$$A = \frac{\mu}{r^2 g^2 \nu^2} \left[\nu^2 \left[rg\,N' + N^2 - 2N(N + g) \right] + \nu \left[rg\,M' - 2Mg \right] - (1 - M^2) \right] \tag{261}$$

$$B = \frac{4\mu}{r^2 g^2 \nu^2} \left[(M - g\nu) \right] \tag{262}$$

where

$$M(r) = r f'(r) \qquad \text{and} \qquad N(r) = r g'(r) \tag{263}$$

On eliminating μ from equations (262) and (263) we get

$$\frac{4A}{B} = -\frac{R_2}{g}\nu + \frac{\left(R_1 + \frac{MR_2}{g} \right) + R_0}{(M - g\nu)} \tag{264}$$

where

$$R_2(r) = rg\,N' + N^2 - 2N(N + g) \tag{265}$$

$$R_1(r) = rg\,M' - 2Mg \tag{266}$$

$$R_0(r) = -(1 - M^2) \tag{267}$$

On setting

$$R_0 = 0 \tag{268}$$

and

$$\left(R_1 + \frac{MR_2}{g} \right) = 0 \tag{269}$$

We can further simplify equation (264) leading to achieve exact solution. Equation $R_0 = 0$, provides

$$M = \pm 1 . \tag{270}$$

When $M = 1$, we found that the equation (196) admits exact solution. For $M = -1$, the temperature distribution equation is not exactly solvable, therefore this case is discarded.

$$r f'(r) = 1 \tag{271}$$

whose solution is

$$f(r) = \ln r + m_1 \tag{272}$$

where m_1 is constant.

Now equation (269) utilizing equation (266) and (267) becomes

$$r^2 K' - r K - 2 = 0 \tag{273}$$

where

$$K = \left(\frac{g'}{g}\right) \tag{274}$$

The equation (273) admits two solutions. One is particular and another is general. The particular and general solutions are

$$g = \frac{c}{r} \tag{275}$$

and

$$g = r\, e^{\frac{1}{2} r^2 m_2 + m_3} \tag{276}$$

where m_2, m_3 are constants

Equation (264) employing equation (260) and (269) becomes

$$B = \frac{-2r}{c\, e^{\psi}} A \tag{277}$$

Inserting equations (274), (275) and (277) in equation (199) we get

$$c\nu A_{rr} + \left(-2\nu + \frac{2c\nu^2}{r} + \frac{2r}{c}\right) A_{r\nu} + \left(-\frac{2r}{c^2} + \frac{2\nu}{c} - \frac{2\nu^2}{r} + \frac{c\nu^3}{r^2}\right) A_{\nu\nu} + \left(\frac{c\nu}{r} - \frac{2r}{c\nu}\right) A_r + \left(-\frac{2\nu}{r} + \frac{c\nu^2}{r^2} + \frac{4r}{c^2\nu}\right) A_\nu - \frac{4r}{c^2\nu^2} A = 0 \tag{278}$$

The variable coefficients of equation (278) suggest to seek solution of the form

$$A(r,\psi) = D\, r^n\, \nu^m \tag{279}$$

where D is constant.
Utilizing equation (279) in (278) we find

$$D\, r^n \nu^m \left[\frac{\nu}{r^2}\{cn(n-1) + 2cn\, m + cm\,(m-1) + nc + cm\} + \frac{1}{\nu}\left\{\frac{2n\, m}{c} + \frac{2m\,(m-1)}{c} - \frac{2n}{c}\right\} + \frac{1}{r}\{-2n\, m - 2m\,(m-1) - 2m\} + \frac{r}{\nu^2}\left\{-\frac{2m\,(m-1)}{c^2} + \frac{4m}{c^2} - \frac{4}{c^2}\right\} \right] = 0 \tag{280}$$

Equation (280) is identically satisfied provided

$$n\,(n-1) + 2n\, m + m\,(m-1) + n + m = 0 \tag{281}$$

$$n\, m + m\,(m-1) - n = 0 \tag{282}$$

$$n\, m + m\,(m-1) + m = 0 \tag{283}$$

$$m\,(m-1) - 2m + 2 = 0 \tag{284}$$

The solution of equations (281-284) is

$m = 1,\ 2$

and

$n = -m$

For $(m,n) = (1,-1)$ the equation (262) and (279) gives

$$\mu = \left(\frac{-D\,c}{2}\right)\left(\frac{r\,\nu^2}{(r - c\nu)}\right) \tag{285}$$

and

$$A = D\left(\frac{\nu}{r}\right) \tag{286}$$

The solution of equation (194) and (195) when $(m,n) = (1,-1)$ is

$$R_e\,L = \left(\frac{R_e}{c^2\,\nu^2}\right) - D\left(\frac{\nu}{r}\right) + \ p_7 \tag{287}$$

where p_7 is constant.
The energy equation is

$$(c\,\nu)^2\ \ T_{rr}\ -2\left(1 - \frac{c\,\nu}{r}\right)\ c\,\nu^2\ T_{\nu r} + \ \nu^2\left\{2 - \frac{2c\,\nu}{r} + \frac{c^2\,\nu^2}{r^2}\right\} T_{\nu\nu}\ + \left(\frac{c^2\,\nu^2}{r} - c\,\nu\,R_e\,P_r\right) T_r\ + \left(-\frac{2c\nu^2}{r} + \frac{c^2\,\nu^3}{r^2}\right) T_\nu$$
$$= \ \frac{2\,D\,E_c\,P_r}{c}\left[1 - \left(\frac{c\,\nu}{r}\right) + \left(\frac{c\,\nu}{r}\right)^2 - \left(\frac{c\,\nu}{r}\right)^3\right] \tag{288}$$

In the light of the coefficient of equation (288) we seek solution of the form

$$T\ = a\ln r + b\ \ln\nu + T_0 + T_1\left(\frac{\nu}{r}\right) + T_2\left(\frac{\nu}{r}\right)^2 \tag{289}$$

Inserting equation (289) in equation (288) we get

$$\left(\frac{\nu}{r}\right)^4\ \left\{c^2(6T_2) + 2c^2(-4T_2) + c^2(2T_2) + c^2(-2T_2) + c^2(2T_2)\right\}$$

$$+\left(\frac{\nu}{r}\right)^3\ \left\{\begin{matrix} c^2(2T_1) + 2c^2(-T_1) - 2c(-4T_2) - 2c(2T_2) \\ +c^2(-T_1) - cR_eP_r(-2T_2) + c^2(T_1) - 2c(2T_2)\end{matrix}\right\}$$

$$+\left(\frac{\nu}{r}\right)^2\ \left\{\begin{matrix} c^2(-a) - 2c(-T_1) + c^2(-b) + 2(2T_2) + c^2(a) - cR_eP_r(-T_1) \\ +c^2(b) - 2c(T_1)\end{matrix}\right\}$$

$$+\left(\frac{\nu}{r}\right)\ \left\{-2c(-b) - cR_eP_r(a) - 2c(b)\right\}\ +\ \left\{2(-b)\right\}\ =\ \frac{2E_cP_rD}{c}\ \left\{1 - \left(\frac{c\nu}{r}\right) + \left(\frac{c\nu}{r}\right)^2 - \left(\frac{c\nu}{r}\right)^3\right\} \tag{290}$$

Equation (290) on comparing the similar coefficients yields

$$a = \frac{2E_cD}{cR_r}\ , \qquad b = -\frac{E_cP_rD}{c}\ , \qquad T_1 = \frac{2E_cP_r\,D\,(R_eP_r + 2)}{(R_eP_r)^2}\ , \qquad T_2 = \frac{-cE_c\,D}{R_e} \tag{291}$$

On inserting equation (291) in equation (289) we get

$$T\ = \frac{2E_cD}{cR_r}\ \ln r\ - \frac{E_cP_rD}{c}\ \ln\nu + T_0\ + \frac{2E_cP_r\,D\,(R_eP_r + 2)}{(R_eP_r)^2}\left(\frac{\nu}{r}\right) - \frac{cE_c\,D}{R_e}\ \left(\frac{\nu}{r}\right)^2 \tag{292}$$

