*International Journal of Engineering & Technology*, 7 (2.15) (2018) 38-40

International Journal of Engineering & Technology

*Website: www.sciencepubco.com/index.php/IJET*

***Research Paper***

# Travelling-Wave Similarity Solution for Gravity-Driven Rivulet of a Non-Newtonian Fluid

**Siti Sabariah Abas[1]*, Yazariah Mohd Yatim[2]**

[1]*Faculty of Informatics & Computing, Universiti Sultan Zainal Abidin, 22200 Besut, Terengganu, Malaysia*
[2]*School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM, Pulau Pinang, Malaysia*
**Corresponding author E-mail: sabariahabas@unisza.edu.my*

**Abstract**

Unsteady travelling-wave similarity solution describing the flow of a slender symmetric rivulet of non-Newtonian power-law fluid down an inclined plane is obtained. The flow is driven by gravity with strong surface-tension effect. The solution predicts that at any time $t$ and position $x$, the rivulet widens or narrows according to $(x-ct)^{1/4}$, where $c$ is velocity of a rivulet, and the film thickens or thins according to a free parameter $F_0$, independent of power-law index $N$. The rivulet also has a quartic transverse profile which always has a global maximum at its symmetrical axis.

***Keywords***: *Power-law fluid, Rivulet, Travelling-wave similarity solution, Thin film.*

## 1. Introduction

Rivulet flows occur in a wide range of practical situations ranging from industrial situation such as coating processes to geophysical situation such as lava flow. There is therefore a considerable literature of both steady and unsteady flows of thin and slender rivulets. Following the approach of Smith[1] for gravity-driven rivulet of a Newtonian fluid, Duffy and Moffat[2] obtained a steady similarity solution for gravity-driven rivulet of a Newtonian fluid with strong surface-tension effect down a near-vertical plane. The similarity solution predicts that the width and the height of rivulet obtained by Smith[1] is modified to $x^{3/13}$ and $x^{-1/13}$, where $x$ is the distance down the plane. Wilson and Burgess[3] obtained a steady similarity solution for gravity-driven rivulet of a non-Newtonian power-law fluid down an inclined plane. The similarity solution indicates that the width and the height of rivulet vary according to $x^{(2N+1)/(5N+2)}$ and $x^{-N/(5N+2)}$, where $N$ is a power-law index. Wilson et. al[4] obtained the steady similarity solutions for rivulet of a non-Newtonian power-law fluid driven by either gravity or constant surface shear stress down an inclined plane, for both weak and strong surface-tension effects. They found that, despite the rather different physical mechanisms driving the flow, the similarity solutions for gravity-driven and shear-stress-driven rivulets are qualitatively similar. Particularly, the solution for gravity-driven flow recovers the solutions of Wilson and Burgess[3], while for shear-stress-driven flow, the width and the height of rivulet vary according to $x^{-1/3}$ and $x^{-1/6}$, respectively, independent of power-law index $N$.

The unsteady similarity solution for gravity-driven rivulet of a non-Newtonian power-law fluid on an inclined plane has been studied by Yatim et. al[5], both for converging sessile rivulet and diverging pendent rivulet. The solution predicts that the evolution of the width and the height of rivulets at any time $t$ vary according to $|x|^{(2N+1)/2(N+1)}$ and $|x|^{N/(N+1)}$, respectively, while at any position $x$ vary according to $|t|^{-N/2(2N+1)}$ and $|t|^{-N/(N+1)}$, respectively, with cross-sectional profiles that are either single-humped or double-humped. More recently, Abas et. al[6] obtained a different type similarity solution namely a travelling-wave similarity solution for the unsteady gravity-driven rivulet of a Newtonian fluid down an inclined plane, with strong surface-tension effect. In this study, the approach of Abas et. al[6] is used to obtain travelling-wave similarity solution describing unsteady gravity-driven rivulet of a non-Newtonian power-law fluid down an inclined plane, with strong surface-tension effect.

## 2. Problem Formulation

Consider the unsteady flow of a thin slender rivulet of a non-Newtonian power-law fluid with constant density $\rho$ and viscosity $\mu = \mu_0 \gamma^{N-1}$, where $\mu_0$ is the consistency coefficient, $\gamma$ is the shear rate and $N(>0)$ is the power-law index on a plane inclined at an angle $\alpha(0<\alpha<\pi/2)$ to the horizontal subject to gravitational acceleration $g$ with strong surface-tension effect $\sigma$. The power-law fluid is characterized as a shear thinning when $0<N<1$ and a shear thickening when $N>1$; when $N=1$, the special case of a Newtonian fluid with constant viscosity $\mu_0$ is recovered.

