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Research paper

A 3-component mixture of inverse Rayleigh distributions: properties and estimation in Bayesian framework

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Abstract

This paper is about studying a 3-component mixture of the inverse Rayleigh distributions under Bayesian perspective. The censored sampling scheme is considered due to its popularity in reliability theory and survival analysis. The expressions for the Bayes estimators and their posterior risks are derived under different loss scenarios. In case, no little prior information is available, elicitation of hyper parameters is given. To examine, numerically, the performance of the Bayes estimators using non-informative and informative priors under different loss functions, we have simulated their statistical properties for different sample sizes and test termination times.

Keywords: *Bayes Estimators; Censoring, Loss Functions; Mixture Models; Posterior Risks.*

1. Introduction

The inverse Rayleigh distribution has many applications in the area of reliability studies. Most of the lifetime distributions used in reliability studies is characterized by a monotone failure rate. However, one parameter inverse Rayleigh distribution has also been used as a failure time distribution Voda [22] mentioned that the distribution of lifetime of several types of experimental units can be approximated by the inverse Rayleigh distribution. Different studies have been used the inverse Rayleigh distribution for various purposes. For example, Gharraph [8] derived five measures of the parameter of inverse Rayleigh distribution and also obtained the estimators of the unknown parameter using different methods of estimation. Abdel-Monem [1] developed some estimation and prediction results for the inverse Rayleigh distribution. Soliman and Al-Aboud [20] used Bayesian and classical techniques for parameter estimation based on a set of upper record values from the Rayleigh distribution. Bayesian estimators have been developed under symmetric and asymmetric loss functions. Howlader et al. [10] used the Bayesian approach to predict the bounds for Rayleigh and inverse Rayleigh lifetime models. Soliman et al. [19] discussed the problems of Bayesian and non-Bayesian estimation of an unknown parameter for an inverse Rayleigh distribution based on lower record values. Maximum likelihood estimate of the unknown parameter and Bayesian analysis was addressed using squared error and zero-one loss functions. The informative prior used to derive these estimates and the predictive intervals were also addressed with a real life data set. Dey [7] obtained Bayesian estimate of an inverse Rayleigh distribution using squared error and linear exponential loss functions.

The mixture models have established great interest for the analysts in the recent era. These models include finite and infinite number of components that can analyze different data sets. A finite mixture of some suitable probability distribution is recommended to study the population that is supposed to comprise a number of sub-populations mixing in an unknown proportion. Finite mixture models have been widely used in almost all fields of statistical sciences to model diverse populations. Fields in which mixture models have been successfully applied includes genetics, astronomy, medicine, engineering, economics; marketing etc. The analysis of mixture models under Bayesian framework has developed a significant interest among statisticians. Sultan et al. [21] discussed the properties of a 2-component mixture of the inverse Weibull distributions using a classical approach and the identifiably property of of the mixture model were also discussed. Kazmi et al. [14] described the Bayesian analysis for the 2-component mixture of Maxwell distributions. Noor and Aslam [17] studied Bayesian inference of the inverse Weibull mixture model using Type I censoring. Sajid Ali [2] described the 2-component mixture of the inverse Rayleigh distributions under Bayesian framework. Aslam and Tahir [4] presented the 3-component mixture of Rayleigh distribution under Bayesian framework.

Several types of data are encountered in everyday life, including simple data, grouped data, truncated data, censored data and progressively censored data. Censoring is an important and valuable aspect of the lifetime data. A valuable account of censoring is given in Gijbles [9] and Kalbfleisch and Prentice [13].

Motivated by above mentioned applications of mixture models, we plan to have Bayesian analysis of a 3-component mixture of the inverse Rayleigh distributions with unknown mixing proportions. The parameters of component distributions are assumed to be unknown. Four different priors and three different loss functions are used for the Bayesian analysis. In addition, we assume an ordinary Type I right censored sampling schemes.

The rest of the paper is organized as follows: The 3-component mixture of the inverse Rayleigh distributions is defined in section 2. The Likelihood function of the inverse Rayleigh mixture model is constructed in section 3. The expressions for posterior distributions using the non-informative and informative priors are derived in section 4. In section 5, the Bayes estimators and their posterior risks using the non-informative and informative priors under the squared error loss function (SELF), precautionary loss function (PLF) and DeGroot loss function (DLF) are presented. The elicitation of hyper parameters, if unknown is given in section 6. The limiting expression of the Bayes estimators and their posterior risks are derived in section 7. The simulation study is presented in section 8. Finally, the conclusion of this study is given in section 9.

2. Component mixture of the inverse Rayleigh distributions

The probability density function (p.d.f) and the cumulative distribution function (c.d.f) of the inverse Rayleigh distribution for a random variable X are given by:

$$f_m(x;\theta_m)=\frac{2\theta_m}{x^3}\exp\left(\frac{-\theta_m}{x^2}\right); x\ge 0, \theta_m>0, m=1,2,3 \quad (1)$$

$$F_m(x)=\exp\left(\frac{-\theta_m}{x^2}\right); m=1,2,3 \quad (2)$$

Where θ_m is the parameter of the inverse Rayleigh distribution.

A finite 3-component mixture model with the unknown mixing proportions p_1 and p_2 is defined as

$$f(x)=p_1f_1(x)+p_2f_2(x)+(1-p_1-p_2)f_3(x),$$
$$p_1,p_2\ge 0, p_1+p_2\le 1 \quad (3)$$

$$f(x,\theta_1,\theta_2,\theta_3,p_1,p_2)=p_1\frac{2\theta_1}{x^3}\exp\left(\frac{-\theta_1}{x^2}\right)+p_2\frac{2\theta_2}{x^3}\exp\left(\frac{-\theta_2}{x^2}\right)+(1-p_1-p_2)\frac{2\theta_3}{x^3}\exp\left(\frac{-\theta_3}{x^2}\right) \quad (4)$$

While the c.d.f of the 3-component mixture of the Inverse Rayleigh distribution is given by:

$$F(x)=p_1F_1(x)+p_2F_2(x)+(1-p_1-p_2)F_3(x) \quad (5)$$

$$F(x)=p_1\exp\left(\frac{-\theta_1}{x^2}\right)+p_2\exp\left(\frac{-\theta_2}{x^2}\right)+(1-p_1-p_2)\exp\left(\frac{-\theta_3}{x^2}\right) \quad (6)$$

3. The likelihood function

Suppose 'n' units from the 3-component mixture of inverse Rayleigh distributions are used in a life testing experiment with fixed test termination time t. Let 'r' units out of 'n' units failed until fixed test termination time't' and the remaining (n-r) units are still working. According to Mendenhall and Hader [16], there are many practical situations in which the failed objects can be pointed out easily as subset of subpopulation-1, subpopulation-II or subpopulation-III. Out of 'r' units, suppose r_1, r_2 and r_3 units belong to subpopulation-1, subpopulation-II or subpopulation-III respectively and such that $r=r_1+r_2+r_3$. Now we define x_{lk}, $0<x_{lk}<t$ be the failure time of k^{th} unit belonging to the l^{th} subpopulation, where $l=1,2,3$ and $k=1,2,...,r_l$. For a 3-component mixture model, the likelihood function can be written as

$$L\langle\varphi|x\rangle\propto\left\{\prod_{k=1}^{r_1}p_1f_1(x_{1k})\right\}\left\{\prod_{k=1}^{r_2}p_2f_2(x_{2k})\right\}\left\{\prod_{k=1}^{r_3}(1-p_1-p_2)f_3(x_{3k})\right\}\left[1-F(t)\right]^{n-r} \quad (7)$$

After simplification, the likelihood function of 3-component mixture of Inverse Rayleigh distribution is given by:

$$L\langle\varphi|x\rangle\propto\theta_1^{r_1}\theta_2^{r_2}\theta_3^{r_3}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\exp\left\{-\theta_1\left(\sum_{k=1}^{r_1}x_{1k}^{-2}+\frac{i-j}{t^2}\right)\right\}\exp\left\{-\theta_2\left(\sum_{k=1}^{r_2}x_{2k}^{-2}+\frac{j-l}{t^2}\right)\right\}\exp\left\{-\theta_3\left(\sum_{k=1}^{r_3}x_{3k}^{-2}+\frac{l}{t^2}\right)\right\}p_1^{i-j+r_1}p_2^{j-l+r_2}(1-p_1-p_2)^{l+r_3} \quad (8)$$

Where

$\varphi=(\theta_1,\theta_2,\theta_3,p_1,p_2)$ and $X=\left(x_{11},...,x_{1r_1},x_{21},...,x_{2r_2},x_{31},...,x_{3r_3}\right)$

Are the observed failure times for the uncensored observations.

4. The posterior distribution using the non-informative and the informative priors

In this section, posterior distributions of parameters given data, say x, are derived using the non-informative (Uniform and Jeffreys') and the informative (Gamma and Exponential) priors.

4.1. The posterior distribution using the uniform prior (UP)

When elicitation of hyper parameters is difficult or little prior information is given, then usually the non-informative prior is assumed to be the UP. Ups over the intervals $(0,\infty)$ and $(0,1)$ are taken for the parameters $(\theta_1,\theta_2 \,\&\, \theta_3)$ of Inverse Rayleigh distribution and for the mixing proportions (p_1,p_2) respectively. With these settings, joint prior distribution of parameters $(\theta_1,\theta_2,\theta_3,p_1,p_2)$, is given by:

$$\pi_1(\varphi)\propto 1; \theta_1,\theta_2,\theta_3>0, p_1,p_2\ge 0, p_1+p_2\le 1 \quad (9)$$

The joint posterior distribution of parameters $\theta_1,\theta_2,\theta_3,p_1$ and p_2 given data x assuming the UP is:

$$g_1\langle\varphi|x\rangle=\frac{L\langle\varphi|x\rangle\pi_1(\varphi)}{\int_{\varphi}L\langle\varphi|x\rangle\pi_1(\varphi)d\varphi} \quad (10)$$

$$g_1\langle\varphi|x\rangle=\Omega_1^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\theta_1^{A_{11}-1}\theta_2^{A_{21}-1}\theta_3^{A_{31}-1}\exp(-\theta_1B_{11})\exp(-\theta_2B_{21})\exp(-\theta_3B_{31})p_1^{A_{01}-1}p_2^{B_{01}-1}(1-p_1-p_2)^{C_{01}-1} \quad (11)$$

Where

$A_{11}=r_1+1, A_{21}=r_2+1, A_{31}=r_3+1, B_{11}=\sum_{k=1}^{r_1}x_{1k}^{-2}+\frac{i-j}{t^2}, B_{21}=\sum_{k=1}^{r_2}x_{2k}^{-2}+\frac{j-l}{t^2},$

$B_{31}=\sum_{k=1}^{r_3}x_{3k}^{-2}+\frac{l}{t^2}, A_{01}=i-j+r_1+1, B_{01}=j-l+r_2+1, C_{01}=l+r_3+1$

$$\Omega_1=\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}B(A_{01},C_{01})$$
$$B(B_{01},A_{01}+C_{01})\frac{\Gamma(A_{11})}{B_{11}^{A_{11}}}\frac{\Gamma(A_{21})}{B_{21}^{A_{21}}}\frac{\Gamma(A_{31})}{B_{31}^{A_{31}}}$$

4.2. The posterior distribution using the Jeffreys' prior (JP)

According to Jeffreys' [11], [12] and Berger [5], the JP is defined as

$$p(\theta_m)\propto\sqrt{|I(\theta_m)|},m=1,2,3\text{ , where }I(\theta_m)=-E\left[\frac{\partial^2 f\langle x|\theta_m\rangle}{\partial\theta_m^2}\right]$$

is the Fisher's information matrix. The prior distributions of the mixing proportions p_1 and p_2 are again taken to be the uniform over the interval $(0,1)$. Under the assumption of independence of all parameters, the joint prior distribution of $(\theta_1,\theta_2,\theta_3,p_1,p_2)$ is:

$$\pi_2(\varphi)\propto\frac{1}{\theta_1\theta_2\theta_3},\theta_1,\theta_2,\theta_3\geq 0,p_1,p_2\geq 0,p_1+p_2\leq 1 \qquad (12)$$

The joint posterior distribution of parameters $\theta_1,\theta_2,\theta_3,p_1$ and p_2 given data x assuming the JP is:

$$g_2\langle\varphi|x\rangle=\frac{L\langle\varphi|x\rangle\pi_2(\varphi)}{\int_\varphi L\langle\varphi|x\rangle\pi_2(\varphi)d\varphi} \qquad (13)$$

$$g_2\langle\varphi|x\rangle=$$
$$\Omega_2^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\theta_1^{A_{12}-1}\theta_2^{A_{22}-1}\theta_3^{A_{32}-1} \qquad (14)$$
$$\exp(-\theta_1B_{12})$$

$$\exp(-\theta_2B_{22})$$
$$\exp(-\theta_3B_{32})p_1^{A_{02}-1}p_2^{B_{02}-1}(1-p_1-p_2)^{C_{02}-1} \qquad (13)$$

Where

$$A_{12}=r_1,A_{22}=r_2,A_{32}=r_3,B_{12}=\sum_{k=1}^{r_1}x_{1k}^{-2}+\frac{i-j}{t^2},B_{22}=\sum_{k=1}^{r_2}x_{2k}^{-2}+\frac{j-l}{t^2},$$

$$B_{32}=\sum_{k=1}^{r_3}x_{3k}^{-2}+\frac{l}{t^2},A_{02}=i-j+r_1+1,B_{02}=j-l+r_2+1,C_{02}=l+r_3+1,$$

$$\Omega_2=\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}B(A_{02},C_{02})$$
$$B(B_{02},A_{02}+C_{02})\frac{\Gamma(A_{12})}{B_{12}^{A_{12}}}\frac{\Gamma(A_{22})}{B_{22}^{A_{22}}}\frac{\Gamma(A_{32})}{B_{32}^{A_{32}}}$$