For $(m,n)=(2,-2)$ the function A , μ and the generalized pressure distribution L are

$$A(r,\psi)=D\left(\frac{v}{r}\right)^2 \tag{293}$$

$$\mu=\left(\frac{-Dc}{2}\right)\left(\frac{v^3}{(r-cv)}\right) \tag{294}$$

$$R_e L=\left(\frac{R_e}{c^2v^2}\right)-\left(\frac{Dv^2}{r^2}\right)+\left(\frac{2Dv}{cr}\right)+\left(\frac{2D}{c^2}\right)\ln r+p_8 \tag{295}$$

The energy equation is

$$(cv)^2 T_{rr} -2\left(1-\frac{cv}{r}\right) cv^2 T_{vr}+v^2\left\{2-\frac{2cv}{r}+\frac{c^2v^2}{r^2}\right\}T_{vv}+\left(\frac{c^2v^2}{r}-cvR_eP_r\right)T_r+\left(-\frac{2cv^2}{r}+\frac{c^2v^3}{r^2}\right)T_v$$
$$=\frac{2DE_cP_r}{c}\left[\left(\frac{v}{r}\right)-c\left(\frac{v}{r}\right)^2+c^2\left(\frac{v}{r}\right)^3-c^3\left(\frac{v}{r}\right)^4\right] \tag{296}$$

which on substituting

$$T=a\ln r+T_0+T_1\left(\frac{v}{r}\right)+T_2\left(\frac{v}{r}\right)^2+T_3\left(\frac{v}{r}\right)^3 \tag{297}$$

yields

$$a=-\frac{2E_cD}{c^2R_e},\qquad T_1=\frac{-E_cD}{cR_e}\left(1+\frac{4}{R_eP_r}-\frac{8}{(R_eP_r)^2}\right),\qquad T_2=\frac{E_cD}{R_e}\left(1+\frac{4}{R_eP_r}\right),\qquad \text{and}\qquad T_3=\frac{-2E_cD}{3R_e} \tag{298}$$

Substituting equation (298) in equation (297) we get

$$T=\left(-\frac{2E_cD}{c^2R_e}\right)\ln r+T_0+\left(\frac{-E_cD}{cR_e}\right)\left(1+\frac{4}{R_eP_r}-\frac{8}{(R_eP_r)^2}\right)\left(\frac{v}{r}\right)+\left(\frac{E_cD}{R_e}\right)\left(1+\frac{4}{R_eP_r}\right)\left(\frac{v}{r}\right)^2+\left(\frac{-2E_cD}{3R_e}\right)\left(\frac{v}{r}\right)^3 \tag{299}$$

4. Results and discussion

For the flows under consideration the streamlines are given by $\frac{\theta-f(r)}{g(r)}=Const.$ for case I, $g(r)=1$ the streamlines are $\theta-f(r)=Const.$ When $f(r)$ is arbitrary we can construct an infinite set of streamlines and also an infinite set of velocity components. This indicates an infinite set of solutions to the flow equations. When $f(r)$ is not arbitrary there are two values of $f(r)$, and therefore, two solutions to flow equations. The streamlines for case I are plotted in Fig. (1–16). The Fig. (1–16) shows the effect of different chosen forms of $f(r)$.

For case II when $f(r)=0$, $g(r)=\frac{-1}{c_0r^2+c_1}$ the streamlines are presented through Fig.(17–25) and the influence of various parameters are also indicated. When $f(r)$ is non-zero and $g(r)=\frac{c}{r}$, we can construct infinite set of velocity components and streamlines since $f(r)$ is arbitrary. This indicates an infinite set of solutions to flow equations. The Fig. (26–50) clearly indicate the effect on streamlines for different forms of $f(r)$.

When $v(\psi)=e^{\psi}$ there are two values of $g(r)$. When $g(r)=\frac{c}{r}$ the function $f(r)=\ln r+m_1$. The streamlines for $g(r)=\frac{c}{r}$ and $f(r)=\ln r+m_1$ are presented through Fig.(51–56).

5. Conclusion

The aim of this paper is to indicate a class of new exact solutions of the equations governing the steady plane flows of incompressible fluid of variable viscosity in the absence of external force. To achieve our aim, we first transformed the flow equation into Martin system (ϕ,ψ), and then setting ϕ defined in equation (34). The exact solutions are determined when $f(r)$ is arbitrary and $f(r)$ is not arbitrary. When $f(r)$ is arbitrary an infinite set of velocity component implying an infinite solution to flow equations. When $f(r)$ is not arbitrary, there are solutions of the flow equation. We see that in case II $f(r)=0$, and there is solution and when $f(r)\neq 0$, we find that $f(r)$ is arbitrary, and therefore, we can construct an infinite set of solutions. The influences of various chosen forms of $f(r)$ on the streamlines are also presented.

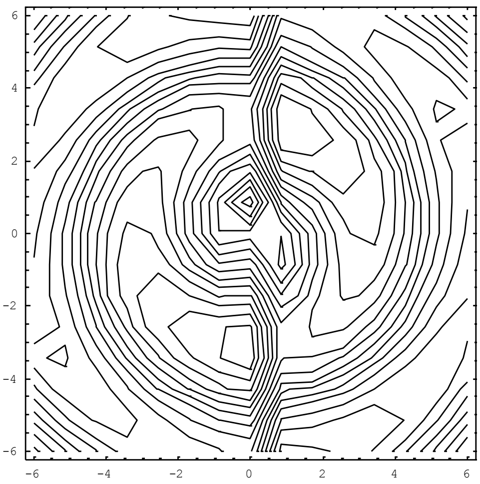

Fig. 1: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right)-2Cos\sqrt{x^2+y^2}-1$

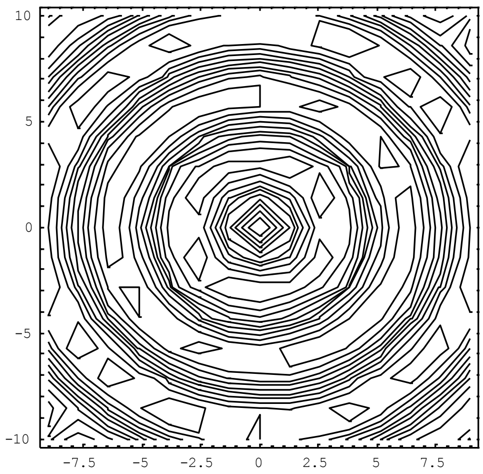

Fig. 2: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right)-28Cos\sqrt{x^2+y^2}-1$

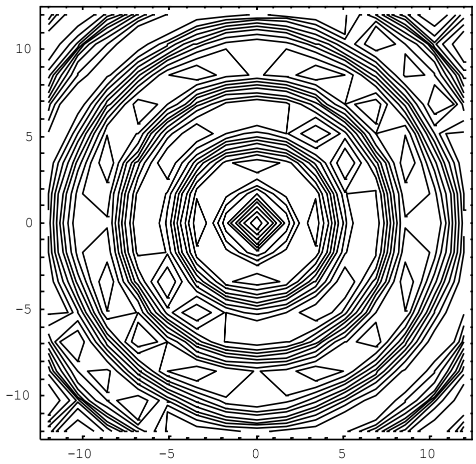

Fig. 3: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right)-280Cos\sqrt{x^2+y^2}-1$

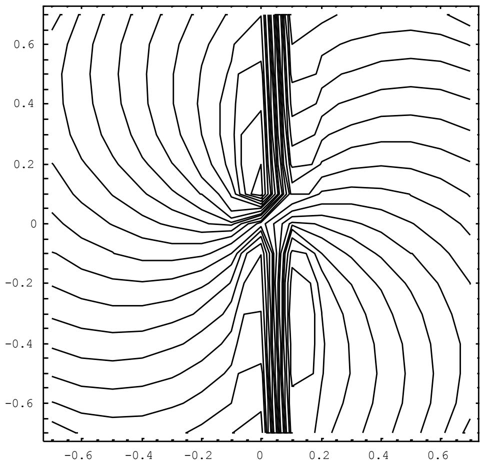

Fig. 4: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right)-Cos^{-1}\sqrt{x^2+y^2}+1$