Cartesian coordinates $Oxyz$ with the $x$-axis down the line of greatest slope and the $z$-axis normal to the substrate, with the substrate at $z=0$ are adopted. The (unknown) free surface of the rivulet is denoted by $z=h(x,y,t)$, where $t$ is time. The rivulet is considered symmetric about $y=0$ (i.e. for which $h$ is even in $y$) with (unknown) semi-width $a=a(x,t)$, so that $h=0$ at the contact lines $y=\pm a$. The geometry of the problem is sketched in Figure 1.

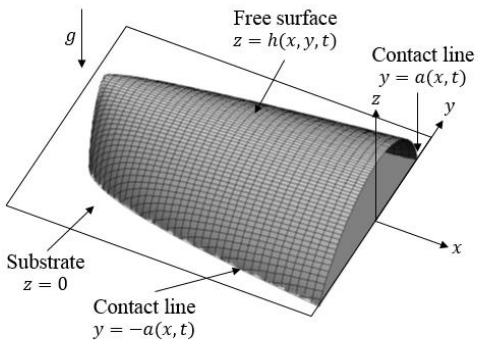



**Fig.1:** Sketch of rivulet

Making the familiar lubrication approximation, the velocity $(u, v, w)$ and pressure $p$ satisfy the governing equations

$$u_x + v_y + w_z = 0, \tag{1}$$

$$(\mu u_z)_z + \rho g \sin\alpha = 0, \tag{2}$$

$$(\mu v_z)_z - p_y = 0, \tag{3}$$

$$-p_z - \rho g \cos\alpha = 0, \tag{4}$$

subject to the boundary conditions of no slip and no penetration on the substrate $z = 0$:

$$u = v = w = 0 \tag{5}$$

and balances of normal and tangential stresses on the free surface $z = h$:

$$p = p_a - \sigma h_{yy}, \quad \mu u_z = \mu v_z = 0, \tag{6}$$

where $p_a$ is atmospheric pressure, together with the kinematic condition on $z = h$:

$$h_t + \bar{u}_x + \bar{v}_y = 0, \tag{7}$$

where $\bar{u} = \bar{u}(x, y, t)$ and $\bar{v} = \bar{v}(x, y, t)$ are the local fluxes of the flow in the longitudinal ($x$-axis) and in the transverse ($y$-axis) direction, respectively, defined by

$$\bar{u} = \int_0^h u\,dz, \qquad \bar{v} = \int_0^h v\,dz \tag{8}$$

and the zero-mass-flux condition at the contact lines $y = \pm a(x, t)$:

$$\bar{v} = \pm a_x \bar{u}. \tag{9}$$

Equations (1) – (4) can readily be solved to yield

$$p = p_a + \rho g \cos\alpha\,(h - z) - \sigma h_{yy}, \tag{10}$$

$$u = \frac{N}{N+1}\left(\frac{\rho g \sin\alpha}{\mu_0}\right)^{\frac{1}{N}}\left[h^{\frac{N+1}{N}} - (h - z)^{\frac{N+1}{N}}\right], \tag{11}$$

$$v = -\frac{N}{N+1}\left(\rho g \cos\alpha\, h - \sigma h_{yy}\right) \left[\frac{(\rho g \sin\alpha)^{1-N}}{\mu_0}\right]^{\frac{1}{N}}\left[h^{\frac{N+1}{N}} - (h - z)^{\frac{N+1}{N}}\right] \tag{12}$$

and substitution of (11) and (12) into (8) gives

$$\bar{u} = \frac{N}{2N+1}\left(\frac{\rho g \sin\alpha}{\mu_0}\right)^{\frac{1}{N}} h^{\frac{2N+1}{N}}, \tag{13}$$

$$\bar{v} = -\frac{N}{2N+1}\left(\rho g \cos\alpha\, h - \sigma h_{yy}\right)_y \left[\frac{(\rho g \sin\alpha)^{1-N}}{\mu_0}\right]^{\frac{1}{N}} h^{\frac{2N+1}{N}}, \tag{14}$$

respectively. Therefore, the kinematic condition (7) yields the governing partial differential equation for $h$:

$$\frac{2N+1}{N}\mu_0\left(\frac{\rho g \sin\alpha}{\mu_0}\right)^{\frac{N-1}{N}} h_t = -\sigma\left(h^{\frac{2N+1}{N}} h_{yyy}\right)_y - \rho g \sin\alpha\left(h^{\frac{2N+1}{N}}\right)_x, \tag{15}$$

with $h$ satisfies the contact-line condition

$$h = 0 \text{ at } y = \pm a, \; h^{\frac{2N+1}{N}} h_{yyy} \to 0 \text{ as } y \to \pm a, \tag{16}$$

where the fluid occupies $|y| \le a$. Once $h$ is determined from (15), the complete solution for $p$, $u$ and $v$ is given by (10) – (12). Note that, in the special case of $N = 1$, equation (15) reduces to the familiar equation describing the unsteady gravity-driven flow of a thin slender rivulet of Newtonian fluid studied by Abas et. al[6]. The draining down the plane driven by gravity is negligible in comparison with the flow down caused by surface tension; this is justified provided that

$$\rho g \cos\alpha\, h \ll \sigma h_{yy}. \tag{17}$$

Consider the unsteady travelling-wave similarity solution of (15) in the form

$$h = bF(\eta), \qquad \eta = \frac{y}{[\ell(x - ct)]^{\frac{1}{4}}}, \tag{18}$$

where the velocity of rivulet $c$ (up or down the substrate) and the dimensionless function $F = F(\eta)(\ge 0)$ of the dimensionless similarity variable $\eta$ are to be determined, and $b(> 0)$ and $\ell$ are constants, which, without loss of generality, can be written as $\ell = 4\sigma b S_\ell / \rho g \sin\alpha$, where $S_\ell = \pm 1$. The rivulet lies in the region where $\ell(x - ct) \ge 0$; along $x = ct$, the fluid thickness $h$ and its derivative $h_y$ are continuous (i.e. so that $u$, $v$ and $p$ are also continuous there), except at the apex of a rivulet, $x = ct$, $y = 0$.

For simplicity in plotting results, the variables are scaled according to

$$x = Xx^*, \qquad h = bh^*, \qquad z = bz^*,$$
$$y = (\ell X)^{\frac{1}{4}} y^*, \qquad t = \frac{X}{U}t^*, \qquad a = (\ell X)^{\frac{1}{4}} a^*, \tag{19}$$
$$h_m = b{h_m}^*, \qquad c = Uc^*,$$

where $X$ is a length scale in the $x$-direction which may be chosen arbitrarily and $U$ is a velocity scale given by

$$U = \frac{N}{2N+1}\left(\frac{\rho g \sin\alpha}{\mu_0}\right)^{\frac{1}{N}}. \tag{20}$$

Then, with asterisks dropped for clarity, the solution (18) takes the simpler form

$$h = F(\eta), \qquad \eta = \frac{y}{(x - ct)^{\frac{1}{4}}} \tag{21}$$

and hence (15) reduces to a fourth-order ordinary differential equation for $F$, namely

$$\left(F^{\frac{2N+1}{N}} F'''\right)' - S_\ell \eta \left(F^{\frac{2N+1}{N}} - cF\right)' = 0, \tag{22}$$

where a prime denotes differentiation with respect to $\eta$. For a symmetric rivulet, regular at $y = 0$, appropriate boundary conditions are

$$F = F_0, \qquad F' = 0, \qquad F'' = F_2, \qquad F''' = 0 \text{ at } \eta = 0, \tag{23}$$

where $F_0(\ge 0)$ and $F_2$ are the free parameters. The position where $F = 0$ is $\eta = \eta_0$ (corresponding to the contact-line position $y = a$), so that

$$F = 0 \text{ at } \eta = \eta_0, \qquad F^{\frac{2N+1}{N}} F''' \to 0 \text{ as } \eta \to \eta_0, \tag{24}$$

where the fluid now lies in $|\eta| \le \eta_0$. The semi-width of the rivulet varies with $x$ and $t$ according to

$$a = (x - ct)^{\frac{1}{4}} \eta_0. \tag{25}$$

In order to satisfy the assumption of thin and slender rivulet, the length scales in $x$, $y$ and $z$ directions (namely $X$, $a$ and $h_m$, respectively) must satisfy $h_m \ll a \ll X$, which requires that

$$\frac{\sigma X S_\ell}{b^3 \rho g \sin\alpha} \gg 1, \qquad \frac{X^3 \rho g \sin\alpha\, S_\ell}{\sigma b} \gg 1, \tag{26}$$

Respectively, showing that $X$ must be sufficiently large and that $\alpha$ cannot be close to 0.