4.3. The posterior distribution using the gamma prior (GP)

As an informative prior, we take the Gamma prior for the component parameters $\theta_1,\theta_2,\theta_3$ and bivariate beta prior for proportion parameters p_1,p_2 .Symbolically, it can be written as: $\theta_1\sim Gamma(a_1,b_1),\theta_2\sim Gamma(a_2,b_2),\theta_3\sim Gamma(a_3,b_3)$ and $p_1,p_2\sim$ Bivariate Beta (a,b,c) .Again assuming independence of all parameters, the joint prior distribution of $(\theta_1,\theta_2,\theta_3,p_1,p_2)$ is given by:

$$\pi_3(\varphi)\propto\theta_1^{a_1-1}\exp(-b_1\theta_1)\theta_2^{a_2-1}\exp(-b_2\theta_2)\theta_3^{a_3-1}$$
$$\exp(-b_3\theta_3)p_1^{a-1}p_2^{b-1}(1-p_1-p_2)^{c-1} \qquad (15)$$

The joint posterior distribution of parameters $\theta_1,\theta_2,\theta_3,p_1$ and p_2 given data x

$$g_3\langle\varphi|x\rangle=\frac{L\langle\varphi|x\rangle\pi_3(\varphi)}{\int_\varphi L\langle\varphi|x\rangle\pi_3(\varphi)d\varphi} \qquad (16)$$

$$g_3\langle\varphi|x\rangle=\Omega_3^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\theta_1^{A_{13}-1}\theta_2^{A_{23}-1}\theta_3^{A3-1}$$
$$\exp(-\theta_1B_{13})\exp(-\theta_2B_{23}) \qquad (17)$$
$$\exp(-\theta_3B_{33})p_1^{A_{03}-1}p_2^{B_{03}-1}(1-p_1-p_2)^{C_{03}-1}$$

Where

$$A_{13}=r_1+a_1,A_{23}=r_2+a_2,A_{33}=r_3+a_3,B_{13}=\sum_{k=1}^{r_1}x_{1k}^{-2}+\frac{i-j}{t^2}+b_1,B_{23}=\sum_{k=1}^{r_2}x_{2k}^{-2}+\frac{j-l}{t^2}+b_2,$$

$$B_{33}=\sum_{k=1}^{r_3}x_{3k}^{-2}+\frac{l}{t^2}+b_3,A_{03}=i-j+r_1+a,B_{03}=j-l+r_2+b,C_{03}=l+r_3+c,$$

$$\Omega_3=\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}B(A_{03},C_{03})$$
$$B(B_{03},A_{03}+C_{03})\frac{\Gamma(A_{13})}{B_{13}^{A_{13}}}\frac{\Gamma(A_{23})}{B_{23}^{A_{23}}}\frac{\Gamma(A_{33})}{B_{33}^{A_{33}}}$$

4.4. The posterior distribution using the exponential prior (EP)

As an informative prior, we take the Exponential prior for the component parameters $\theta_1,\theta_2,\theta_3$ and bivariate beta prior for proportion parameters p_1,p_2 .Symbolically, it can be written $\theta_1\sim Exponential(w_1),\theta_2\sim Exponential(w_2),\theta_3\sim Exponential(w_3)$ and $p_1,p_2\sim$ Bivariate Beta (a,b,c) .Again assuming independence of all parameters, the joint prior distribution of $(\theta_1,\theta_2,\theta_3,p_1,p_2)$ is given by:

$$\pi_4(\varphi)\propto w_1\exp(-w_1\theta_1)w_2\exp(-w_2\theta_2)$$
$$w_3\exp(-w_3\theta_3)p_1^{a-1}p_2^{b-1}(1-p_1-p_2)^{c-1} \qquad (18)$$

The joint posterior distribution of parameters $\theta_1,\theta_2,\theta_3,p_1$ and p_2 given data x

$$g_4\langle\varphi|x\rangle=\frac{L\langle\varphi|x\rangle\pi_4(\varphi)}{\int_\varphi L\langle\varphi|x\rangle\pi_4(\varphi)d\varphi} \qquad (19)$$

$$g_4\langle\varphi|x\rangle = \Omega_4^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\theta_1^{A_{14}-1}\theta_2^{A_{24}-1}\theta_3^{A_{34}-1} \exp(-\theta_1 B_{14}) \exp(-\theta_2 B_{24})\exp(-\theta_3 B_{34})p_1^{A_{04}-1}p_2^{B_{04}-1}(1-p_1-p_2)^{C_{04}-1} \quad (20)$$

Where

$$A_{14}=r_1+1, A_{24}=r_2+1, A_{34}=r_3+1, B_{14} = \sum_{k=1}^{r_1}x_{1k}^{-2}+\frac{i-j}{t^2}+w_1, B_{24}=\sum_{k=1}^{r_2}x_{2k}^{-2}+\frac{j-l}{t^2}+w_2,$$

$$B_{34} = \sum_{k=1}^{r_3}x_{3k}^{-2}+\frac{l}{t^2}+w_3, A_{04}=i-j+r_1+a, B_{04} = j-l+r_2+b, C_{04}=l+r_3+c,$$

$$\Omega_4=\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}B(A_{04},C_{04}) B(B_{04},A_{04}+C_{04})\frac{\Gamma(A_{14})}{B_{14}^{A_{14}}}\frac{\Gamma(A_{24})}{B_{24}^{A_{24}}}\frac{\Gamma(A_{34})}{B_{34}^{A_{34}}}$$

5. Bayes estimators and posterior risks using the UP, the JP, the gamma and the exponential prior under SELF, PLF and DLF

If $\hat{d}$ is a Bayes estimator then $\rho(\hat{d})$ is called posterior risk and is defined as: $\rho(\hat{d})=E_{\theta|x}\{L(\theta,\hat{d})\}$. Our purpose, in this study, is to look for efficient Bayes estimators of the different parameters. For this purpose, three different loss functions, namely SELF, PLF and DLF used to obtain Bayes estimators and their posterior risks. The SELF, defined as $L(\theta,d)=(\theta-d)^2$,was introduced by Legendre [15] to develop the least squares theory. Norstrom [18] discussed an asymmetric PLF and also introduced a special case of general class of PLFs, which is defined as $L(\theta,d)=\frac{(\theta-d)^2}{d}$. While the DLF is presented by DeGroot [6] and is defined as $L(\theta,d)=\left(\frac{\theta-d}{d}\right)^2$.

For a given prior, the Bayes estimator and posterior risk under SELF are calculated as: $\hat{d}=E_{\theta|x}(\theta)$ and $\rho(\hat{d})=E_{\theta|x}(\theta^2)-\{E_{\theta|x}(\theta)\}^2$, respectively. Similarly, the Bayes estimators and posterior risks with PLF and DLF are given by:

$$\hat{d}=\left\{E_{\theta|x}(\theta^2)\right\}^{\frac{1}{2}}, \rho(\hat{d})=2\left\{E_{\theta|x}(\theta^2)\right\}^{\frac{1}{2}}-2E_{\theta|x}(\theta)$$

And

$$\hat{d}=\frac{E_{\theta|x}(\theta^2)}{E_{\theta|x}(\theta)}, \rho(\hat{d})=1-\frac{\{E_{\theta|x}(\theta)\}^2}{E_{\theta|x}(\theta^2)}, \text{ respectively.}$$

5.1. The bayes estimators and posterior risks using the UP, the JP, the GP and the EP under SELF

The Bayes estimators and posterior risks using the UP, the JP and IP for parameters $\theta_1,\theta_2,\theta_3,p_1$ and p_2 under SELF are obtained with their respective marginal posterior distributions are given below:

$$\hat{\theta}_{1\omega}=\Omega_\omega^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega}+1)}{B_{1\omega}^{A_{1\omega}+1}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}} B(A_{0\omega},C_{0\omega})B(B_{0\omega},A_{0\omega}+C_{0\omega}) \quad (21)$$

$$\hat{\theta}_{2\omega}=\Omega_\omega^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1\omega}}}\frac{\Gamma(A_{2\omega}+1)}{B_{2\omega}^{A_{2\omega}+1}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}}B(A_{0\omega},C_{0\omega})B(B_{0\omega},A_{0\omega}+C_{0\omega}) \quad (22)$$

$$\hat{\theta}_{3\omega}=\Omega_\omega^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1\omega}}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega}+1)}{B_{3\omega}^{A_{3\omega}+1}}B(A_{0\omega},C_{0\omega})B(B_{0\omega},A_{0\omega}+C_{0\omega}) \quad (23)$$

$$\hat{p}_{1\omega}=\Omega_\omega^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1\omega}}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}}B(B_{0\omega},C_{0\omega})B(A_{0\omega}+1,B_{0\omega}+C_{0\omega}) \quad (24)$$

$$\hat{p}_{2\omega}=\Omega_\omega^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1\omega}}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}}B(A_{0\omega},C_{0\omega})B(B_{0\omega}+1,A_{0\omega}+C_{0\omega}) \quad (25)$$

$$\rho(\hat{\theta}_{1\omega})=\Omega_\omega^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega}+2)}{B_{1\omega}^{A_{1\omega}+2}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}}B(A_{0\omega},C_{0\omega}) B(B_{0\omega},A_{0\omega}+C_{0\omega})-(\hat{\theta}_{1\omega})^2 \quad (26)$$

$$\rho(\hat{\theta}_{2\omega})=\Omega_\omega^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1\omega}}}\frac{\Gamma(A_{2\omega}+2)}{B_{2\omega}^{A_{2\omega}+2}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}}B(A_{0\omega},C_{0\omega}) B(B_{0\omega},A_{0\omega}+C_{0\omega})-(\hat{\theta}_{2\omega})^2 \quad (27)$$

$$\rho(\hat{\theta}_{3\omega})=\Omega_\omega^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1\omega}}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega}+2)}{B_{3\omega}^{A_{3\omega}+2}} B(A_{0\omega},C_{0\omega})B(B_{0\omega},A_{0\omega}+C_{0\omega})-(\hat{\theta}_{3\omega})^2 \quad (28)$$

$$\rho(\hat{p}_{1\omega})=\Omega_\omega^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1\omega}}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}}B(B_{0\omega},C_{0\omega})B(A_{0\omega}+2,B_{0\omega}+C_{0\omega})-(\hat{p}_{1\omega})^2 \quad (29)$$

$$\rho(\hat{p}_{2\omega})=\Omega_\omega^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1\omega}}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}}B(A_{0\omega},C_{0\omega})B(B_{0\omega}+2,A_{0\omega}+C_{0\omega})-(\hat{p}_{2\omega})^2 \quad (30)$$

Where $\omega=1$ for the UP, $\omega=2$ for the JP, $\omega=3$ for the Gamma prior and $\omega=4$ for the Exponential prior.

5.2. The bayes estimators and posterior risks using the UP, the JP, the GP and the EP under PLF

Norstrom discussed an asymmetric PLF and also introduced a special case of general class of PLFs, which is defined as $L(\theta,d)=\frac{(\theta-d)^2}{d}$. The Bayes estimator and posterior risk are: $\hat{d}=\left\{E_{\theta|x}\left(\theta^2\right)\right\}^{\frac{1}{2}}$, $\rho(\hat{d})=2\left\{E_{\theta|x}\left(\theta^2\right)\right\}^{\frac{1}{2}}-2E_{\theta|x}(\theta)$, respectively. The respective marginal posterior distribution yields the Bayes estimators and posterior risk using the UP, the JP and the IP for parameters $\theta_1,\theta_2,\theta_3,p_1$ and p_2 under PLF as:

$$\hat{\theta}_{1\omega}=\left\{\Omega_\omega^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega}+2)}{B_{1\omega}^{A_{1\omega}+2}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}}B(A_{0\omega},C_{0\omega})B(B_{0\omega},A_{0\omega}+C_{0\omega})\right\}^{\frac{1}{2}} \quad (31)$$

$$\hat{\theta}_{2\omega}=\left\{\Omega_\omega^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1\omega}}}\frac{\Gamma(A_{2\omega}+2)}{B_{2\omega}^{A_{2\omega}+2}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}}B(A_{0\omega},C_{0\omega})B(B_{0\omega},A_{0\omega}+C_{0\omega})\right\}^{\frac{1}{2}} \quad (32)$$

$$\hat{\theta}_{3\omega}=\left\{\Omega_\omega^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1\omega}}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega}+2)}{B_{3\omega}^{A_{3\omega}+2}}B(A_{0\omega},C_{0\omega})B(B_{0\omega},A_{0\omega}+C_{0\omega})\right\}^{\frac{1}{2}} \quad (33)$$

$$\hat{p}_{1\omega}=\left\{\Omega_\omega^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1\omega}}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}}B(B_{0\omega},C_{0\omega})B(A_{0\omega}+2,B_{0\omega}+C_{0\omega})\right\}^{\frac{1}{2}} \quad (34)$$

$$\hat{p}_{2\omega}=\left\{\Omega_\omega^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1\omega}}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}}B(A_{0\omega},C_{0\omega})B(B_{0\omega}+2,A_{0\omega}+C_{0\omega})\right\}^{\frac{1}{2}} \quad (35)$$

$$\rho(\hat{\theta}_{1\omega})=2\left\{\Omega_\omega^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega}+2)}{B_{1\omega}^{A_{1\omega}+2}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}}B(A_{0\omega},C_{0\omega})B(B_{0\omega},A_{0\omega}+C_{0\omega})\right\}^{\frac{1}{2}}$$
$$-2\left\{\Omega_\omega^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega}+1)}{B_{1\omega}^{A_{1\omega}+1}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}}B(A_{0\omega},C_{0\omega})B(B_{0\omega},A_{0\omega}+C_{0\omega})\right\} \quad (36)$$

$$\rho(\hat{\theta}_{2\omega})=2\left\{\Omega_\omega^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1\omega}}}\frac{\Gamma(A_{2\omega}+2)}{B_{2\omega}^{A_{2\omega}+2}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}}B(A_{0\omega},C_{0\omega})B(B_{0\omega},A_{0\omega}+C_{0\omega})\right\}^{\frac{1}{2}}$$
$$-2\left\{\Omega_\omega^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1\omega}}}\frac{\Gamma(A_{2\omega}+1)}{B_{2\omega}^{A_{2\omega}+1}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}}B(A_{0\omega},C_{0\omega})B(B_{0\omega},A_{0\omega}+C_{0\omega})\right\} \quad (37)$$

$$\rho(\hat{\theta}_{3\omega})=2\left\{\Omega_\omega^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1\omega}}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega}+2)}{B_{3\omega}^{A_{3\omega}+2}}B(A_{0\omega},C_{0\omega})B(B_{0\omega},A_{0\omega}+C_{0\omega})\right\}^{\frac{1}{2}}$$
$$-2\left\{\Omega_\omega^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1\omega}}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega}+1)}{B_{3\omega}^{A_{3\omega}+1}}B(A_{0\omega},C_{0\omega})B(B_{0\omega},A_{0\omega}+C_{0\omega})\right\} \quad (38)$$

$$\rho(\hat{p}_{1\omega})=$$

$$2\left\{\Omega_{\omega}^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^{i}\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1\omega}}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}}B(B_{0\omega},C_{0\omega})B(A_{0\omega}+2,B_{0\omega}+C_{0\omega})\right\}^{\frac{1}{2}} \quad (39)$$

$$-2\left\{\Omega_{\omega}^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^{i}\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1\omega}}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}}B(B_{0\omega},C_{0\omega})B(A_{0\omega}+1,B_{0\omega}+C_{0\omega})\right\}$$

$$\rho(\hat{p}_{2\omega})=$$

$$2\left\{\Omega_{\omega}^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^{i}\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1\omega}}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}}B(A_{0\omega},C_{0\omega})B(B_{0\omega}+2,A_{0\omega}+C_{0\omega})\right\}^{\frac{1}{2}} \quad (40)$$

$$-2\left\{\Omega_{\omega}^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^{i}\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1\omega}}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}}B(A_{0\omega},C_{0\omega})B(B_{0\omega}+1,A_{0\omega}+C_{0\omega})\right\}$$

Where $\omega=1$ for the UP, $\omega=2$ for the JP, $\omega=3$ for the Gamma prior and $\omega=4$ for the Exponential prior.