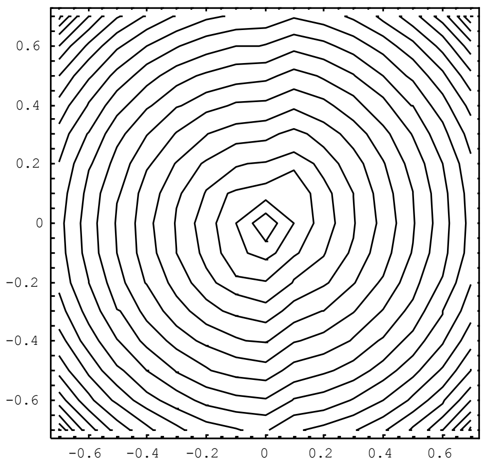

Fig. 5: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right)+50Cos^{-1}\sqrt{x^2+y^2}-200$

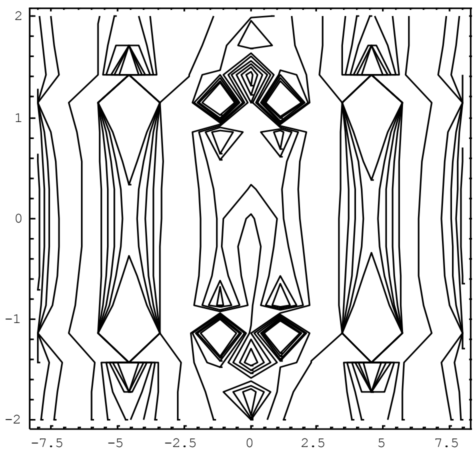

Fig. 6: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right)-Tan\left(\sqrt{x^2+y^2}\right)-1$

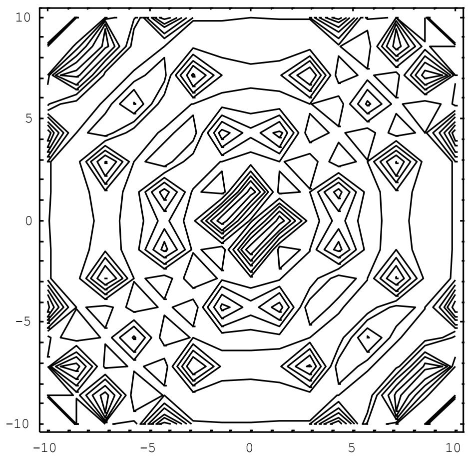

Fig. 7: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right) - 15\,Tan\left(\sqrt{x^2+y^2}\right) - 1$

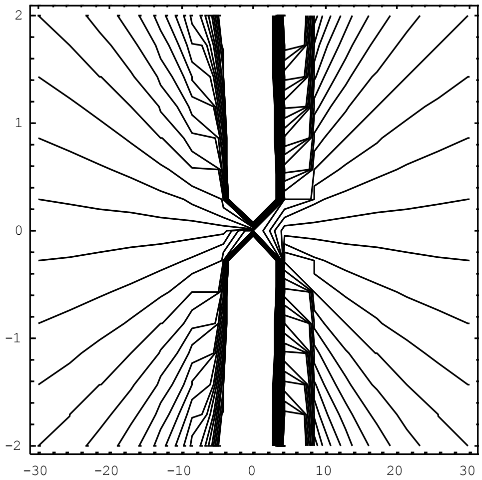

Fig. 8: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right) - (0.001)Tan^{-1}\left(\sqrt{x^2+y^2}\right) - 5$

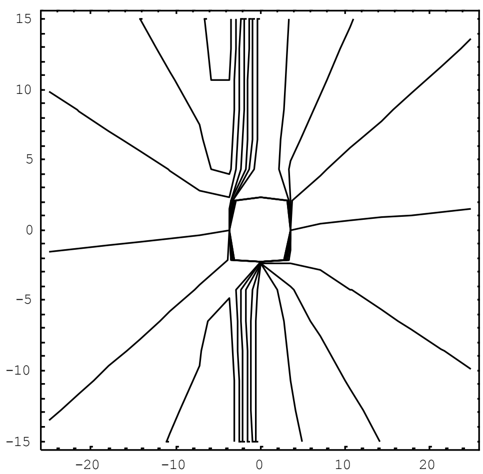

Fig. 9: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right) - 500Tanh\sqrt{x^2+y^2} - 1$

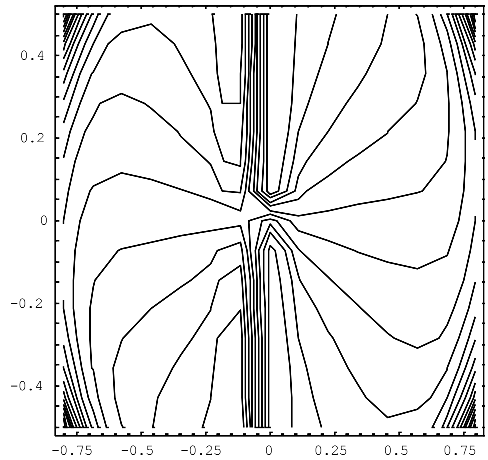

Fig. 10: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right) - 8Tanh^{-1}\sqrt{x^2+y^2} - 5$

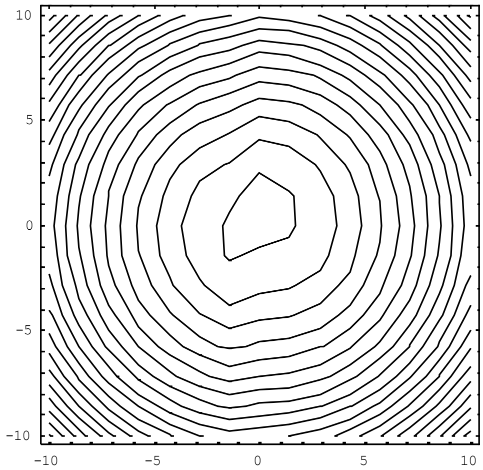

Fig. 11: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}(x^2+y^2) - 1$

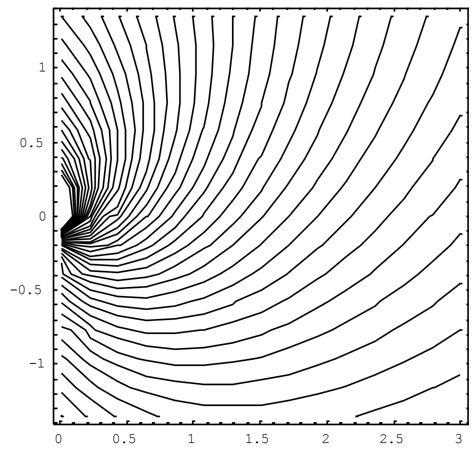

Fig. 12: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right) - \ln\sqrt{x^2+y^2} - \frac{1}{120}$

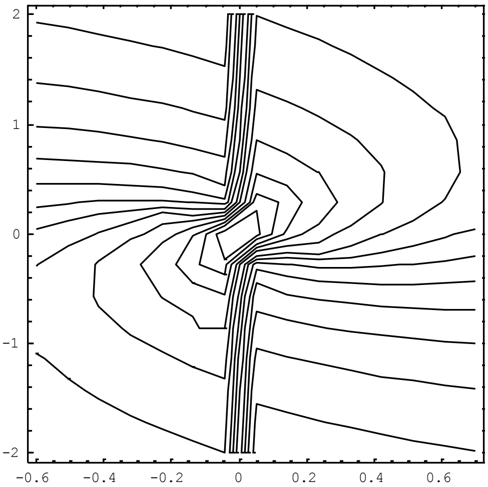

Fig. 13: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right) - \ln\sqrt{x^2+y^2} - \frac{1}{50}$