## 3. Results and Conclusion

Since a closed-form solution of (22) is not available, so it must be solved numerically for $F$ subject to the boundary conditions (23) and (24), where $c$ and $\eta_0$ are parameters to be determined. There are four cases to consider, namely case 1: $S_\ell = 1$, $c > 0$, $0 < N < 1$, case 2: $S_\ell = -1$, $c < 0$, $0 < N < 1$, case 3: $S_\ell = -1$, $c > 0$, $N > 1$ and case 4: $S_\ell = 1$, $c < 0$, $N > 1$ however, it turns out that the system (22) – (24) has solutions only in case 1, case 3 and case

4. Therefore, from now on only these three cases shall be considered.

Equation (22) was solved numerically for $F$ subject to (23) for a given value of $F_0$ and $F_2(<0)$ by using a shooting technique via *Mathematica* 9.0 software, the value of $c$ and $\eta_0$ being determined as the point where $F=0$. It was found that there are solutions when $0<c\le c_{max}$ for case 1, $c\ge c_{min}$ for case 3 and $c<0$ for case 4, where the value of $c_{min}$ and $c_{max}$ vary according to $F_0$ and $F_2$.

In case 1, the relation between $F_0$ and $\eta_0$ is monotonic; for any value of $F_0(\ge F_{0c})$ there is a corresponding unique solution of $\eta_0$, but for any value of $\eta_0$, there is no solution when $\eta_0<\eta_{0c}$ while there is a unique solution of $F_0$ when $\eta_0\ge\eta_{0c}$. In case 3 and case 4, the relation between $F_0$ and $\eta_0$ is also monotonic; with a unique solution occurs for any choice of $F_0(>0)$ and $\eta_0$.

Also, it is found that $F$ satisfies

$$F=F_0+\frac{F_2}{2}\eta^2+\frac{F_2S_\ell}{360F_0^{\frac{2N+1}{N}}}\left[\frac{(2N+1)}{N}F_0^{\frac{N+1}{N}}-c\right]\eta^6+O(\eta^8) \tag{27}$$

near $\eta\to 0$ and

$$F\sim\left[-\frac{(2N+1)^3}{3N(N-1)(N+2)}S_\ell c\eta_0(\eta_0-\eta)^3\right]^{\frac{N}{2N+1}} \tag{28}$$

if either $S_\ell c>0$, $0<N<1$ or $S_\ell c<0$, $N>1$

at the leading order, as $\eta\to\eta_0$. Therefore, so far, the family of solutions of (22) parameterized by $F_0$ and $F_2$ are obtained, with $c$ and $\eta_0$ are determined in terms of $F_0$ and $F_2$. Figure 2 shows a plot of $\eta_0$ as a function of $F_0$, together with $F_{0c}$ and $\eta_{0c}$ (shown as dots), while Figure 3 shows three-dimensional plots of the free-surface profile $z=h$ at different times $t$ which demonstrate that the rivulets become narrower as time elapse while maintaining their cross-sectional shapes and their thickness.