5.3. The bayes estimators and posterior risks using the UP, the JP, the GP and the EP under DLF

DeGroot (2005) introduced the asymmetric loss function, $L(\theta,d)=\left(\frac{\theta-d}{d}\right)^{2}$ known as DLF. The Bayes estimator and its posterior risk under DLF are: $\hat{d}=\frac{E_{\theta|x}(\theta^{2})}{E_{\theta|x}(\theta)}$ and $\rho(\hat{d})=1-\frac{\{E_{\theta|x}(\theta)\}^{2}}{E_{\theta|x}(\theta^{2})}$, respectively. The Bayes estimators and posterior risks using the UP,the JP and the IP for parameters $\theta_1,\theta_2,\theta_3,p_1$ and p_2 under DLF are:

$$\hat{\theta}_{1\omega}=\frac{\left\{\Omega_{\omega}^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^{i}\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega}+2)}{B_{1\omega}^{A_{1\omega}+2}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}}B(A_{0\omega},C_{0\omega})B(B_{0\omega},A_{0\omega}+C_{0\omega})\right\}}{\left\{\Omega_{\omega}^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^{i}\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega}+1)}{B_{1\omega}^{A_{1w}+1}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}}B(A_{0\omega},C_{0\omega})B(B_{0\omega},A_{0\omega}+C_{0\omega})\right\}} \quad (41)$$

$$\hat{\theta}_{2\omega}=\frac{\left\{\Omega_{\omega}^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^{i}\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1\omega}}}\frac{\Gamma(A_{2\omega}+2)}{B_{2\omega}^{A_{2\omega}+2}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}}B(A_{0\omega},C_{0\omega})B(B_{0\omega},A_{0\omega}+C_{0\omega})\right\}}{\left\{\Omega_{\omega}^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^{i}\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1w}}}\frac{\Gamma(A_{2\omega}+1)}{B_{2\omega}^{A_{2\omega}+1}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}}B(A_{0\omega},C_{0\omega})B(B_{0\omega},A_{0\omega}+C_{0\omega})\right\}} \quad (42)$$

$$\hat{\theta}_{3\omega}=\frac{\left\{\Omega_{\omega}^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^{i}\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1\omega}}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega}+2)}{B_{3\omega}^{A_{3\omega}+2}}B(A_{0\omega},C_{0\omega})B(B_{0\omega},A_{0\omega}+C_{0\omega})\right\}}{\left\{\Omega_{\omega}^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^{i}\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1w}}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega}+1)}{B_{3\omega}^{A_{3\omega}+1}}B(A_{0\omega},C_{0\omega})B(B_{0\omega},A_{0\omega}+C_{0\omega})\right\}} \quad (43)$$

$$\hat{p}_{1\omega}=\frac{\left\{\Omega_{\omega}^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^{i}\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1\omega}}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}}B(B_{0\omega},C_{0\omega})B(A_{0\omega}+2,B_{0\omega}+C_{0\omega})\right\}}{\left\{\Omega_{\omega}^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^{i}\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1w}}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}}B(B_{0\omega},C_{0\omega})B(A_{0\omega}+1,B_{0\omega}+C_{0\omega})\right\}} \quad (44)$$

$$\hat{p}_{2\omega}=\frac{\left\{\Omega_{\omega}^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^{i}\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1\omega}}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}}B(A_{0\omega},C_{0\omega})B(B_{0\omega}+2,A_{0\omega}+C_{0\omega})\right\}}{\left\{\Omega_{\omega}^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^{i}\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1w}}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}}B(A_{0\omega},C_{0\omega})B(B_{0\omega}+1,A_{0\omega}+C_{0\omega})\right\}} \quad (45)$$

$$\rho\left(\hat{\theta}_1\right)=1-\frac{\left\{\Omega_{\omega}^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}\frac{(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega}+1)}{B_{1\omega}^{A_{1\omega}+1}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}}}{B(A_{0\omega},C_{0\omega})B(B_{0\omega},A_{0\omega}+C_{0\omega})}\right\}^2}{\left\{\Omega_{\omega}^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}\frac{(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega}+2)}{B_{1\omega}^{A_{1\omega}+2}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}}}{B(A_{0\omega},C_{0\omega})B(B_{0\omega},A_{0\omega}+C_{0\omega})}\right\}} \quad (46)$$

$$\rho\left(\hat{\theta}_2\right)=1-\frac{\left\{\Omega_{\omega}^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}\frac{(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1\omega}}}\frac{\Gamma(A_{2\omega}+1)}{B_{2\omega}^{A_{2\omega}+1}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}}}{B(A_{0\omega},C_{0\omega})B(B_{0\omega},A_{0\omega}+C_{0\omega})}\right\}^2}{\left\{\Omega_{\omega}^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}\frac{(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1\omega}}}\frac{\Gamma(A_{2\omega}+2)}{B_{2\omega}^{A_{2\omega}+2}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}}}{B(A_{0\omega},C_{0\omega})B(B_{0\omega},A_{0\omega}+C_{0\omega})}\right\}} \quad (47)$$

$$\rho\left(\hat{\theta}_3\right)=1-\frac{\left\{\Omega_{\omega}^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}\frac{(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1\omega}}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega}+1)}{B_{3\omega}^{A_{3\omega}+1}}}{B(A_{0\omega},C_{0\omega})B(B_{0\omega},A_{0\omega}+C_{0\omega})}\right\}^2}{\left\{\Omega_{\omega}^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}\frac{(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1\omega}}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega}+2)}{B_{3\omega}^{A_{3\omega}+2}}}{B(A_{0\omega},C_{0\omega})B(B_{0\omega},A_{0\omega}+C_{0\omega})}\right\}} \quad (48)$$

$$\rho\left(\hat{p}_1\right)=1-\frac{\left\{\Omega_{\omega}^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}\frac{(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1\omega}}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}}}{B(B_{0\omega},C_{0\omega})B(A_{0\omega}+1,B_{0\omega}+C_{0\omega})}\right\}^2}{\left\{\Omega_{\omega}^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}\frac{(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1\omega}}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}}}{B(B_{0\omega},C_{0\omega})B(A_{0\omega}+2,B_{0\omega}+C_{0\omega})}\right\}} \quad (49)$$

$$\rho\left(\hat{p}_2\right)=1-\frac{\left\{\Omega_{\omega}^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}\frac{(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1\omega}}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}1}}}{B(A_{0\omega},C_{0\omega})B(B_{0\omega}+1,A_{0\omega}+C_{0\omega})}\right\}^2}{\left\{\Omega_{\omega}^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}\frac{(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1\omega})}{B_{1\omega}^{A_{1\omega}}}\frac{\Gamma(A_{2\omega})}{B_{2\omega}^{A_{2\omega}}}\frac{\Gamma(A_{3\omega})}{B_{3\omega}^{A_{3\omega}}}}{B(A_{0\omega},C_{0\omega})B(B_{0\omega}+2,A_{0\omega}+C_{0\omega})}\right\}} \quad (50)$$

6. Elicitation of hyper-parameters

Elicitation is the key task for subjective Bayesian. The whole procedure for quantifying the prior information in the form of prior distribution is precisely known as the elicitation. Aslam [3] proposed different methods of elicitation based on prior predictive distribution for the elicitation of the hyper-parameters. In this study, we use the method of elicitation using prior predictive distribution based on the predictive probabilities. In this method, confidence levels of the prior predictive are obtained for the particular intervals of the random variable's'. The set of hyper parameters, for which the difference between the elicited probabilities and the expert predictive probabilities is minimum, is considered.

6.1. Elicitation of hyper-parameters using the gamma prior

For eliciting the hyper-parameters, prior predictive distribution (PPD) is used. The PPD for a random variable X is:

$$p(x)=\int_{\varphi} p\langle x|\varphi\rangle \pi_3(\varphi)\,d\varphi \quad (51)$$

$$p(x)=\frac{2}{(a+b+c)x^3}\left[\frac{aa_1b_1^{a_1}}{\left(b_1+x^{-2}\right)^{a_1+1}}+\frac{ba_2b_2^{a_2}}{\left(b_2+x^{-2}\right)^{a_2+1}}+\frac{ca_3b_3^{a_3}}{\left(b_3+x^{-2}\right)^{a_3+1}}\right] \quad (52)$$

We choose the prior predictive probabilities, satisfying the laws of probability, to elicit the hyper parameters of the prior density. Using the prior predictive distribution given in (51), we consider the nine intervals (0,1), (1,2), (2,3), (3,4), (4,5), (5,6), (6,7), (7,8) and (8,9) with probabilities 0.18, 0.29, 0.20, 0.11, 0.07, 0.04, 0.03, 0.02 and 0.01 respectively, given an expert opinion. The following nine equations are derived from the given information using (51) as:

$$\frac{2}{(a+b+c)x^3}\int_0^1\left[\frac{aa_1b_1^{a_1}}{\left(b_1+x^{-2}\right)^{a_1+1}}+\frac{ba_2b_2^{a_2}}{\left(b_2+x^{-2}\right)^{a_2+1}}+\frac{ca_3b_3^{a_3}}{\left(b_3+x^{-2}\right)^{a_3+1}}\right]dx=0.18 \quad (53)$$

$$\frac{2}{(a+b+c)x^3}\int_1^2\left[\frac{aa_1b_1^{a_1}}{\left(b_1+x^{-2}\right)^{a_1+1}}+\frac{ba_2b_2^{a_2}}{\left(b_2+x^{-2}\right)^{a_2+1}}+\frac{ca_3b_3^{a_3}}{\left(b_3+x^{-2}\right)^{a_3+1}}\right]dx=0.29 \quad (54)$$

$$\frac{2}{(a+b+c)x^3}\int_2^3\left[\frac{aa_1b_1^{a_1}}{\left(b_1+x^{-2}\right)^{a_1+1}}+\frac{ba_2b_2^{a_2}}{\left(b_2+x^{-2}\right)^{a_2+1}}+\frac{ca_3b_3^{a_3}}{\left(b_3+x^{-2}\right)^{a_3+1}}\right]dx=0.20 \quad (55)$$

$$\frac{2}{(a+b+c)x^3}\int_3^4\left[\frac{aa_1b_1^{a_1}}{\left(b_1+x^{-2}\right)^{a_1+1}}+\frac{ba_2b_2^{a_2}}{\left(b_2+x^{-2}\right)^{a_2+1}}+\frac{ca_3b_3^{a_3}}{\left(b_3+x^{-2}\right)^{a_3+1}}\right]dx=0.11 \quad (56)$$

$$\frac{2}{(a+b+c)x^3}\int_4^5\left[\frac{aa_1b_1^{a_1}}{\left(b_1+x^{-2}\right)^{a_1+1}}+\frac{ba_2b_2^{a_2}}{\left(b_2+x^{-2}\right)^{a_2+1}}+\frac{ca_3b_3^{a_3}}{\left(b_3+x^{-2}\right)^{a_3+1}}\right]dx=0.07 \quad (57)$$

$$\frac{2}{(a+b+c)x^3}\int_5^6\left[\frac{aa_1b_1^{a_1}}{\left(b_1+x^{-2}\right)^{a_1+1}}+\frac{ba_2b_2^{a_2}}{\left(b_2+x^{-2}\right)^{a_2+1}}+\frac{ca_3b_3^{a_3}}{\left(b_3+x^{-2}\right)^{a_3+1}}\right]dx=0.04 \quad (58)$$

$$\frac{2}{(a+b+c)x^3}\int_6^7\left[\frac{aa_1b_1^{a_1}}{\left(b_1+x^{-2}\right)^{a_1+1}}+\frac{ba_2b_2^{a_2}}{\left(b_2+x^{-2}\right)^{a_2+1}}+\frac{ca_3b_3^{a_3}}{\left(b_3+x^{-2}\right)^{a_3+1}}\right]dx=0.03 \quad (59)$$

$$\frac{2}{(a+b+c)x^3}\int_7^8\left[\frac{aa_1b_1^{a_1}}{\left(b_1+x^{-2}\right)^{a_1+1}}+\frac{ba_2b_2^{a_2}}{\left(b_2+x^{-2}\right)^{a_2+1}}+\frac{ca_3b_3^{a_3}}{\left(b_3+x^{-2}\right)^{a_3+1}}\right]dx=0.02 \quad (60)$$

$$\frac{2}{(a+b+c)x^3}\int_8^9\left[\frac{aa_1b_1^{a_1}}{\left(b_1+x^{-2}\right)^{a_1+1}}+\frac{ba_2b_2^{a_2}}{\left(b_2+x^{-2}\right)^{a_2+1}}+\frac{ca_3b_3^{a_3}}{\left(b_3+x^{-2}\right)^{a_3+1}}\right]dx=0.01 \quad (61)$$

For eliciting the hyper parameters $a_1, a_2, a_3, b_1, b_2, b_3, a, b$ and c, the equations are simultaneously solved through the computer program developed in SAS package using the 'PROC SYSLIN' command, the values of the hyper parameters are found to be 0.9965,2.1454,3,0.4334,0.2554,0.7450,2.4727,1.5 and 1.0269 respectively.