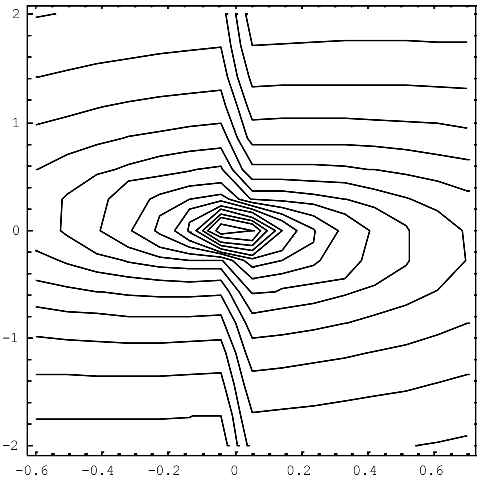

Fig. 14: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right) + 4\ln\sqrt{x^2+y^2} - 1$

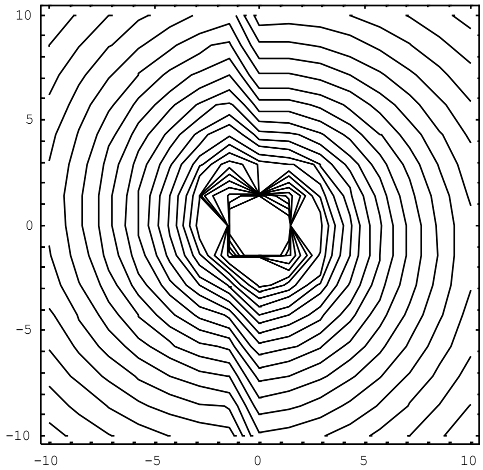

Fig. 15: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right) - 10\ln\sqrt{x^2+y^2} - 100$

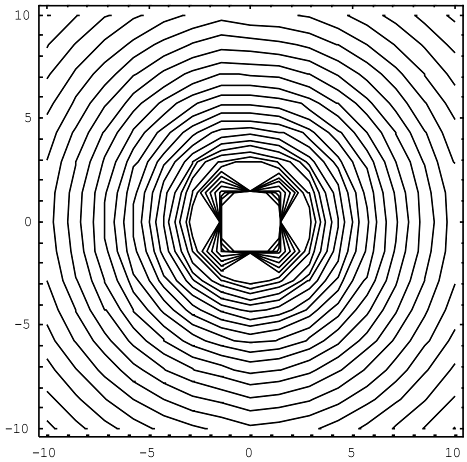

Fig. 16: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right) + 100\ln\sqrt{x^2+y^2} - 1$

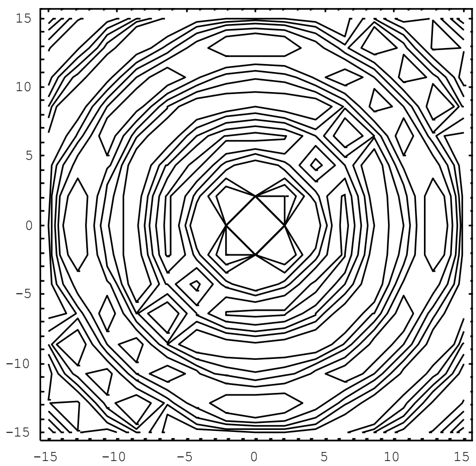

Fig. 26: Streamlines Pattern for
$Tan^{-1}\left(\frac{y}{x}\right) + 300Cos\sqrt{x^2+y^2} + \frac{500}{\sqrt{x^2+y^2}}$

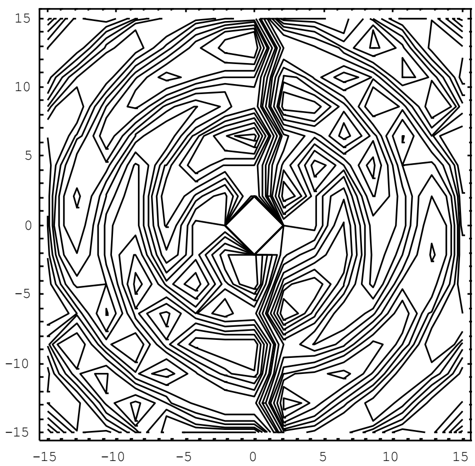

Fig. 27: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right) + Cos\sqrt{x^2+y^2} - \frac{1}{\sqrt{x^2+y^2}}$

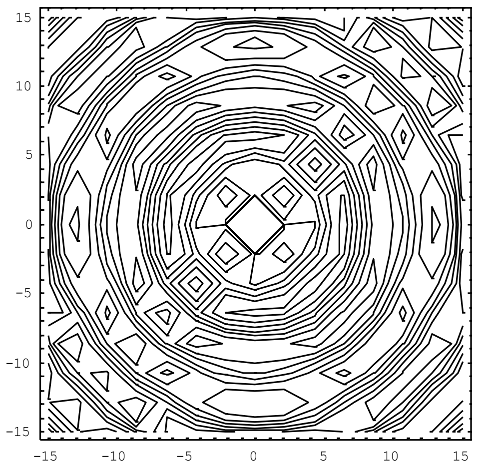

Fig. 28: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right)+100Cos\sqrt{x^2+y^2}-\frac{100}{\sqrt{x^2+y^2}}$

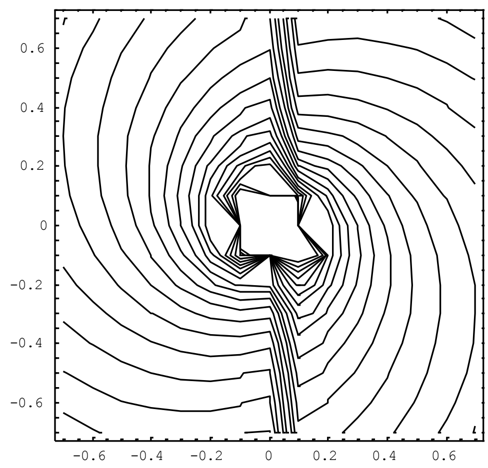

Fig. 29: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right)-Cos^{-1}\sqrt{x^2+y^2}-\frac{1}{\sqrt{x^2+y^2}}$

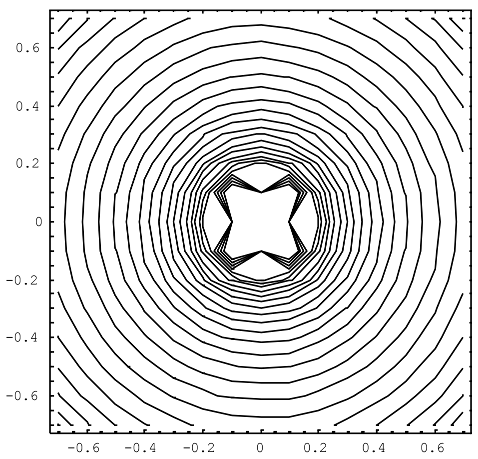

Fig. 30: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right)-200Cos^{-1}\sqrt{x^2+y^2}-\frac{100}{\sqrt{x^2+y^2}}$

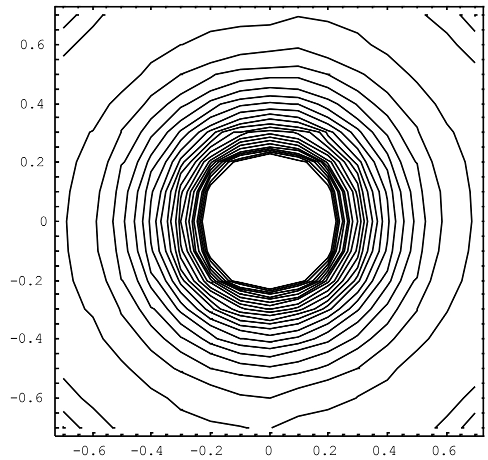

Fig. 31: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right)-100Cos^{-1}\sqrt{x^2+y^2}+\frac{100}{\sqrt{x^2+y^2}}$