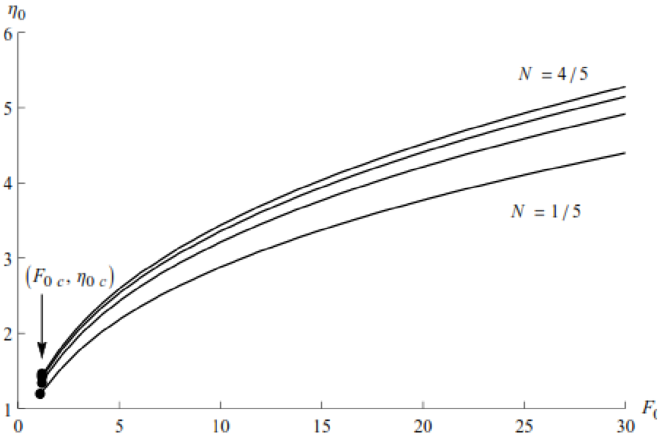
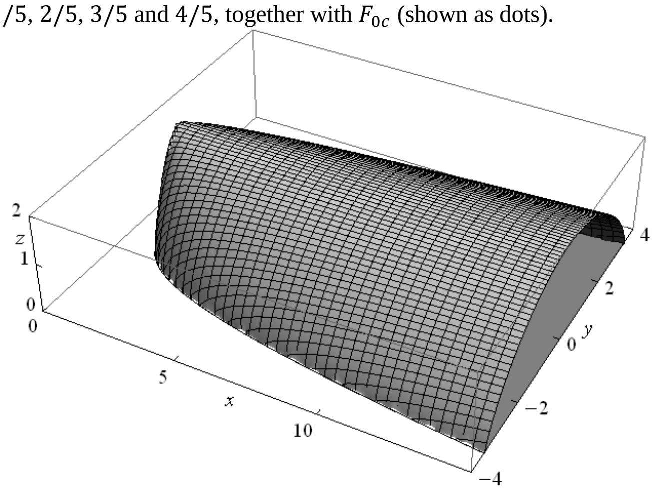
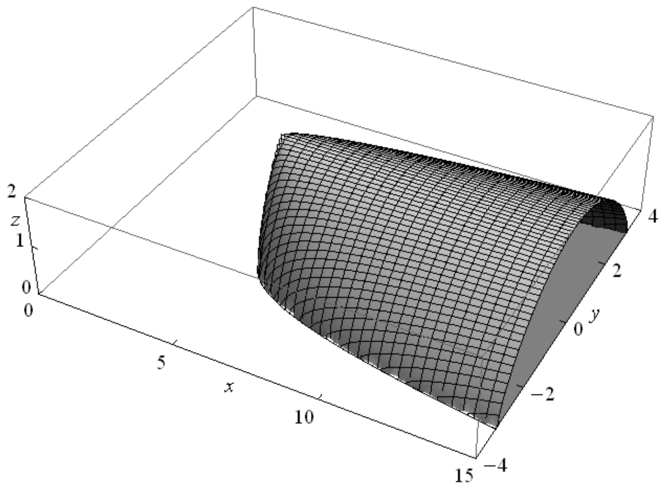



**Fig. 2:** Plot of $\eta_0$ as a function of $F_0$ for $F_2=-1$, $c=1$ and $N=1/5$, $2/5$, $3/5$ and $4/5$, together with $F_{0c}$ (shown as dots).




(a)




(b)

**Fig. 3:** Three-dimensional plots for $F_0=2$, $F_2=-1$, $c=1$ at times (a) $t=1$ and (b) $t=5$ with $N=1/2$.

The travelling-wave similarity solutions describing the unsteady gravity-driven flow of a thin slender rivulet of a non-Newtonian power-law fluid down an inclined plane are obtained. The velocity and pressure are given by (10) – (12) in terms of free surface profile $h$, where $h$ is given by (21). There were four cases to consider (labelled as case 1, case 2, case 3 and case 4), but there is no solution found in case 2. Numerical analysis showed that the rivulet has a quartic shape which always has a maximum thickness at $y=0$. This work also generalized the work of Abas et. al[6] when $N=1$.

# Acknowledgement

The authors gratefully acknowledged the financial support of Fundamental Research Grant Scheme (FRGS) account no: 203/PMATHS/6711432 and Research management, Innovation and Commercialization Centre (RMIC), UniSZA.

# References

[1] P. C. Smith. A similarity solution for slow viscous flow down an inclined plane. Journal of Fluid Mechanics, 58 (1973) 275-288.
[2] B. R. Duffy, H. K. Moffatt. A similarity solution for viscous source flow on a vertical plane. European Journal of Applied Mathematics, 8 (1997) 37-47.
[3] S. D. R. Wilson, S. L. Burgess. The steady, spreading flow of a rivulet of mud. Journal of Non-Newtonian Fluid Mechanics, 79 (1998) 77-85.
[4] S. K. Wilson, B. R. Duffy, R. Hunt. A slender rivulet of a power-law fluid driven by either gravity or a constant shear stress at the free surface. The Quarterly Journal of Mechanics and Applied Mathematics, 55 (2002) 385-408.
[5] Y. M. Yatim, S. K. Wilson, B. R. Duffy. Unsteady gravity-driven slender rivulets of a power-law fluid. Journal of Non-Newtonian Fluid Mechanics, 165 (2010) 1423-1430.
[6] S. S. Abas, Y. M. Yatim. Travelling-wave similarity solution for gravity-driven rivulet of a Newtonian fluid with strong surface-tension effect. AIP Conference Proceedings, 1870 (2017) 40037-1-40037-7.