6.2. Elicitation of hyper-parameters using the Exponential Prior

The PPD using Exponential prior for a random variable X is given by:

$$p(x)=\int_{\varphi} p\langle x|\varphi\rangle \pi_4(\varphi)d\varphi \quad (62)$$

$$p(x)=\frac{2}{x^3(a+b+c)}\left[\frac{aw_1}{\left(w_1+x^{-2}\right)^2}+\frac{bw_2}{\left(w_2+x^{-2}\right)^2}+\frac{cw_3}{\left(w_3+x^{-2}\right)^2}\right] \quad (63)$$

Using similar criteria defined as above for exponential prior, the values of the hyper-parameters w_1, w_2, w_3, a, b and c are 2.9607, 1.9215, 0.9942, 1.1177, 0.7033 and 0.50.

7. Limiting expressions

Letting $t \to \infty$, all the observations that are incorporated in our analysis are uncensored and therefore r tends to n, r_1 tends to the unknown n_1, r_2 tends to the unknown n_2 and r_3 tends to the unknown n_3. As a result, the amount of information contained in the sample is increasing, which consequently results in the reduction of the variances of the estimates. The limiting (complete sample) expressions for Bayes estimators and posterior risks using the UP, the JP and the IP under SELF, PLF and DLF are given in the Tables 1-7.

Table 1: Limiting Expressions for the Bayes Estimators as $t \to \infty$ Using the UP, the JP and the IP under SELF

	Bayes Estimators			
Parameters	UP	JP	Gamma prior	Exponential Prior
θ_1	$\frac{n_1+1}{\sum_{k=1}^{n_1} x_{1k}^{-2}}$	$\frac{n_1}{\sum_{k=1}^{n_1} x_{1k}^{-2}}$	$\frac{n_1+a_1}{\sum_{k=1}^{n_1} x_{1k}^{-2}+b_1}$	$\frac{n_1+1}{\left(\sum_{k=1}^{n_1} x_{1k}^{-2}+w_1\right)}$
θ_2	$\frac{n_2+1}{\sum_{k=1}^{n_2} x_{2k}^{-2}}$	$\frac{n_2}{\sum_{k=1}^{n_2} x_{2k}^{-2}}$	$\frac{n_2+a_2}{\sum_{k=1}^{n_2} x_{2k}^{-2}+b_2}$	$\frac{n_2+1}{\left(\sum_{k=1}^{n_2} x_{2k}^{-2}+w_2\right)}$
θ_3	$\frac{n_3+1}{\sum_{k=1}^{n_3} x_{3k}^{-2}}$	$\frac{n_3}{\sum_{k=1}^{n_3} x_{3k}^{-2}}$	$\frac{n_3+a_3}{\sum_{k=1}^{n_3} x_{3k}^{-2}+b_3}$	$\frac{n_3+1}{\left(\sum_{k=1}^{n_3} x_{3k}^{-2}+w_3\right)}$
p_1	$\frac{n_1+1}{n+3}$	$\frac{n_1+1}{n+3}$	$\frac{n_1+a}{n+a+b+c}$	$\frac{n_1+a}{n+a+b+c}$

p_2	$\frac{n_2+1}{n+3}$	$\frac{n_2+1}{n+3}$	$\frac{n_2+b}{n+a+b+c}$	$\frac{n_2+b}{n+a+b+c}$

Table 2: Limiting Expressions for the Bayes Estimators as $t \to \infty$ Using the UP and the JP under PLF

Parameters	Bayes Estimators UP	JP
θ_1	$\frac{\left[(n_1+1)(n_1+2)\right]^{1/2}}{\sum_{k=1}^{n_1} x_{1k}^{-2}}$	$\frac{\left[n_1(n_1+1)\right]^{1/2}}{\sum_{k=1}^{n_1} x_{1k}^{-2}}$
θ_2	$\frac{\left[(n_2+1)(n_2+2)\right]^{1/2}}{\sum_{k=1}^{n_2} x_{2k}^{-2}}$	$\frac{\left[n_2(n_2+2)\right]^{1/2}}{\sum_{k=1}^{n_2} x_{2k}^{-2}}$
θ_3	$\frac{\left[(n_3+1)(n_3+2)\right]^{1/2}}{\sum_{k=1}^{n_3} x_{3k}^{-2}}$	$\frac{\left[n_3(n_3+2)\right]^{1/2}}{\sum_{k=1}^{n_3} x_{3k}^{-2}}$
p_1	$\left[\frac{(n_1+1)(n_1+2)}{(n+4)}\right]^{1/2}$	$\left[\frac{(n_1+1)(n_1+2)}{(n+4)}\right]^{1/2}$
p_2	$\left[\frac{(n_2+1)(n_2+2)}{(n+4)}\right]^{1/2}$	$\left[\frac{(n_2+1)(n_2+2)}{(n+4)}\right]^{1/2}$

Table 3: Limiting Expressions for the Bayes Estimators as $t \to \infty$ Using the GP and the EP under PLF

Parameters	Bayes Estimators Gamma Prior	Exponential Prior
θ_1	$\frac{\left[(n_1+a_1)(n_1+a_1+1)\right]^{1/2}}{\sum_{k=1}^{n_1} x_{1k}^{-2}+b_1}$	$\frac{\left[(n_1+1)(n_1+2)\right]^{1/2}}{\sum_{k=1}^{n_1} x_{1k}^{-2}+w_1}$
θ_2	$\frac{\left[(n_2+a_2)(n_2+a_2+1)\right]^{1/2}}{\sum_{k=1}^{n_2} x_{2k}^{-2}+b_2}$	$\frac{\left[(n_2+1)(n_2+2)\right]^{1/2}}{\sum_{k=1}^{n_2} x_{2k}^{-2}+w_2}$
θ_3	$\frac{\left[(n_3+a_3)(n_3+a_3+1)\right]^{1/2}}{\sum_{k=1}^{n_3} x_{3k}^{-2}+b_3}$	$\frac{\left[(n_3+1)(n_3+2)\right]^{1/2}}{\sum_{k=1}^{n_3} x_{3k}^{-2}+w_3}$
p_1	$\left[\frac{(n_1+a)(n_1+a+1)}{(n+a+b+c)(n+a+b+c+1)}\right]^{1/2}$	$\left[\frac{(n_1+a)(n_1+a+1)}{(n+a+b+c)(n+a+b+c+1)}\right]^{1/2}$
p_2	$\left[\frac{(n_2+b)(n_2+b+1)}{(n+a+b+c)(n+a+b+c+1)}\right]^{1/2}$	$\left[\frac{(n_2+b)(n_2+b+1)}{(n+a+b+c)(n+a+b+c+1)}\right]^{1/2}$

Table 4: Limiting Expressions for the Bayes Estimators as $t \to \infty$ Using the UP and the JP, GP and EP under DLF

Parameters	Bayes Estimators UP	JP	Gamma prior	Exponential Prior
θ_1	$\frac{n_1+2}{\sum_{k=1}^{n_1} x_{1k}^{-2}}$	$\frac{n_1}{\sum_{k=1}^{n_1} x_{1k}^{-2}}$	$\frac{n_1+a_1+1}{\sum_{k=1}^{n_1} x_{1k}^{-2}+b_1}$	$\frac{n_1+2}{\left(\sum_{k=1}^{n_1} x_{1k}^{-2}+w_1\right)}$
θ_2	$\frac{n_2+2}{\sum_{k=1}^{n_2} x_{2k}^{-2}}$	$\frac{n_2}{\sum_{k=1}^{n_2} x_{2k}^{-2}}$	$\frac{n_2+a_2+1}{\sum_{k=1}^{n_2} x_{2k}^{-2}+b_2}$	$\frac{n_2+2}{\left(\sum_{k=1}^{n_2} x_{2k}^{-2}+w_2\right)}$
θ_3	$\frac{n_3+2}{\sum_{k=1}^{n_3} x_{3k}^{-2}}$	$\frac{n_3}{\sum_{k=1}^{n_3} x_{3k}^{-2}}$	$\frac{n_3+a_3+1}{\sum_{k=1}^{n_3} x_{3k}^{-2}+b_3}$	$\frac{n_3+2}{\left(\sum_{k=1}^{n_3} x_{3k}^{-2}+w_3\right)}$
p_1	$\frac{n_1+2}{n+4}$	$\frac{n_1+2}{n+4}$	$\frac{n_1+a+1}{n+a+b+c}$	$\frac{n_1+a+1}{n+a+b+c}$
p_2	$\frac{n_2+2}{n+4}$	$\frac{n_2+2}{n+4}$	$\frac{n_2+b+1}{n+a+b+c}$	$\frac{n_2+b+1}{n+a+b+c}$

Table 5: Limiting Expressions for the Posterior Risks as $t \to \infty$ Using the UP and the JP, GP and EP under SELF

Parameters	UP	JP	Gamma prior	Exponential Prior
θ_1	$\frac{n_1+1}{\left(\sum_{k=1}^{n_1} x_{1k}^{-2}\right)^2}$	$\frac{n_1}{\left(\sum_{k=1}^{n_1} x_{1k}^{-2}\right)^2}$	$\frac{n_1+a_1}{\left(\sum_{k=1}^{n_1} x_{1k}^{-2}+b_1\right)^2}$	$\frac{n_1+1}{\left(\sum_{k=1}^{n_1} x_{1k}^{-2}+w_1\right)^2}$
θ_2	$\frac{n_2+1}{\left(\sum_{k=1}^{n_2} x_{2k}^{-2}\right)^2}$	$\frac{n_2}{\left(\sum_{k=1}^{n_2} x_{2k}^{-2}\right)^2}$	$\frac{n_2+a_2}{\left(\sum_{k=1}^{n_2} x_{2k}^{-2}+b_2\right)^2}$	$\frac{n_2+1}{\left(\sum_{k=1}^{n_2} x_{2k}^{-2}+w_2\right)^2}$
θ_3	$\frac{n_3+1}{\left(\sum_{k=1}^{n_3} x_{3k}^{-2}\right)^2}$	$\frac{n_3}{\left(\sum_{k=1}^{n_3} x_{3k}^{-2}\right)^2}$	$\frac{n_3+a_3}{\left(\sum_{k=1}^{n_3} x_{3k}^{-2}+b_3\right)^2}$	$\frac{n_3+1}{\left(\sum_{k=1}^{n_3} x_{3k}^{-2}+w_3\right)^2}$
p_1	$\frac{(n_1+1)(n_2+n_3+2)}{(n+3)^2(n+4)}$	$\frac{(n_1+1)(n_2+n_3+2)}{(n+3)^2(n+4)}$	$\frac{(n_1+a)(n_2+n_3+b+c)}{(n+a+b+c)^2(n+a+b+c+1)}$	$\frac{(n_1+a)(n_2+n_3+b+c)}{(n+a+b+c)^2(n+a+b+c+1)}$
p_2	$\frac{(n_2+1)(n_1+n_3+2)}{(n+3)^2(n+4)}$	$\frac{(n_2+1)(n_1+n_3+2)}{(n+3)^2(n+4)}$	$\frac{(n_2+b)(n_1+n_3+b+c)}{(n+a+b+c)^2(n+a+b+c+1)}$	$\frac{(n_2+b)(n_1+n_3+b+c)}{(n+a+b+c)^2(n+a+b+c+1)}$

Table 6: Limiting Expressions for the Posterior Risks as $t \to \infty$ Using the UP and the JP, GP and EP under PLF