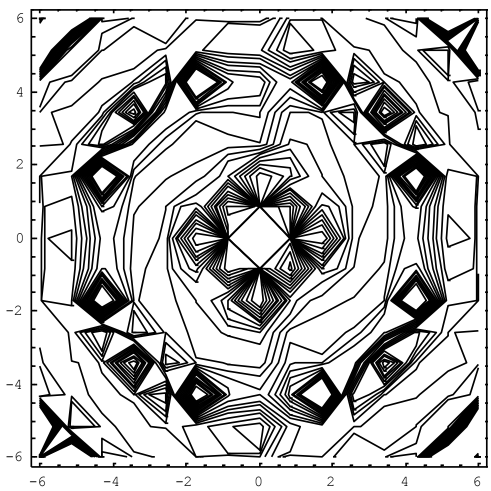

Fig. 32: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right)-Tan\sqrt{x^2+y^2}-\frac{1}{\sqrt{x^2+y^2}}$

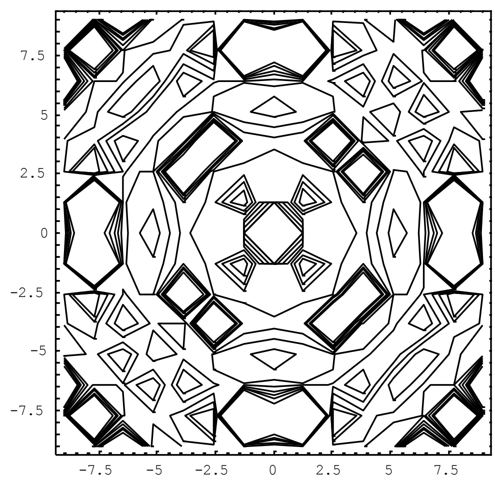

Fig. 33: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right)-200Tan\sqrt{x^2+y^2}-\frac{100}{\sqrt{x^2+y^2}}$

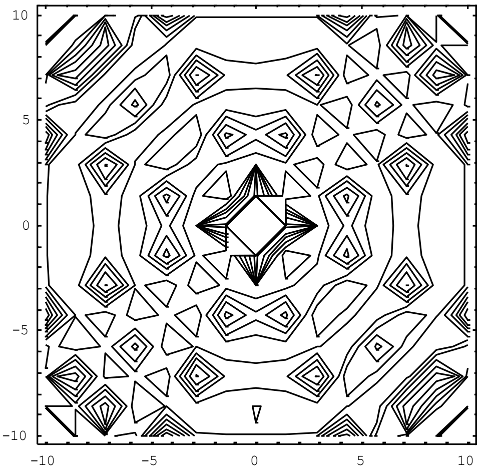

Fig. 34: Streamlines Pattern for

$$Tan^{-1}\left(\frac{y}{x}\right)+100Tan\sqrt{x^2+y^2}-\frac{100}{\sqrt{x^2+y^2}}$$

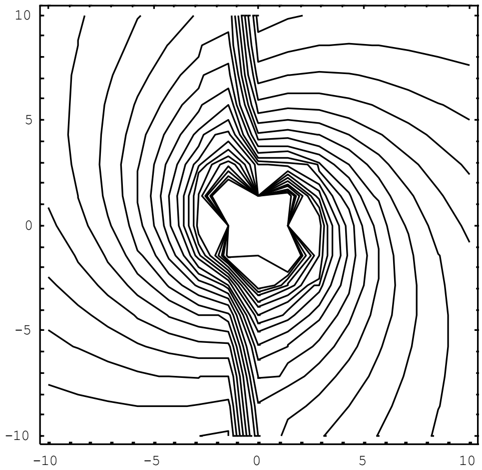

Fig. 35: Streamlines Pattern for

$$Tan^{-1}\left(\frac{y}{x}\right)-Tan^{-1}\sqrt{x^2+y^2}-\frac{20}{\sqrt{x^2+y^2}}$$

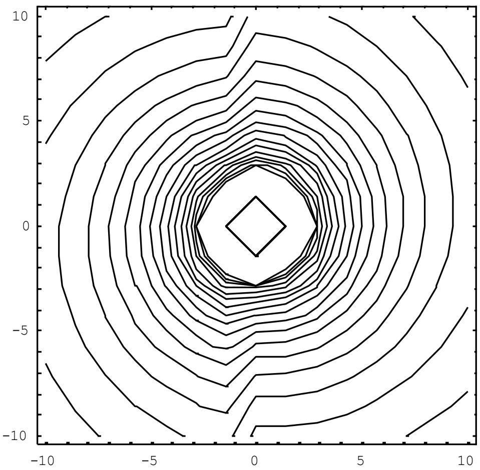

Fig. 36: Streamlines Pattern for

$$Tan^{-1}\left(\frac{y}{x}\right)-200Tan^{-1}\sqrt{x^2+y^2}-\frac{100}{\sqrt{x^2+y^2}}$$

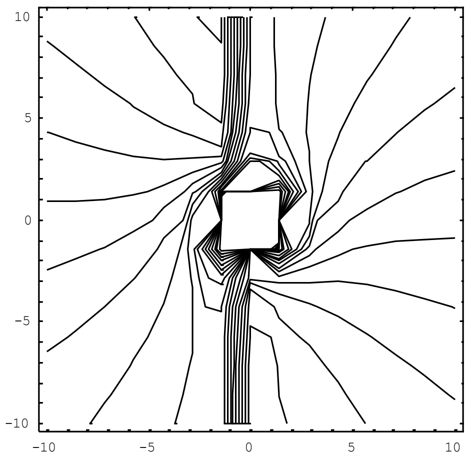

Fig. 37: Streamlines Pattern for

$$Tan^{-1}\left(\frac{y}{x}\right)-100Tan^{-1}\sqrt{x^2+y^2}+\frac{100}{\sqrt{x^2+y^2}}$$

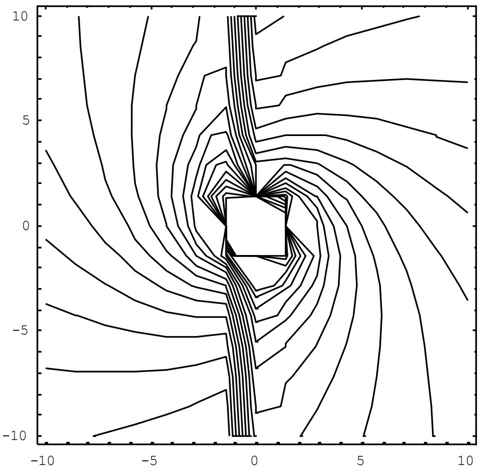

Fig. 38: Streamlines Pattern for

$$Tan^{-1}\left(\frac{y}{x}\right)-Tanh\sqrt{x^2+y^2}-\frac{10}{\sqrt{x^2+y^2}}$$

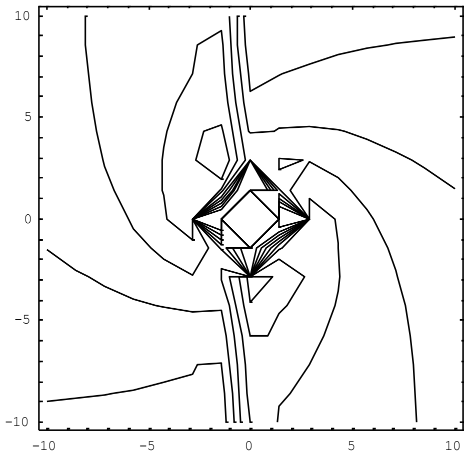

Fig. 39: Streamlines Pattern for

$$Tan^{-1}\left(\frac{y}{x}\right)-100Tanh\sqrt{x^2+y^2}-\frac{10}{\sqrt{x^2+y^2}}$$

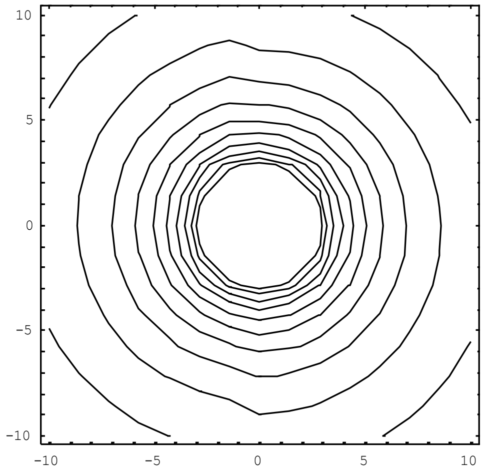

Fig. 40: Streamlines Pattern for

$$Tan^{-1}\left(\frac{y}{x}\right)-2Tanh\sqrt{x^2+y^2}-\frac{400}{\sqrt{x^2+y^2}}$$