Parameters	UP	JP	Gamma prior	Exponential Prior
θ_1	$\frac{2(n_1+1)}{\left(\sum_{k=1}^{n_1} x_{1k}^{-2}\right)}\left\{\frac{(n_1+2)^{1/2}}{(n_1+1)^{1/2}}-1\right\}$	$\frac{2n_1}{\left(\sum_{k=1}^{n_1} x_{1k}^{-2}\right)}\left\{\frac{(n_1+1)^{1/2}}{(n_1)^{1/2}}-1\right\}$	$\frac{2(n_1+a_1)}{\left(\sum_{k=1}^{n_1} x_{1k}^{-2}+b_1\right)}\left\{\frac{(n_1+a_1+1)^{1/2}}{(n_1+a_1)^{1/2}}-1\right\}$	$\frac{2(n_1+1)}{\left(\sum_{k=1}^{n_1} x_{1k}^{-2}+w_1\right)}\left\{\frac{(n_1+2)^{1/2}}{(n_1+1)^{1/2}}-1\right\}$
θ_2	$\frac{2(n_2+1)}{\left(\sum_{k=1}^{n_2} x_{2k}^{-2}\right)}\left\{\frac{(n_2+2)^{1/2}}{(n_2+1)^{1/2}}-1\right\}$	$\frac{2n_2}{\left(\sum_{k=1}^{n_2} x_{2k}^{-2}\right)}\left\{\frac{(n_2+1)^{1/2}}{(n_2)^{1/2}}-1\right\}$	$\frac{2(n_2+a_2)}{\left(\sum_{k=1}^{n_2} x_{2k}^{-2}+b_2\right)}\left\{\frac{(n_2+a_2+1)^{1/2}}{(n_2+a_2)^{1/2}}-1\right\}$	$\frac{2(n_2+1)}{\left(\sum_{k=1}^{n_2} x_{2k}^{-2}+w_2\right)}\left\{\frac{(n_2+2)^{1/2}}{(n_2+1)^{1/2}}-1\right\}$
θ_3	$\frac{2(n_3+1)}{\left(\sum_{k=1}^{n_3} x_{3k}^{-2}\right)}\left\{\frac{(n_3+2)^{1/2}}{(n_3+1)^{1/2}}-1\right\}$	$\frac{2n_3}{\left(\sum_{k=1}^{n_3} x_{3k}^{-2}\right)}\left\{\frac{(n_3+1)^{1/2}}{(n_3)^{1/2}}-1\right\}$	$\frac{2(n_3+a_3)}{\left(\sum_{k=1}^{n_3} x_{3k}^{-2}+b_3\right)}\left\{\frac{(n_3+a_3+1)^{1/2}}{(n_3+a_3)^{1/2}}-1\right\}$	$\frac{2(n_3+1)}{\left(\sum_{k=1}^{n_3} x_{3k}^{-2}+w_3\right)}\left\{\frac{(n_3+2)^{1/2}}{(n_3+1)^{1/2}}-1\right\}$
p_1	$\frac{2(n_1+1)}{(n+3)}\left\{\frac{\left(\frac{(n_1+2)}{(n_1+1)}\right)^{1/2}}{\left(\frac{(n+4)}{(n+3)}\right)^{1/2}}-1\right\}$	$\frac{2(n_1+1)}{(n+3)}\left\{\frac{\left(\frac{(n_1+2)}{(n_1+1)}\right)^{1/2}}{\left(\frac{(n+4)}{(n+3)}\right)^{1/2}}-1\right\}$	$\frac{2(n_1+a)}{(n+a+b+c)}\left\{\frac{\left(\frac{(n_1+a+1)}{(n_1+a)}\right)^{1/2}}{\left(\frac{(n+a+b+c+1)}{(n+a+b+c)}\right)^{1/2}}\right\}$	$\frac{2(n_1+a)}{(n+a+b+c)}\left\{\frac{\left(\frac{(n_1+a+1)}{(n_1+a)}\right)^{1/2}}{\left(\frac{(n+a+b+c+1)}{(n+a+b+c)}\right)^{1/2}}-1\right\}$
p_2	$\frac{2(n_2+1)}{(n+3)}\left\{\frac{\left(\frac{(n_2+2)}{(n_2+1)}\right)^{1/2}}{\left(\frac{(n+4)}{(n+3)}\right)^{1/2}}-\right\}$	$\frac{2(n_2+1)}{(n+3)}\left\{\frac{\left(\frac{(n_2+2)}{(n_2+1)}\right)^{1/2}}{\left(\frac{(n+4)}{(n+3)}\right)^{1/2}}-\right\}$	$\frac{2(n_2+b)}{(n+a+b+c)}\left\{\frac{\left(\frac{(n_2+b+1)}{(n_1+b)}\right)^{1/2}}{\left(\frac{(n+a+b+c+1)}{(n+a+b+c)}\right)^{1/2}}\right\}$	$\frac{2(n_2+b)}{(n+a+b+c)}\left\{\frac{\left(\frac{(n_2+b+1)}{(n_1+b)}\right)^{1/2}}{\left(\frac{(n+a+b+c+1)}{(n+a+b+c)}\right)^{1/2}}-1\right\}$

Table 7: Limiting Expressions for the Posterior Risks as $t \to \infty$ Using the UP and the JP, GP and EP under DLF

Parameters	Posterior Risks UP	JP	Gamma prior	Exponential Prior
θ_1	$\frac{1}{n_1+2}$	$\frac{1}{n_1+1}$	$\frac{1}{n_1+a_1+1}$	$\frac{1}{n_1+2}$
θ_2	$\frac{1}{n_2+2}$	$\frac{1}{n_2+1}$	$\frac{1}{n_2+a_2+1}$	$\frac{1}{n_2+2}$
θ_3	$\frac{1}{n_3+2}$	$\frac{1}{n_3+1}$	$\frac{1}{n_3+a_3+1}$	$\frac{1}{n_3+2}$
p_1	$\frac{(n_2+n_3+2)}{(n_1+2)(n+3)}$	$\frac{(n_2+n_3+2)}{(n_1+2)(n+3)}$	$\frac{(n_2+n_3+b+c)}{(n_1+a+1)(n+a+b+c)}$	$\frac{(n_2+n_3+b+c)}{(n_1+a+1)(n+a+b+c)}$

p_2	$\frac{(n_1+n_3+2)}{(n_2+2)(n+3)}$	$\frac{(n_1+n_3+2)}{(n_2+2)(n+3)}$	$\frac{(n_1+n_3+a+c)}{(n_2+b+1)(n+a+b+c)}$	$\frac{(n_1+n_3+a+c)}{(n_2+b+1)(n+a+b+c)}$

8. Simulation study

A thorough simulation study was conducted in order to investigate the performance of the Bayes estimators, impact of sample size and censoring rate in the fit of the model. Samples of sizes n=25, 50,100,200 are generated from a 3-component mixture of the inverse Rayleigh distributions with different set of the parametric values $\theta_1, \theta_2, \theta_3$, $p_1 \& p_2$ fixed as $(\theta_1, \theta_2, \theta_3, p_1, p_2) =$ (0.25, 0.50, 0.75, 0.20, 0.65), (0.75, 0.50, 0.25, 0.65, 0.20), (0.50, 0.50, 0.50, 0.40, 0.40).For fixed sample size, test termination time and set of parameters, the simulation is repeated 1000 times and the results are then averaged. Sample of sizes $p_1 n, p_2 n$ and $(1-p_1-p_2)n$ are chosen randomly from first component density $f_1(x;\theta_1)$, second component density $f_2(x;\theta_2)$ and third component density $f_3(x;\theta_3)$, respectively. The observations which are greater than a fixed t are declared as censored observations. For each t only failures are identified either as a member of subpopulation-I or subpopulation-II or subpopulation-III. Based on each sample size, the Bayes estimators (BEs) and Posterior risks are computed using the non-informative and informative priors (IP) under SELF, PLF and DLF. In order to evaluate the impact of test termination time on Bayes estimators, the Type-I right censoring scheme is used for fixed test termination time t=15 and 20. For For each of the 1000 samples, the Bayes estimators and Posterior risks were computed using a routine in Mathematica 10.0 and the results are presented in Table 8-19 given below. The simulation study (appendix) provides us some interesting properties of the Bayes estimates. The properties are highlighted in terms of sample sizes, size of mixing proportion parameters, different loss functions and censoring rates. It is observed that due to censoring, the posterior risks of all the parameters are reduced with an increase in sample size.

It is also observed that Posterior risks using the informative priors (IP) are smaller than the Posterior risks using the UP and the JP for different sample sizes and test termination times. Also, the Posterior risks using the JP are smaller than using the UP for different sample sizes and test termination times. It is also observed that in estimating the component parameters $\theta_1, \theta_2 \& \theta_3$ and Posterior risks are smaller under DLF than under SELF and PLF at different sample sizes and test termination times considered in the study. However, for estimating the mixing proportions, SELF yields smaller Posterior risks than PLF and DLF, at different sample sizes and test termination times. Thus, DLF is more suitable for estimating component parameters and SELF is preferable choice for estimating the proportion parameters.

Table 8: Bayes Estimates (BEs) and Posterior Risks (PRs) of 3-component Mixture of Inverse Rayleigh Distributions Using the UP under SELF, PLF and DLF with θ_1 =0.25, θ_2 =0.50, θ_3 =0.75, p_1 =0.20, p_2 =0.65 and t=15, 20

t	n	Loss Functions		UP				
				$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
15	25	SELF	BE	0.36999	0.55842	1.24979	0.21437	0.60670
			PR	0.30547	0.019635	0.494425	0.005815	0.008244
		PLF	BE	0.40052	0.58594	1.37124	0.22742	0.61358
			PR	0.059500	0.033061	0.23896	0.026320	0.013509
		DLF	BE	0.43951	0.59933	1.49256	0.24147	0.62040
			PR	0.142912	0.055616	0.166733	0.112302	0.011133
	50	SELF	BE	0.30588	0.52882	0.97946	0.20760	0.62252
			PR	0.009512	0.008756	0.125443	0.003050	0.004359
		PLF	BE	0.31619	0.54671	1.00714	0.01474	0.62610
			PR	0.026952	0.016213	0.103557	0.014442	0.006982
		DLF	BE	0.33304	0.54652	1.0667	0.22236	0.62946
			PR	0.083379	0.029445	0.100213	0.066085	0.111326
	100	SELF	BE	0.27924	0.51788	0.85678	0.20391	0.64076
			PR	0.003934	0.004129	0.049580	0.001563	0.002218
		PLF	BE	0.28595	0.51917	0.88042	0.20775	0.64238
			PR	0.013158	0.007785	0.052689	0.007594	0.003458
		DLF	BE	0.29101	0.51902	0.90183	0.21157	0.64416
			PR	0.045492	0.014938	0.058942	0.036225	0.005374
	200	SELF	BE	0.26387	0.50864	0.79626	0.20201	0.64531
			PR	0.001746	0.001993	0.021177	0.000791	0.001124
		PLF	BE	0.26981	0.50931	0.80425	0.20398	0.64608
			PR	0.006468	0.003869	0.025390	0.003898	0.001742
		DLF	BE	0.26756	0.51363	0.82888	0.20591	0.64701
			PR	0.023830	0.007582	0.031322	0.019021	0.002693
20	25	SELF	BE	0.37718	0.57244	0.12547	0.21431	0.60720
			PR	0.032674	0.020692	0.451357	0.005812	0.008236
		PLF	BE	0.40654	0.57876	1.36121	0.22738	0.61381
			PR	0.060386	0.032632	0.237261	0.026308	0.013489
		DLF	BE	0.43914	0.60399	1.46604	0.24405	0.62066
			PR	0.14294	0.055588	0.166887	0.112331	0.021859
	50	SELF	BE	0.30138	0.52995	0.97660	0.20755	0.62248
			PR	0.009266	0.008782	0.1226	0.003048	0.004357
		PLF	BE	0.31873	0.54206	1.025	0.21476	0.62602
			PR	0.027152	0.016072	0.105259	0.014434	0.006979
		DLF	BE	0.32664	0.55207	1.08674	0.22224	0.62963
			PR	0.083373	0.029424	0.100125	0.066078	0.011111
	100	SELF	BE	0.27636	0.51432	0.86003	0.20923	0.64073
			PR	0.003826	0.004073	0.049861	0.001562	0.002216
		PLF	BE	0.28315	0.52107	0.88695	0.20773	0.64243
			PR	0.013023	0.007810	0.053023	0.007588	0.003454
		DLF	BE	0.29275	0.52194	0.9132	0.21158	0.64412
			PR	0.045473	0.014935	0.058878	0.0366206	0.005371
	200	SELF	BE	0.26457	0.50856	0.80037	0.20196	0.64534

			PR	0.001754	0.001989	0.021523	0.000791	0.001123
		PLF	BE	0.26517	0.50953	0.80874	0.20395	0.64613
			PR	0.006355	0.003869	0.025508	0.003896	0.001740
		DLF	BE	0.26633	0.51107	0.82657	0.20588	0.64705
			PR	0.023824	0.007579	0.031293	0.019015	0.002691

Table 9: Bayes Estimates (BEs) and Posterior Risks (PRs) of 3-component Mixture of Inverse Rayleigh Distributions Using the JP under SELF, PLF and DLF with θ_1 =0.25, θ_2 =0.50, θ_3 =0.75, p_1 =0.20, p_2 =0.65 and t=15,20

T	n	Loss Functions		JP $\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
15	25	SELF	BE	0.30963	0.53263	1.01429	0.021421	0.60717
			PR	0.06215	0.019036	0.393191	0.005812	0.008241
		PLF	BE	0.37834	0.56682	1.2653	0.24136	0.62062
			PR	0.16693	0.058886	0.200378	0.112441	0.021883
		DLF	BE	0.33951	0.54990	1.14907	0.22745	0.61383
			PR	0.059287	0.032871	0.243465	0.026324	0.013503
	50	SELF	BE	0.28071	0.52157	0.85413	0.20761	0.62240
			PR	0.008876	0.008796	0.105511	0.003050	0.004362
		PLF	BE	0.30399	0.53541	0.97765	0.22218	0.62954
			PR	0.091043	0.030334	0.111295	0.066151	0.011128
		DLF	BE	0.29111	0.51809	0.90623	0.21471	0.62620
			PR	0.027128	0.015831	0.10391	0.014442	0.006981
	100	SELF	BE	0.26338	0.50380	0.81073	0.20386	0.64071
			PR	0.003687	0.003969	0.047572	0.001563	0.002218
		PLF	BE	0.27716	0.51597	0.85778	0.21161	0.64417
			PR	0.047653	0.015165	0.062668	0.036219	0.005375
		DLF	BE	0.26402	0.51463	0.82832	0.20771	0.64250
			PR	0.012736	0.007833	0.052743	0.007593	0.003456
	200	SELF	BE	0.25768	0.50311	0.77881	0.20201	0.64527
			PR	0.001704	0.001962	0.020925	0.000791	0.001125
		PLF	BE	0.26292	0.50793	0.80484	0.20585	0.64709
			PR	0.024419	0.007639	0.032341	0.019028	0.002692
		DLF	BE	0.26077	0.50452	0.78655	0.20393	0.64613
			PR	0.006405	0.003862	0.025635	0.003898	0.001741
20	25	SELF	BE	0.31395	0.53233	0.98164	0.21431	0.60710
			PR	0.027012	0.018976	0.351554	0.005810	0.008234
		PLF	BE	0.37627	0.56269	1.2435	0.24156	0.62036
			PR	0.16664	0.058886	0.200054	0.112225	0.021883
		DLF	BE	0.33418	0.54161	1.09453	0.22753	0.61382
			PR	0.058255	0.032368	0.231823	0.026308	0.013495
	50	SELF	BE	0.27811	0.51494	0.86103	0.20759	0.62265
			PR	0.008795	0.008562	0.110235	0.003049	0.004357
		PLF	BE	0.30469	0.52987	0.97616	0.22222	0.62958
			PR	0.090966	0.030318	0.11121	0.066086	0.011114
		DLF	BE	0.29218	0.52419	0.91168	0.21477	0.62607
			PR	0.027211	0.016015	0.010437	0.014434	0.006977
	100	SELF	BE	0.26350	0.50559	0.81023	0.20390	0.64074
			PR	0.003676	0.003998	0.047006	0.001562	0.002216
		PLF	BE	0.27701	0.51380	0.85305	0.21160	0.64420
			PR	0.047635	0.015159	0.062601	0.036201	0.005369
		DLF	BE	0.26966	0.51164	0.82294	0.20767	0.64251
			PR	0.01301	0.007785	0.052332	0.007590	0.003457
	200	SELF	BE	0.25647	0.50298	0.77572	0.20199	0.64524
			PR	0.001690	0.001962	0.020734	0.000791	0.001124
		PLF	BE	0.26369	0.50642	0.8008	0.20592	0.64743
			PR	0.022338	0.008160	0.032398	0.019576	0.003383
		DLF	BE	0.25990	0.5061	0.77873	0.20395	0.64618
			PR	0.006381	0.003872	0.025364	0.003896	0.001739

Table 10: Bayes Estimates (BEs) and Posterior Risks (PRs) of 3-component Mixture of Inverse Rayleigh Distributions Using the UP under SELF, PLF and DLF with θ_1 =0.75, θ_2 =0.50, θ_3 =0.25, p_1 =0.65, p_2 =0.20 and t=15,20

T	n	Loss Functions		UP $\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
15	25	SELF	BE	0.85579	0.73238	0.52775	0.60699	0.21448
			PR	0.406235	0.112065	0.058728	0.008243	0.005822
		PLF	BE	0.86855	0.80514	0.44510	0.61371	0.22753
			PR	0.048992	0.119603	0.077605	0.013504	0.026338
		DLF	BE	0.89841	0.87655	0.50184	0.62052	0.24160
			PR	0.055616	0.142993	0.166805	0.021888	0.112386
	50	SELF	BE	0.81877	0.63138	0.33623	0.63637	0.21126
			PR	0.020696	0.041191	0.016930	0.004361	0.003145
		PLF	BE	0.83278	0.66050	0.35934	0.63979	0.21952
			PR	0.024302	0.056073	0.041069	0.006836	0.014639
		DLF	BE	0.83440	0.68578	0.38514	0.64312	0.22619
			PR	0.028985	0.083150	0.111127	0.010664	0.065855
	100	SELF	BE	0.77465	0.55633	0.28541	0.64078	0.20388
			PR	0.009245	0.015625	0.005493	0.002218	0.001565
		PLF	BE	0.78571	0.56710	0.29378	0.64241	0.20775

			PR	0.011773	0.026127	0.017561	0.003464	0.007609
		DLF	BE	0.78725	0.57699	0.30415	0.64412	0.21165
			PR	0.014939	0.045515	0.058878	0.005375	0.036248
	200	SELF	BE	0.75863	0.52218	0.26727	0.64529	0.20199
			PR	0.004430	0.006854	0.002391	0.001124	0.000792
		PLF	BE	0.76387	0.53600	0.27137	0.64661	0.20391
			PR	0.00580	0.012865	0.008554	0.001741	0.003902
		DLF	BE	0.76939	0.53901	0.27719	0.64697	0.20596
			PR	0.007575	0.023839	0.031272	0.002707	0.019089
20	25	SELF	BE	0.84632	0.75231	0.43192	0.60696	0.21438
			PR	0.045045	0.128126	0.060103	0.008235	0.005814
		PLF	BE	0.89032	0.82139	0.45736	0.61371	0.22758
			PR	0.050209	0.121871	0.079713	0.013495	0.026318
		DLF	BE	0.90351	0.87620	0.50183	0.62051	0.24152
			PR	0.055609	0.14299	0.166711	0.021880	0.112383
	50	SELF	BE	0.82254	0.62653	0.33300	0.63648	0.21117
			PR	0.020891	0.040124	0.016222	0.004356	0.003141
		PLF	BE	0.82539	0.65125	0.35165	0.63980	0.21855
			PR	0.024079	0.055257	0.040193	0.007829	0.014623
		DLF	BE	0.84585	0.69106	0.37787	0.64328	0.22601
			PR	0.028962	0.08312	0.111019	0.010640	0.065822
	100	SELF	BE	0.77246	0.55071	0.28603	0.64077	0.20392
			PR	0.009179	0.015270	0.005504	0.002216	0.001563
		PLF	BE	0.78018	0.56269	0.29621	0.64248	0.20770
			PR	0.011693	0.025901	0.017699	0.003453	0.007594
		DLF	BE	0.78523	0.58050	0.30725	0.64423	0.21158
			PR	0.014932	0.045494	0.058865	0.005368	0.036227
	200	SELF	BE	0.75959	0.52537	0.26744	0.64531	0.20198
			PR	0.004443	0.006893	0.002385	0.001123	0.000791
		PLF	BE	0.76560	0.53235	0.26889	0.64619	0.20393
			PR	0.005813	0.012764	0.008475	0.001740	0.003898
		DLF	BE	0.76811	0.54007	0.27919	0.64701	0.20592
			PR	0.007580	0.023832	0.031265	0.002691	0.019023

Table 11: Bayes Estimates (BEs) and Posterior Risks (PRs) of 3-component Mixture of Inverse Rayleigh Distributions Using the JP Under SELF, PLF and DLF with θ_1 =0.75, θ_2 =0.50, θ_3 =0.25, p_1 =0.65, p_2 =0.20 and t=15,20

t	n	Loss Functions		JP $\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
15	25	SELF	BE	0.79757	0.61784	0.33901	0.60705	0.21427
			PR	0.428113	0.099036	0.041886	0.008241	0.005817
		PLF	BE	0.82919	0.66490	0.36461	0.61395	0.22745
			PR	0.049535	0.116068	0.077096	0.013490	0.026329
		DLF	BE	0.83832	0.73276	0.41934	0.62059	0.24159
			PR	0.058887	0.166893	0.200269	0.021884	0.112413
	50	SELF	BE	0.78818	0.56316	0.28992	0.63671	0.21098
			PR	0.019662	0.035417	0.014356	0.004356	0.003140
		PLF	BE	0.79900	0.60870	0.31550	0.64004	0.21838
			PR	0.024006	0.056540	0.040743	0.006828	0.014636
		DLF	BE	0.81257	0.62042	0.34540	0.64351	0.22588
			PR	0.029823	0.090776	0.124992	0.010639	0.065927
	100	SELF	BE	0.76217	0.52993	0.26547	0.64077	0.20386
			PR	0.009076	0.014802	0.005040	0.002218	0.001564
		PLF	BE	0.76933	0.53281	0.28053	0.64245	0.20778
			PR	0.011711	0.025713	0.017839	0.003457	0.007600
		DLF	BE	0.76953	0.55396	0.28056	0.64412	0.21163
			PR	0.015166	0.047694	0.062553	0.005376	0.036256
	200	SELF	BE	0.75546	0.51540	0.25853	0.64533	0.20198
			PR	0.004426	0.006824	0.002304	0.001124	0.000792
		PLF	BE	0.76123	0.52745	0.26227	0.64649	0.20399
			PR	0.005826	0.012962	0.008540	0.001741	0.003901
		DLF	BE	0.76297	0.52294	0.26751	0.64702	0.20591
			PR	0.007639	0.024432	0.032285	0.002693	0.019041
20	25	SELF	BE	0.81188	0.62087	0.32657	0.60709	0.21436
			PR	0.044288	0.102377	0.038562	0.008233	0.005813
		PLF	BE	0.82381	0.69815	0.36981	0.61398	0.22747
			PR	0.049212	0.121764	0.078306	0.013485	0.026319
		DLF	BE	0.83634	0.75245	0.40867	0.62071	0.24136
			PR	0.058842	0.166832	0.200084	0.021842	0.112368
	50	SELF	BE	0.79473	0.56875	0.29461	0.63691	0.21079
			PR	0.020005	0.036401	0.015122	0.004351	0.003135
		PLF	BE	0.80552	0.60134	0.31311	0.64007	0.21839
			PR	0.024192	0.055791	0.040429	0.006822	0.014623
		DLF	BE	0.80681	0.61676	0.33511	0.64360	0.22584
			PR	0.029804	0.090682	0.124966	0.010620	0.065846
	100	SELF	BE	0.76200	0.52783	0.2661	0.64074	0.20393
			PR	0.009087	0.014761	0.005080	0.002216	0.001563
		PLF	BE	0.77182	0.54028	0.27744	0.64253	0.20768
			PR	0.011743	0.026069	0.017637	0.003453	0.007594
		DLF	BE	0.77924	0.55162	0.28773	0.64426	0.21154

			PR	0.015159	0.047682	0.062555	0.005369	0.036245
		SELF	BE	0.75675	0351312	0.25800	0.64533	0.20197
			PR	0.004440	0.006774	0.002299	0.001123	0.000791
	200	PLF	BE	0.75775	0.52059	0.26232	0.64624	0.20391
			PR	0.005797	0.012790	0.008539	0.001739	0.003898
		DLF	BE	0.76023	0.52633	0.26704	0.64700	0.20590
			PR	0.007637	0.024413	0.032268	0.002691	0.019023

Table 12: Bayes Estimates (BEs) and Posterior Risks (PRs) of 3-component Mixture of Inverse Rayleigh Distributions Using the UP under SELF, PLF and DLF with θ_1 =0.50, θ_2 =0.50, θ_3 =0.50, p_1 =0.40, p_2 =0.40 and t=15,20

t	n	Loss Functions		UP $\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
15	25	SELF	BE	0.61245	0.60466	0.74088	0.39267	0.39294
			PR	0.03900	0.037474	0.119946	0.008241	0.008243
		PLF	BE	0.64979	0.63962	0.81558	0.40309	0.40323
			PR	0.055428	0.054540	0.121156	0.020713	0.020707
		DLF	BE	0.66753	0.66165	0.85625	0.41389	0.41376
			PR	0.083414	0.083448	0.143066	0.050679	0.050714
	50	SELF	BE	0.55239	0.55340	0.61168	0.39619	0.39618
			PR	0.015402	0.015402	0.038175	0.004440	0.004440
		PLF	BE	0.56319	0.56128	0.64479	0.40178	0.40165
			PR	0.025934	0.025849	0.054938	0.011126	0.011128
		DLF	BE	0.57492	0.58277	0.65920	0.40744	0.40742
			PR	0.045513	0.045516	0.083490	0.027504	0.027507
	100	SELF	BE	0.52296	0.52524	0.54667	0.39799	0.39808
			PR	0.006848	0.006913	0.015003	0.002308	0.002308
		PLF	BE	0.53250	0.53182	0.56370	0.40097	0.40097
			PR	0.012771	0.012754	0.025973	0.005779	0.005779
		DLF	BE	0.54348	0.53928	0.58453	0.40379	0.40398
			PR	0.023850	0.023839	0.045562	0.014374	0.014362
	200	SELF	BE	0.51282	0.51582	0.52255	0.39901	0.39890
			PR	0.003294	0.003335	0.006842	0.001178	0.001178
		PLF	BE	0.51640	0.51630	0.52982	0.4004	0.40048
			PR	0.006327	0.006325	0.012712	0.002948	0.002948
		DLF	BE	0.51501	0.51951	0.53939	0.40203	0.40193
			PR	0.012207	0.012211	0.023849	0.007342	0.007345
20	25	SELF	BE	0.60417	0.60427	0.72072	0.39283	0.39276
			PR	0.037499	0.036972	0.11148	0.008235	0.008235
		PLF	BE	0.62746	0.64000	0.85174	0.40339	0.40293
			PR	0.053446	0.054572	0.126424	0.020686	0.020702
		DLF	BE	0.67759	0.65516	0.84778	0.41387	0.41383
			PR	0.083367	0.083373	0.143013	0.050631	0.050637
	50	SELF	BE	0.55542	0.55120	0.61711	0.39611	0.39632
			PR	0.015552	0.015283	0.039071	0.004435	0.004436
		PLF	BE	0.56801	0.56662	0.64425	0.40186	0.40174
			PR	0.026134	0.026078	0.054935	0.011116	0.011185
		DLF	BE	0.57743	0.58061	0.67661	0.40740	0.40744
			PR	0.045482	0.045478	0.083402	0.027472	0.027469
	100	SELF	BE	0.52417	0.52474	0.5566	0.39808	0.39800
			PR	0.006891	0.006910	0.015519	0.002307	0.002306
		PLF	BE	0.53159	0.53399	0.56688	0.40104	0.40084
			PR	0.012740	0.012804	0.026092	0.005773	0.005775
		DLF	BE	0.53997	0.54307	0.57771	0.40389	0.40384
			PR	0.023822	0.023825	0.045498	0.014345	0.014348
	200	SELF	BE	0.51318	0.51296	0.53037	0.39897	0.39902
			PR	0.003294	0.003292	0.007042	0.001177	0.001177
		PLF	BE	0.51696	0.51482	0.53228	0.40050	0.40047
			PR	0.006328	0.006302	0.012761	0.002944	0.002944
		DLF	BE	0.51761	0.51734	0.54254	0.40201	0.40191
			PR	0.012204	0.012207	0.023836	0.007338	0.007341

Table 13: Bayes Estimates (BEs) and Posterior Risks (PRs) of 3-component Mixture of Inverse Rayleigh Distributions Using the JP under SELF, PLF and DLF with θ_1 =0.50, θ_2 =0.50, θ_3 =0.50, p_1 =0.40, p_2 =0.40 and t=15,20

t	n	Loss Functions		JP $\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
15	25	SELF	BE	0.55202	0.55143	0.61820	0.39262	0.39312
			PR	0.304288	0.033994	0.099941	0.008236	0.008240
		PLF	BE	0.57768	0.58651	0.66873	0.40319	0.40335
			PR	0.053838	0.054654	0.116811	0.020700	0.020697
		DLF	BE	0.60700	0.60329	0.75817	0.41384	0.41388
			PR	0.091001	0.090991	0.166995	0.050675	0.050666
0	50	SELF	BE	0.5207	0.52197	0.54642	0.39629	0.39626
			PR	0.014284	0.014391	0.033356	0.004439	0.004439
		PLF	BE	0.53929	0.53277	0.57225	0.40174	0.40167
			PR	0.026033	0.025724	0.053315	0.011128	0.011130
		DLF	BE	0.55118	0.55527	0.60131	0.40725	0.40754
			PR	0.047703	0.047666	0.091037	0.027522	0.027488