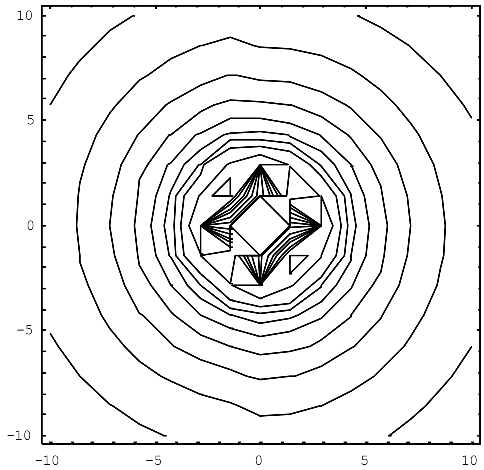

Fig. 41: Streamlines Pattern for

$$Tan^{-1}\left(\frac{y}{x}\right)-2000Tanh\sqrt{x^2+y^2}-\frac{400}{\sqrt{x^2+y^2}}$$

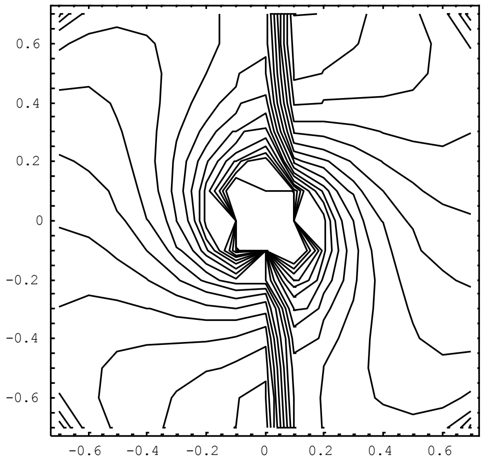

Fig. 42: Streamlines Pattern for

$$Tan^{-1}\left(\frac{y}{x}\right)-Tanh^{-1}\sqrt{x^2+y^2}-\frac{1}{\sqrt{x^2+y^2}}$$

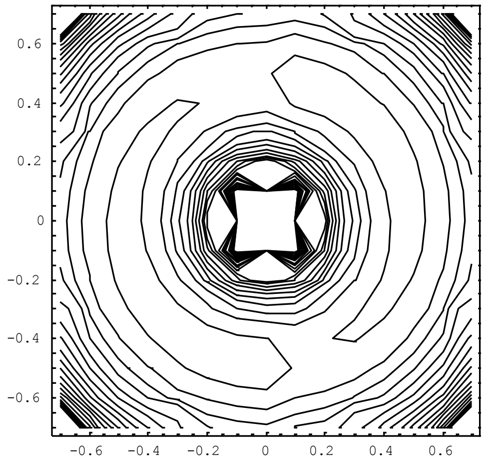

Fig. 43: Streamlines Pattern for

$$Tan^{-1}\left(\frac{y}{x}\right)-100Tanh^{-1}\sqrt{x^2+y^2}-\frac{30}{\sqrt{x^2+y^2}}$$

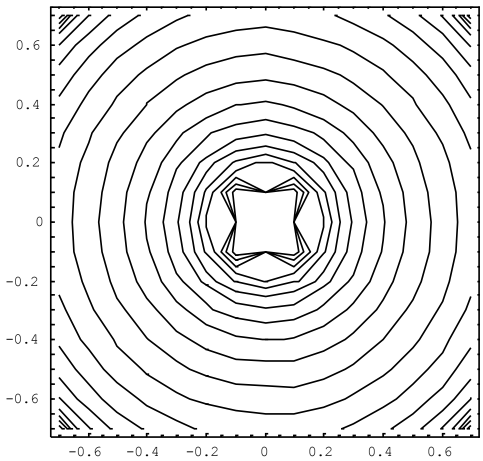

Fig. 44: Streamlines Pattern for

$$Tan^{-1}\left(\frac{y}{x}\right)+100Tanh^{-1}\sqrt{x^2+y^2}-\frac{40}{\sqrt{x^2+y^2}}$$

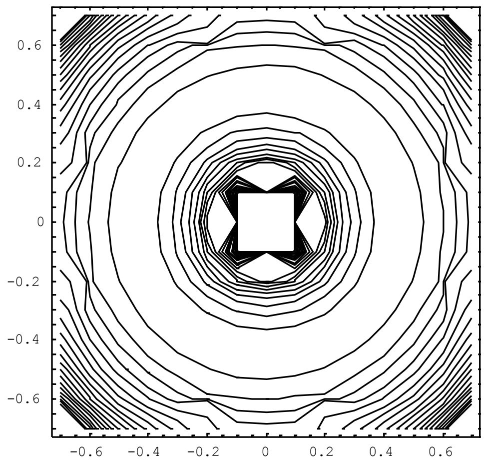

Fig. 45: Streamlines Pattern for

$$Tan^{-1}\left(\frac{y}{x}\right)+2000Tanh^{-1}\sqrt{x^2+y^2}-\frac{500}{\sqrt{x^2+y^2}}$$

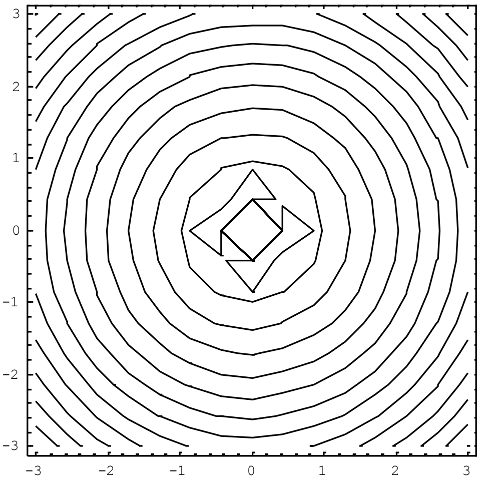

Fig. 46: Streamlines Pattern for
$$Tan^{-1}\left(\frac{y}{x}\right)+10(x^2+y^2)-50\sqrt{(x^2+y^2)}-10-\frac{1}{\sqrt{x^2+y^2}}$$

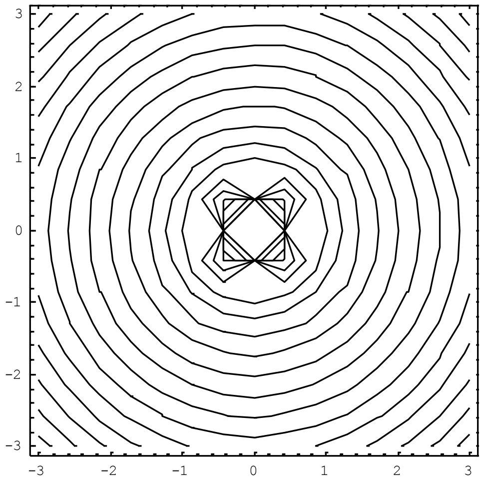

Fig. 47: Streamlines Pattern for
$$Tan^{-1}\left(\frac{y}{x}\right)-10(x^2+y^2)-50\sqrt{(x^2+y^2)}-10+\frac{100}{\sqrt{x^2+y^2}}$$

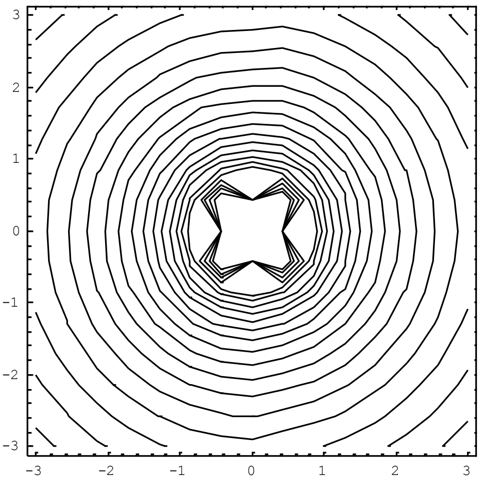

Fig. 48: Streamlines Pattern for
$$Tan^{-1}\left(\frac{y}{x}\right)-10[\ln\sqrt{(x^2+y^2)}]^2-50\ln\sqrt{(x^2+y^2)}-10+\frac{100}{\sqrt{x^2+y^2}}$$