	100	SELF	BE	0.51324	0.51360	0.52275	0.39813	0.39796
			PR	0.006770	0.006782	0.014446	0.002309	0.002308
		PLF	BE	0.51773	0.51549	0.54627	0.40083	0.40091
			PR	0.012727	0.012670	0.026357	0.005781	0.005780
		DLF	BE	0.52333	0.52557	0.55651	0.40382	0.40385
			PR	0.024425	0.024423	0.047693	0.014365	0.014363
	200	SELF	BE	0.50578	0.50577	0.51098	0.39893	0.39908
			PR	0.003242	0.003243	0.006724	0.001178	0.001178
		PLF	BE	0.50892	0.50961	0.51746	0.40024	0.40055
			PR	0.006311	0.006319	0.012721	0.002947	0.002947
		DLF	BE	0.51269	0.51429	0.52613	0.40186	0.40206
			PR	0.012366	0.012360	0.024433	0.007349	0.007343
20	25	SELF	BE	0.55534	0.55329	0.62183	0.39287	0.39269
			PR	0.034468	0.034484	0.100995	0.008235	0.008233
		PLF	BE	0.58354	0.57825	0.68740	0.40313	0.40310
			PR	0.054358	0.053876	0.119805	0.020686	0.020687
		DLF	BE	0.60611	0.61666	0.7374	0.41376	0.41386
			PR	0.091008	0.090984	0.166865	0.050681	0.060660
	50	SELF	BE	0.52536	0.52419	0.55471	0.39626	0.39607
			PR	0.014659	0.014491	0.034555	0.004435	0.004434
		PLF	BE	0.54212	0.53221	0.58980	0.40187	0.40173
			PR	0.026139	0.025669	0.054962	0.011112	0.011147
		DLF	BE	0.54967	0.55470	0.60039	0.40736	0.40744
			PR	0.047657	0.047648	0.090974	0.027480	0.027471
	100	SELF	BE	0.50954	0.51243	0.52142	0.39806	0.39804
			PR	0.006642	0.006740	0.014293	0.002306	0.002306
		PLF	BE	0.51867	0.51672	0.52864	0.40099	0.40091
			PR	0.012735	0.012689	0.025503	0.005773	0.005773
		DLF	BE	0.52224	0.52642	0.55384	0.40387	0.40383
			PR	0.024406	0.024410	0.047664	0.014347	0.014351
	200	SELF	BE	0.50402	0.50327	0.51236	0.39896	0.39904
			PR	0.003218	0.003208	0.006754	0.001177	0.001177
		PLF	BE	0.50618	0.50933	0.51227	0.40051	0.40047
			PR	0.006273	0.006312	0.012584	0.002944	0.002944
		DLF	BE	0.51371	0.51172	0.52289	0.40196	0.40195
			PR	0.012355	0.012356	0.024412	0.007338	0.007339

Table 14: Bayes Estimates (BEs) and Posterior Risks (PRs) of 3-component Mixture of Inverse Rayleigh Distributions Using the GP under SELF, PLF and DLF with θ_1 =0.25, θ_2 =0.50, θ_3 =0.75, p_1 =0.20, p_2 =0.65 and t=15,20

t	n	Loss Functions		GP				
				$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
15	25	SELF	BE	0.37422	0.60723	1.40427	0.24916	0.58307
			PR	0.031562	0.021855	0.348811	0.006041	0.007855
		PLF	BE	0.39114	0.61177	1.50023	0.26108	0.58975
			PR	0.058080	0.032423	0.193981	0.023685	0.013401
		DLF	BE	0.42524	0.62819	1.63483	0.27317	0.59691
			PR	0.143209	0.052262	0.125178	0.088819	0.022555
	50	SELF	BE	0.30387	0.55035	1.07914	0.22686	0.60883
			PR	0.009356	0.009167	0.120313	0.003136	0.004261
		PLF	BE	0.31326	0.55804	1.11408	0.23359	0.61241
			PR	0.026700	0.016007	0.094904	0.013616	0.006975
		DLF	BE	0.32731	0.56343	1.20517	0.24070	0.61586
			PR	0.083404	0.028485	0.083444	0.057429	0.011364
	100	SELF	BE	0.27417	0.52197	0.90843	0.21406	0.63326
			PR	0.003773	0.004124	0.049299	0.001589	0.002195
		PLF	BE	0.28075	0.52459	0.95500	0.21778	0.63483
			PR	0.012921	0.007735	0.051004	0.007360	0.003464
		DLF	BE	0.28967	0.53477	0.97652	0.22150	0.63685
			PR	0.045501	0.014685	0.052786	0.033519	0.005443
	200	SELF	BE	0.26387	0.51071	0.84892	0.20718	0.64142
			PR	0.001741	0.001991	0.022665	0.000798	0.001118
		PLF	BE	0.26313	0.51320	0.85375	0.20913	0.64230
			PR	0.006309	0.003865	0.025358	0.003835	0.001742
		DLF	BE	0.26914	0.51263	0.85354	0.21100	0.64324
			PR	0.02384	0.007516	0.029477	0.018263	0.002710
20	25	SELF	BE	0.36371	0.60520	1.4207	0.24906	0.58327
			PR	0.028910	0.021631	0.353566	0.006036	0.007848
		PLF	BE	0.39854	0.61388	1.50819	0.26086	0.58992
			PR	0.059221	0.032513	0.194804	0.023677	0.013380
		DLF	BE	0.42139	0.63853	1.62286	0.27332	0.59667
			PR	0.14299	0.052258	0.125013	0.088669	0.022551
	50	SELF	BE	0.29891	0.54622	1.08893	0.22681	0.60899
			PR	0.009092	0.009032	0.123026	0.003134	0.004257
		PLF	BE	0.31529	0.55940	1.14757	0.23359	0.61256
			PR	0.026866	0.016039	0.097847	0.013612	0.006969
		DLF	BE	0.32893	0.56474	1.16836	0.24061	0.61607
			PR	0.083396	0.028465	0.084219	0.057423	0.011343
	100	SELF	BE	0.27448	0.52211	0.92324	0.21404	0.63333
			PR	0.003790	0.004128	0.050646	0.001588	0.002193

t	n	Loss Functions		$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
		PLF	BE	0.27966	0.52263	0.95315	0.21774	0.63499
			PR	0.012867	0.007702	0.050910	0.007356	0.003459
		DLF	BE	0.28802	0.53025	0.98208	0.22151	0.63668
			PR	0.045476	0.014684	0.052686	0.033497	0.005442
	200	SELF	BE	0.26560	0.51228	0.83382	0.20718	0.64147
			PR	0.001765	0.002003	0.021861	0.000798	0.001117
		PLF	BE	0.26611	0.51329	0.85924	0.20913	0.64225
			PR	0.006377	0.003865	0.025489	0.003834	0.001742
		DLF	BE	0.26807	0.51419	0.85609	0.21102	0.64318
			PR	0.023826	0.007514	0.029443	0.018248	0.002705

Table 15: Bayes Estimates (BEs) and Posterior Risks (PRs) of 3-component Mixture of Inverse Rayleigh Distributions Using the GP under SELF, PLF and DLF with θ_1 =0.75, θ_2 =0.50, θ_3 =0.25, p_1 =0.65, p_2 =0.20 and t=15,20

t	n	Loss Functions		GP				
				$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
15	25	SELF	BE	0.82927	0.85261	0.55109	0.61556	0.21679
			PR	0.043317	0.128387	0.058983	0.007650	0.005490
		PLF	BE	0.86677	0.90082	0.58593	0.62168	0.22907
			PR	0.048914	0.114347	0.075699	0.012366	0.024618
		DLF	BE	0.88344	0.99082	0.62287	0.62794	0.24218
			PR	0.055646	0.122907	0.125097	0.019803	0.104616
	50	SELF	BE	0.80923	0.67780	0.41199	0.63963	0.21278
			PR	0.020189	0.041850	0.019987	0.004192	0.003050
		PLF	BE	0.82296	0.70081	0.42421	0.64297	0.21985
			PR	0.024038	0.054215	0.039479	0.006535	0.014099
		DLF	BE	0.82619	0.72671	0.45190	0.64627	0.22710
			PR	0.029006	0.075950	0.090939	0.010139	0.063139
	100	SELF	BE	0.76898	0.57705	0.31313	0.64245	0.20487
			PR	0.009131	0.015844	0.005828	0.002172	0.001541
		PLF	BE	0.77271	0.58619	0.32796	0.64420	0.20853
			PR	0.011587	0.025648	0.017506	0.003384	0.007453
		DLF	BE	0.77709	0.59771	0.33184	0.64588	0.21233
			PR	0.014940	0.043272	0.052668	0.005233	0.035415
	200	SELF	BE	0.76255	0.54010	0.28430	0.64604	0.20250
			PR	0.004479	0.007109	0.002533	0.001194	0.000793
		PLF	BE	0.75876	0.55233	0.28834	0.64706	0.20443
			PR	0.005764	0.012897	0.008555	0.001719	0.003862
		DLF	BE	0.76583	0.55220	0.29357	0.64807	0.20632
			PR	0.007581	0.023218	0.029450	0.002874	0.018880
20	25	SELF	BE	0.83230	0.87476	0.53722	0.61567	0.21677
			PR	0.043653	0.137201	0.056175	0.007643	0.005485
		PLF	BE	0.86782	0.91090	0.55457	0.62168	0.22913
			PR	0.048958	0.115548	0.071635	0.012357	0.024603
		DLF	BE	0.88076	0.94978	0.61573	0.62803	0.24208
			PR	0.055606	0.122815	0.125043	0.019767	0.104502
	50	SELF	BE	0.81359	0.67782	0.40399	0.63965	0.21279
			PR	0.020373	0.042506	0.019598	0.004186	0.003046
		PLF	BE	0.81672	0.70551	0.41586	0.64299	0.21984
			PR	0.023848	0.054547	0.038681	0.006528	0.014088
		DLF	BE	0.83242	0.73539	0.45425	0.64618	0.22715
			PR	0.029000	0.075877	0.090880	0.010133	0.063055
	100	SELF	BE	0.76954	0.57767	0.31850	0.64254	0.20482
			PR	0.009119	0.015944	0.006063	0.002169	0.001538
		PLF	BE	0.77946	0.58893	0.32588	0.64430	0.20847
			PR	0.011684	0.025756	0.017395	0.003372	0.007446
		DLF	BE	0.77680	0.60440	0.33650	0.64597	0.21227
			PR	0.014933	0.043251	0.052658	0.005226	0.035393
	200	SELF	BE	0.76233	0.53906	0.28019	0.64618	0.20244
			PR	0.004475	0.007068	0.002459	0.001111	0.000785
		PLF	BE	0.76201	0.54534	0.28941	0.64703	0.20438
			PR	0.005787	0.012728	0.008579	0.001716	0.003858
		DLF	BE	0.76821	0.54953	0.28884	0.64810	0.20634
			PR	0.007580	0.023203	0.029429	0.003042	0.018948

Table 16: Bayes Estimates (BEs) and Posterior Risks (PRs) of 3-component Mixture of Inverse Rayleigh Distributions Using the GP under SELF, PLF and DLF with θ_1 =0.50, θ_2 =0.50, θ_3 =0.50, p_1 =0.40, p_2 =0.40 and t=15,20

t	n	Loss Functions		JP				
				$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
15	25	SELF	BE	0.59691	0.66389	0.88846	0.41582	0.38306
			PR	0.036413	0.040213	0.118906	0.007847	0.007634
		PLF	BE	0.62452	0.68879	0.95756	0.42524	0.39301
			PR	0.053247	0.053525	0.109718	0.018668	0.019684
		DLF	BE	0.65209	0.71000	1.02008	0.43478	0.40319
			PR	0.083426	0.076165	0.11128	0.043420	0.049439
	50	SELF	BE	0.55224	0.57438	0.69741	0.40844	0.39101
			PR	0.01542	0.015714	0.041181	0.004322	0.004260

		PLF	BE	0.55567	0.59775	0.72718	0.41385	0.39626
			PR	0.025586	0.026147	0.052968	0.010514	0.010825
		DLF	BE	0.57452	0.6057	0.73432	0.41932	0.40182
			PR	0.045503	0.043256	0.071580	0.025232	0.027125
	100	SELF	BE	0.52408	0.53393	0.59090	0.40443	0.39520
			PR	0.006894	0.006952	0.015919	0.002277	0.002259
		PLF	BE	0.52885	0.53772	0.60945	0.40730	0.39807
			PR	0.012683	0.012551	0.025697	0.005609	0.005696
		DLF	BE	0.53497	0.55109	0.61497	0.41015	0.40091
			PR	0.023844	0.023212	0.041736	0.013729	0.014266
	200	SELF	BE	0.51150	0.51996	0.54766	0.40217	0.39752
			PR	0.003280	0.003336	0.007156	0.001227	0.001221
		PLF	BE	0.51437	0.52515	0.55177	0.40376	0.39903
			PR	0.006301	0.006344	0.012634	0.002955	0.002977
		DLF	BE	0.51586	0.52453	0.56354	0.40528	0.40040
			PR	0.012211	0.012047	0.022755	0.007323	0.007469
20	25	SELF	BE	0.59363	0.66657	0.87214	0.41563	0.38325
			PR	0.035591	0.040578	0.116424	0.007842	0.007632
		PLF	BE	0.63018	0.69875	0.96372	0.42514	0.39304
			PR	0.053693	0.054258	0.110251	0.018650	0.019660
		DLF	BE	0.65012	0.73387	1.0077	0.43465	0.40322
			PR	0.083419	0.076133	0.111207	0.043415	0.049403
	50	SELF	BE	0.54177	0.57817	0.68669	0.40854	0.39093
			PR	0.014743	0.015925	0.040049	0.004302	0.004257
		PLF	BE	0.57065	0.58725	0.71168	0.41382	0.39641
			PR	0.026262	0.025659	0.051830	0.010504	0.010810
		DLF	BE	0.57144	0.60592	0.74213	0.41913	0.40181
			PR	0.045491	0.043227	0.071468	0.025222	0.027094
	100	SELF	BE	0.51785	0.53517	0.59717	0.40457	0.39515
			PR	0.006709	0.006995	0.016428	0.002275	0.002258
		PLF	BE	0.52183	0.54576	0.60954	0.40730	0.39806
			PR	0.022509	0.012732	0.025690	0.005605	0.005692
		DLF	BE	0.53445	0.54983	0.61245	0.41012	0.40101
			PR	0.023828	0.023195	0.041695	0.013776	0.014549
	200	SELF	BE	0.50891	0.52149	0.55130	0.40232	0.39754
			PR	0.003238	0.003355	0.007263	0.001168	0.001163
		PLF	BE	0.51656	0.5212	0.55008	0.40380	0.39897
			PR	0.006323	0.006296	0.012584	0.002898	0.002921
		DLF	BE	0.51679	0.52561	0.56103	0.40522	0.40051
			PR	0.012204	0.012035	0.022753	0.007167	0.007309