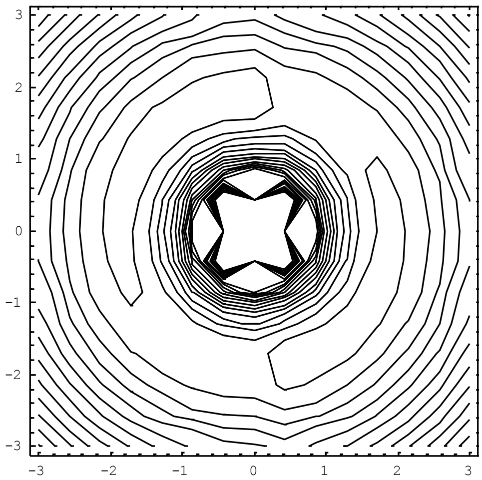

Fig. 49: Streamlines Pattern for
$$Tan^{-1}\left(\frac{y}{x}\right)+60[\ln\sqrt{(x^2+y^2)}]^2-50\ln\sqrt{(x^2+y^2)}-1+\frac{50}{\sqrt{x^2+y^2}}$$

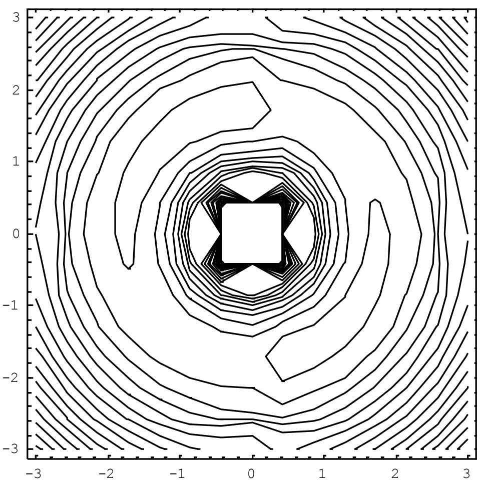

Fig. 50: Streamlines Pattern for
$$Tan^{-1}\left(\frac{y}{x}\right)-60[\ln\sqrt{(x^2+y^2)}]^2+50\ln\sqrt{(x^2+y^2)}-1+\frac{50}{\sqrt{x^2+y^2}}$$

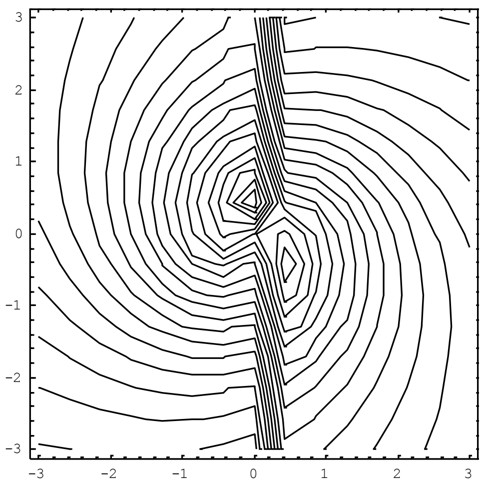

Fig. 17: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right)-\frac{17.8}{(x^2+y^2)+3}$

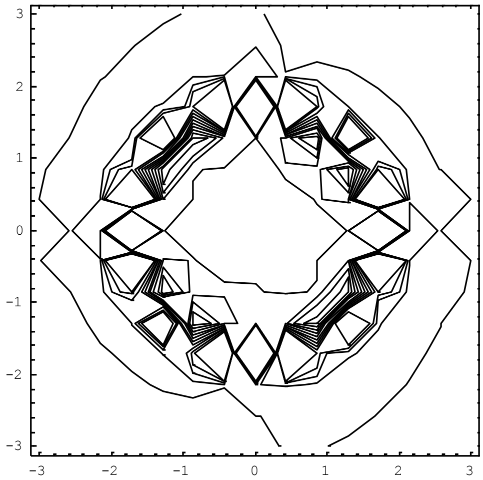

Fig. 18: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right)-\frac{17.8}{(x^2+y^2)-3}$

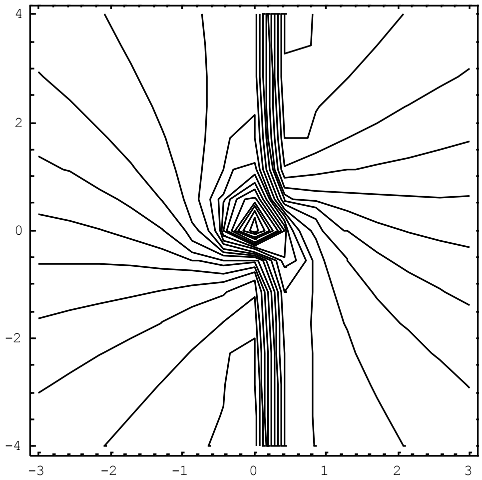

Fig. 19: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right)-\frac{17.8}{20(x^2+y^2)+3}$

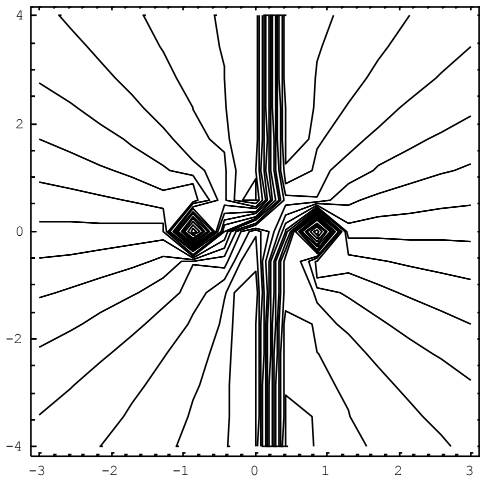

Fig. 20: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right)-\frac{17.8}{-400(x^2+y^2)+300}$

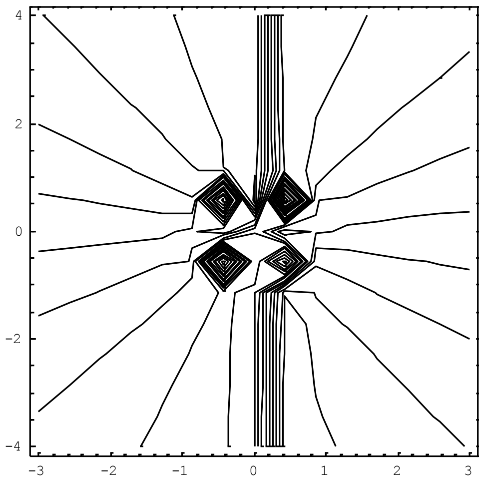

Fig. 21: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right)-\frac{17.8}{-400(x^2+y^2)+200}$

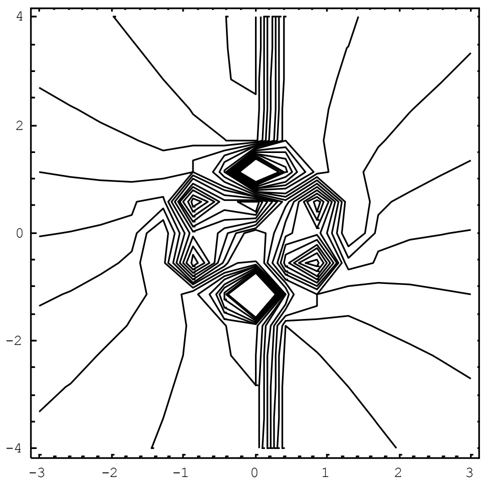

Fig. 22: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right)-\frac{200}{-400(x^2+y^2)+500}$

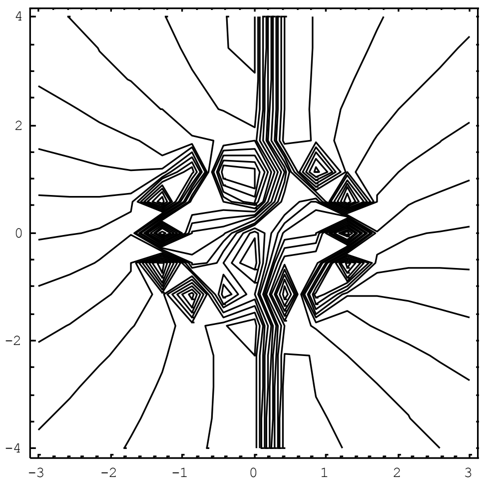

Fig. 23: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right)-\frac{200}{-500(x^2+y^2)+900}$

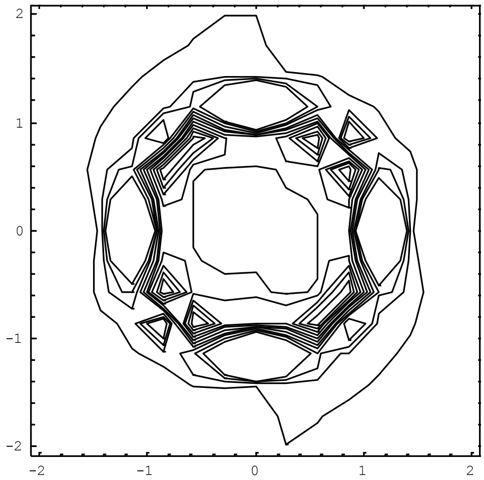

Fig. 24: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right)+\frac{510}{80(x^2+y^2)-100}$

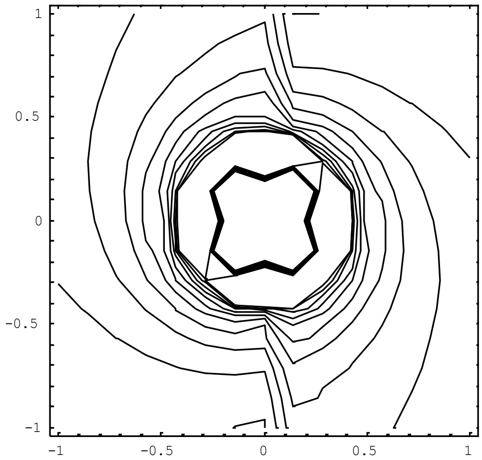

Fig. 25: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right)-\frac{210}{180(x^2+y^2)-10}$

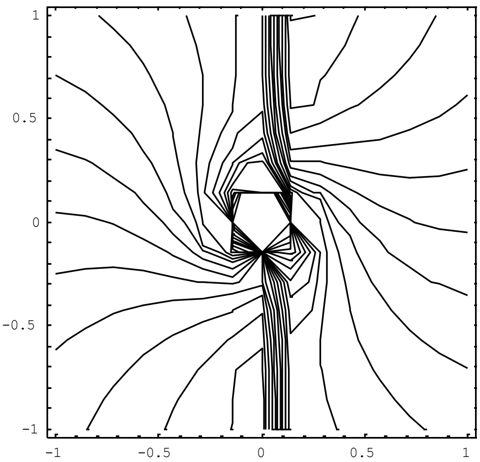

Fig. 51: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right)-\ln\sqrt{(x^2+y^2)}-\frac{1}{\sqrt{x^2+y^2}}$

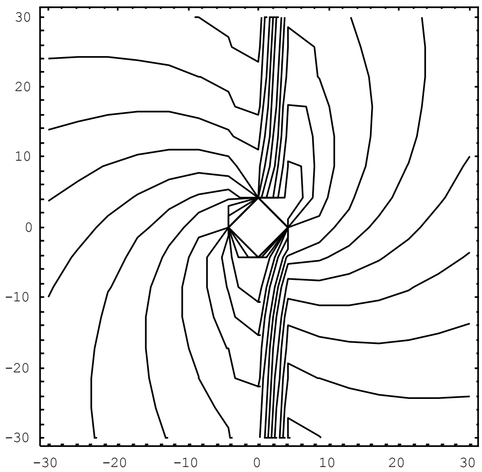

Fig. 52: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right)-\ln\sqrt{(x^2+y^2)}+\frac{0.69}{\sqrt{x^2+y^2}}$

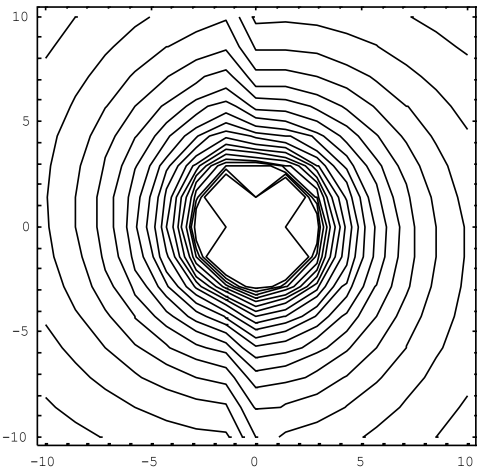

Fig. 53: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right)-\ln\sqrt{(x^2+y^2)}-\frac{100}{\sqrt{x^2+y^2}}$

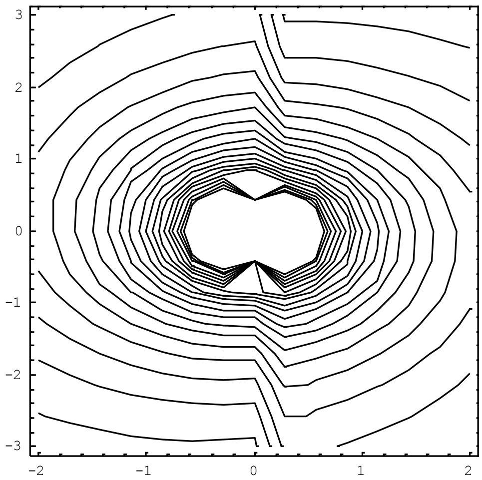

Fig. 54: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right)-\ln\sqrt{(x^2+y^2)}-\frac{20.1606}{\sqrt{x^2+y^2}}$

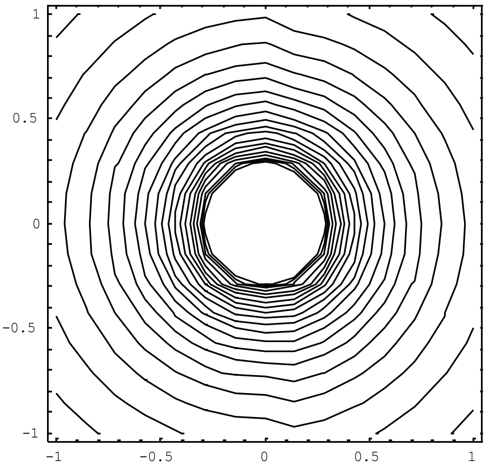

Fig. 55: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right)-\ln\sqrt{(x^2+y^2)}-\frac{50}{\sqrt{x^2+y^2}}$

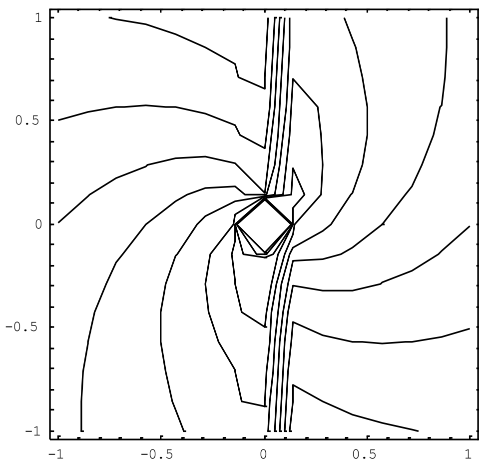

Fig. 56: Streamlines Pattern for $Tan^{-1}\left(\frac{y}{x}\right)-\ln\sqrt{(x^2+y^2)}-\frac{4.26652\,x\,10^{-15}}{\sqrt{x^2+y^2}}$

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