Table 17: Bayes Estimates (BEs) and Posterior Risks (PRs) of 3-component Mixture of Inverse Rayleigh Distributions Using the EP under SELF, PLF and DLF with $\theta_1 = 0.25$, $\theta_2 = 0.50$, $\theta_3 = 0.75$, $p_1 = 0.20$, $p_2 = 0.65$ and t = 15, 20

t	n	Loss Functions		EP				
				$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
15	25	SELF	BE	0.30920	0.53506	0.94410	0.22393	0.61115
			PR	0.018811	0.017950	0.215536	0.006144	0.008407
		PLF	BE	0.33646	0.54228	1.02548	0.23731	0.61804
			PR	0.049928	0.030586	0.179001	0.026637	0.013675
		DLF	BE	0.35373	0.56103	1.14205	0.25149	0.62497
			PR	0.142908	0.055614	0.167032	0.109087	0.022015
	50	SELF	BE	0.27944	0.51824	0.86138	0.21248	0.62504
			PR	0.007833	0.008386	0.092381	0.003142	0.004404
		PLF	BE	0.29122	0.52165	0.91900	0.21980	0.62857
			PR	0.024814	0.015467	0.094515	0.014536	0.007025
		DLF	BE	0.30798	0.53791	0.95542	0.22734	0.63191
			PR	0.083403	0.029445	0.100168	0.065058	0.011160
	100	SELF	BE	0.26453	0.50554	0.80709	0.20643	0.64205
			PR	0.003496	0.003936	0.043761	0.001588	0.002229
		PLF	BE	0.26945	0.51066	0.83838	0.21018	0.64393
			PR	0.012403	0.007655	0.050203	0.007620	0.003466
		DLF	BE	0.27792	0.51443	0.85020	0.21409	0.64554
			PR	0.045494	0.014937	0.058942	0.035921	0.005377
	200	SELF	BE	0.25762	0.50567	0.77852	0.20323	0.64595
			PR	0.001660	0.001968	0.020278	0.0008000	0.001156
		PLF	BE	0.26188	0.50663	0.79073	0.20520	0.64686
			PR	0.006279	0.003849	0.024970	0.003905	0.001744
		DLF	BE	0.26499	0.50673	0.80782	0.20718	0.6478
			PR	0.023828	0.007581	0.031331	0.019018	0.002855
20	25	SELF	BE	0.30068	0.52915	0.92535	0.22403	0.61126
			PR	0.018110	0.017517	0.20643	0.006142	0.008398
		PLF	BE	0.32422	0.54462	1.05716	0.23736	0.61807
			PR	0.048095	0.030704	0.184347	0.026624	0.013663
		DLF	BE	0.35530	0.56231	1.15575	0.25141	0.62502
			PR	0.142893	0.055587	0.166851	0.109071	0.021987
	50	SELF	BE	0.28290	0.51573	0.85127	0.21259	0.62505
			PR	0.008015	0.008310	0.092115	0.003142	0.004400
		PLF	BE	0.29107	0.52410	0.91085	0.21981	0.62848
			PR	0.024789	0.015537	0.093592	0.014529	0.007021

		DLF	BE	0.30182	0.53129	0.94785	0.22734	0.63192
			PR	0.08335	0.029432	0.100063	0.065005	0.011146
	100	SELF	BE	0.26483	0.50942	0.80936	0.20640	0.64207
			PR	0.003506	0.003996	0.044051	0.001586	0.002227
		PLF	BE	0.27277	0.51334	0.82514	0.21024	0.64376
			PR	0.012548	0.007696	0.049343	0.007617	0.003465
		DLF	BE	0.27916	0.51736	0.87226	0.21404	0.64551
			PR	0.045494	0.014935	0.058873	0.035920	0.005375
	200	SELF	BE	0.25557	0.50155	0.77322	0.20321	0.64602
			PR	0.001634	0.001936	0.019953	0.000802	0.001074
		PLF	BE	0.25803	0.50727	0.79692	0.20519	0.64684
			PR	0.006184	0.003852	0.025128	0.003903	0.001742
		DLF	BE	0.26682	0.50702	0.79592	0.20715	0.64773
			PR	0.023824	0.007579	0.03129	0.018932	0.002692

Table 18: Bayes Estimates (BEs) and Posterior Risks (PRs) of 3-component Mixture of Inverse Rayleigh Distributions Using the EP under SELF, PLF and DLF with θ_1 =0.75, θ_2 =0.50, θ_3 =0.25, p_1 =0.65, p_2 =0.20 and t=15,20

t	n	Loss Functions		EP $\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
15	25	SELF	BE	0.72983	0.57671	03644	0.62625	0.20890
			PR	0.032956	0.064501	0.034564	0.008285	0.005850
		PLF	BE	0.75550	0.62751	0.39546	0.63269	0.22256
			PR	0.042634	0.093138	0.068944	0.013163	0.02712
		DLF	BE	0.77976	0.66437	0.46093	0.63952	0.23693
			PR	0.055629	0.143026	0.166836	0.020689	0.118232
	50	SELF	BE	0.75671	0.55750	0.31944	0.64716	0.20806
			PR	0.017587	0.030906	0.015289	0.004356	0.003150
		PLF	BE	0.76907	0.58461	0.34425	0.65051	0.21542
			PR	0.022429	0.049696	0.039350	0.006717	0.014872
		DLF	BE	0.78276	0.60127	0.36283	0.65393	0.22311
			PR	0.028957	0.083222	0.111097	0.010298	0.067850
	100	SELF	BE	0.75378	0.52320	0.28048	0.64614	0.20234
			PR	0.008745	0.013675	0.005290	0.002218	0.001566
		PLF	BE	0.75045	0.54207	0.29064	0.64779	0.20623
			PR	0.011253	0.024963	0.017371	0.003428	0.007665
		DLF	BE	0.75518	0.55713	0.30221	0.64951	0.21013
			PR	0.014939	0.045529	0.058875	0.005286	0.036831
	200	SELF	BE	0.75104	0.51285	0.26437	0.64810	0.20114
			PR	0.004340	0.006588	0.002332	0.001129	0.000792
		PLF	BE	0.75014	0.51851	0.26842	0.64887	0.20318
			PR	0.005698	0.012441	0.008463	0.001766	0.003928
		DLF	BE	0.75355	0.52350	0.27113	0.64933	0.20497
			PR	0.007584	0.023851	0.031285	0.008053	0.024395
20	25	SELF	BE	0.73456	0.58691	0.36558	0.62636	0.20887
			PR	0.033389	0.066069	0.035018	0.008275	0.005843
		PLF	BE	0.75572	0.62879	0.40432	0.63304	0.22224
			PR	0.042611	0.093456	0.070453	0.013139	0.027102
		DLF	BE	0.77022	0.68694	0.4559	0.63964	0.23677
			PR	0.055587	0.14296	0.16672	0.020646	0.118156
	50	SELF	BE	0.76140	0.56112	0.32881	0.64716	0.20811
			PR	0.0178	0.031315	0.016079	0.004354	0.003147
		PLF	BE	0.77057	0.58541	0.34377	0.65061	0.21539
			PR	0.022462	0.049723	0.039293	0.006708	0.014856
		DLF	BE	0.78026	0.59844	0.36791	0.65398	0.22306
			PR	0.028944	0.083163	0.11106	0.010284	0.067786
	100	SELF	BE	0.74789	0.52533	0.28246	0.64614	0.20240
			PR	0.008601	0.013836	0.005374	0.002215	0.001564
		PLF	BE	0.74895	0.53425	0.29004	0.64785	0.20615
			PR	0.011226	0.024595	0.017324	0.003424	0.007658
		DLF	BE	0.76235	0.55042	0.30071	0.64961	0.21004
			PR	0.014932	0.045511	0.058853	0.005280	0.036813
	200	SELF	BE	0.74814	0.51680	0.26383	0.64801	0.20117
			PR	0.004307	0.006687	0.002326	0.001160	0.000795
		PLF	BE	0.74991	0.52032	0.27001	0.64891	0.20314
			PR	0.005695	0.012478	0.008509	0.001732	0.003916
		DLF	BE	0.75577	0.52231	0.27534	0.64978	0.20517
			PR	0.007579	0.023830	0.031278	0.002667	0.019176

Table 19: Bayes Estimates (BEs) and Posterior Risks (PRs) of 3-component Mixture of Inverse Rayleigh Distributions Using the EP under SELF, PLF and DLF with θ_1 =0.50, θ_2 =0.50, θ_3 =0.50, p_1 =0.40, p_2 =0.40 and t=15,20

t	n	Loss Functions		EP $\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
15	25	SELF	BE	0.52369	0.54327	0.66756	0.40686	0.39198
			PR	0.026892	0.029591	0.090404	0.008536	0.008431
		PLF	BE	0.53896	0.56863	0.70688	0.41732	0.40254
			PR	0.045930	0.04846	0.105175	0.020713	0.021223
		DLF	BE	0.56524	0.59731	0.74852	0.42773	0.41342

			PR	0.083476	0.083407	0.143065	0.049087	0.052053
		SELF	BE	0.51684	0.52714	0.56392	0.40368	0.39560
			PR	0.013381	0.013906	0.031975	0.004524	0.004494
	50	PLF	BE	0.51828	0.53389	0.59830	0.40927	0.40124
			PR	0.023857	0.024586	0.051033	0.011130	0.011279
		DLF	BE	0.53835	0.54483	0.63163	0.41483	0.40691
			PR	0.045508	0.045525	0.083413	0.027018	0.027918
		SELF	BE	0.50363	0.51318	0.53330	0.40179	0.39789
			PR	0.006345	0.006587	0.014284	0.002331	0.002324
	100	PLF	BE	0.51363	0.51756	0.55201	0.40477	0.40064
			PR	0.012318	0.012415	0.025418	0.005781	0.005820
		DLF	BE	0.51658	0.52621	0.55887	0.40757	0.40365
			PR	0.023843	0.023838	0.045519	0.014160	0.014379
		SELF	BE	0.50464	0.50705	0.52160	0.40104	0.39886
			PR	0.003189	0.003220	0.006827	0.001184	0.001182
	200	PLF	BE	0.50495	0.51181	0.52761	0.40232	0.40039
			PR	0.006187	0.006269	0.012655	0.002949	0.002959
		DLF	BE	0.50942	0.51322	0.53306	0.40284	0.40074
			PR	0.012215	0.012216	0.023854	0.004143	0.005802
		SELF	BE	0.51997	0.55017	0.62601	0.40675	0.39178
			PR	0.026725	0.030354	0.078076	0.008530	0.008424
	25	PLF	BE	0.53874	0.577624	0.69072	0.41743	0.40197
			PR	0.045887	0.049264	0.102463	0.020699	0.021230
		DLF	BE	0.56148	0.60244	0.75242	0.42788	0.41322
			PR	0.083389	0.083395	0.142955	0.04900	0.052039
		SELF	BE	0.50353	0.51020	0.57678	0.40366	0.39575
			PR	0.012657	0.013011	0.033254	0.004519	0.004490
	50	PLF	BE	0.53874	0.57762	0.69072	0.41743	0.40197
			PR	0.045887	0.049264	0.102463	0.020699	0.021230
		DLF	BE	0.56148	0.60244	0.75242	0.42788	0.41322
20			PR	0.083389	0.083395	0.142955	0.048999	0.052039
		SELF	BE	0.50529	0.51234	0.54411	0.40181	0.39783
			PR	0.006389	0.006558	0.014889	0.002329	0.002321
	100	PLF	BE	0.51955	0.51864	0.54214	0.40468	0.40077
			PR	0.012456	0.012430	0.024954	0.005776	0.005814
		DLF	BE	0.51394	0.52369	0.56323	0.40774	0.40358
			PR	0.023821	0.023830	0.045505	0.014214	0.014461
		SELF	BE	0.50160	0.50895	0.52211	0.40099	0.39880
			PR	0.003146	0.003243	0.006826	0.001183	0.0011810
	200	PLF	BE	0.50979	0.50773	0.52238	0.40238	0.40040
			PR	0.006241	0.006215	0.012525	0.002946	0.002956
		DLF	BE	0.50754	0.51126	0.53351	0.40395	0.40178
			PR	0.012202	0.012206	0.023834	0.007304	0.007371

9. Conclusion

In this study, we have considered the Bayesian analysis of 3-component mixture of inverse Rayleigh distributions using the non-informative (Uniform and Jeffreys') and the informative (Gamma and Exponential) priors under SELF, PLF and DLF. The purpose of this paper is to find out the appropriate combinations of prior distributions and loss functions to estimate the parameters of the 3-component mixture of the inverse Rayleigh distributions. We conducted a comprehensive simulation study to determine the relative performance of the Bayes estimators. From simulated results, we observed that an increase in the sample size and test termination time provides better Bayes estimators. Furthermore, as sample size increases (decreases) the posterior risks of Bayes estimator's decreases (increase) for a fixed test termination time. Also, the DLF is observed as a suitable choice for estimating component parameters and SELF is preferable for estimating the proportion parameters. Finally, we conclude that the GP is suitable prior in order to estimate the component parameters. When SELF is used, the GP is an appropriate prior for proportion parameters. The same pattern is observed for the JP when non-informative priors are considered.

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