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Research paper

Unsteady solute dispersion in blood rheology with reversible phase exchange at the artery wall

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Abstract

The effect of reversible phase exchange between the flowing fluid and wall tissues of arteries in the unsteady dispersion of solute in blood flow through a narrow artery is analysed mathematically, modelling the blood as Casson fluid. The resulting convective diffusion equation along with the initial and boundary conditions is solved analytically using the derivative series expansion method. The expressions for the negative asymptotic phase exchange, negative asymptotic convection, longitudinal diffusion coefficient and mean concentration are obtained. It is noted that when the solute disperses in blood flow through a narrow artery, the negative exchange coefficient, the negative convection coefficient increase and the longitudinal diffusion coefficient decreases with the increase of the Damköhler number and partition coefficient.

Keywords: *Blood Flow; Narrow Artery; Casson Fluid; Unsteady Solute Dispersion; Reversible Phase Exchange.*

1. Introduction

Taylor [1] was the first researcher who theoretically analysed the solute dispersion in a laminar flow through a circular pipe. Several researchers [2-5] extended his study by incorporating various physical parameters. Gill and Sankarasubramanian [5] introduced generalized dispersion model (GDM) to demonstrate the whole dispersion process. Reversible phase exchange is an exchange process between the fluid and thin tissue layer at the vessel wall which is an important physical phenomenon. The studies pertaining to reversible reactions were taken up by several researchers [6], [7]. It is well known that the phenomena of the solute dispersion in a solvent flow in the presence of phase exchange between the flowing fluid and vessel wall has several physical applications, one such application is the dispersion of medicines in blood flow in arteries. The previous literatures used Newtonian fluid modelling blood when it flows through larger arteries at high shear rates [8]. Casson fluid is a non-Newtonian fluid model which is suitable for representing blood when it flows through narrow arteries at low shear rates [9]. Hence, in this paper, we mathematically analyze the solute dispersion in blood flow through narrow arteries in the presence of phase exchange between the flowing fluid (blood miscible with solute) and the thin tissue layer at the wall of the artery, treating the blood as Casson fluid model.

2. Mathematical formulation

Consider the unsteady dispersion of solute in the axi-symmetric, steady, laminar and fully developed unidirectional flow of blood through a circular pipe (narrow artery), treating the blood as Casson fluid. The geometry of fluid flow through a circular pipe is shown in Fig. 1. Cylindrical polar coordinate system $(\bar{r}, \bar{\psi}, \bar{z})$ is used to analyze the flow in a circular pipe, where $\bar{\psi}$ is the azimuthal angle, $\bar{r}$ and $\bar{z}$ are the coordinates in the radial and axial directions, respectively.

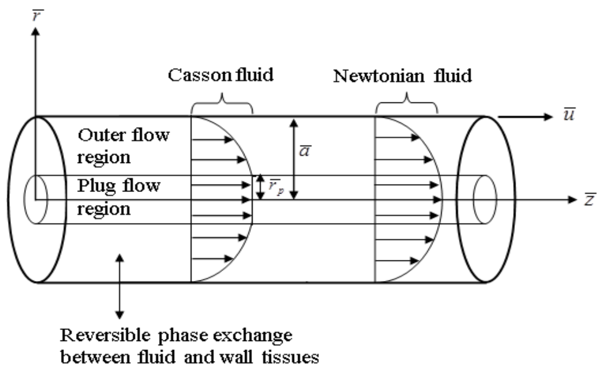

Fig. 1: The Geometry of Fluid Flow in A Circular Pipe.

2.1. Governing equations

The unsteady convective-diffusion equation for flow in a circular pipe is given by [9]

$$\frac{\partial \bar{C}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{C}}{\partial \bar{z}} = \bar{D}_m \left(\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left(\bar{r}\frac{\partial}{\partial \bar{r}}\right) + \frac{\partial^2}{\partial \bar{z}^2}\right)\bar{C}, \quad (1)$$

where $C(\bar{r}, \bar{z}, \bar{t})$ is the concentration of the solute and $\bar{t}$ is the time variable with the non-uniform initial condition [10]

$$\bar{C}(\bar{r}, \bar{z}, 0) = \begin{cases} \bar{C}_0 \bar{Y}_1(\bar{z})\bar{Y}_2(\bar{r}) \text{ if } |\bar{z}| \le \bar{z}_s/2, \\ 0 \text{ if } |\bar{z}| > \bar{z}_s/2, \end{cases} \quad (2)$$

Where $\overline{C}_0$ is the reference concentration, $\overline{Y}_1(\overline{z})$ and $\overline{Y}_2(\overline{z})$ are the separable function of $\overline{z}$ and $\overline{r}$ as given by

$$\overline{Y}_1(\overline{z}) = = \overline{a}\overline{\delta}(\overline{z})/\overline{d}^2, \quad (3)$$

$$\overline{Y}_2(\overline{r}) = \begin{cases} 1 & \text{if } 0 \le \overline{r} \le \overline{a}\overline{d}, \\ 0 & \text{if } \overline{a}\overline{d} < \overline{r} \le \overline{a}, \end{cases} \quad (4)$$

Where $\overline{\delta}(\overline{z})$ is the Dirac delta function and $\overline{d}$ is the cross-section of circle of radius concentric with the pipe. The boundary conditions of (1) with the reaction at the wall are

$$-\overline{D}_m \, \partial\overline{C}/\partial\overline{r} = \partial\overline{C}_s/\partial\overline{t} = \overline{k}\left(\overline{\sigma}\overline{C} - \overline{C}_s\right) \text{ at } \overline{r} = \overline{a}, \quad (5)$$

$$\partial\overline{C}/\partial\overline{r} = 0 \text{ at } \overline{r} = 0, \quad (6)$$

$$\overline{C}(\overline{r},\infty,\overline{t}) = \left[\partial\overline{C}/\partial\overline{z}\right](\overline{r},\infty,\overline{t}) = 0, \quad (7)$$

$$\overline{C}(0,\overline{z},\overline{t}) = \text{finite}, \quad (8)$$

Where $\overline{k}$ is the reaction rate and $\overline{\sigma} = (\overline{C}_s/\overline{C})_{\text{equilibrium}}$ is the equilibrium partition coefficient between the blood and wall.

2.2. Non-dimensional variables

Let us introduce the following non-dimensional variables:

$$C = \overline{C}/\overline{C}_0,\ u = \overline{u}/\overline{u}_0,\ r = \overline{r}/\overline{a},\ r_p = \overline{r}_p/\overline{a},\ z = \overline{D}_m\,\overline{z}/\overline{a}^2\overline{u}_0,\ t = \overline{D}_m\,\overline{t}/\overline{a}^2,$$

$$\tau = \overline{\tau}/\left(\overline{\eta}_H\overline{u}_0/\overline{a}\right)^{1/n},\ \tau_y = \overline{\tau}_y/\left(\overline{\eta}_H\overline{u}_0/\overline{a}\right)^{1/n},\ Da = \overline{k}\overline{a}^2/\overline{D}_m,\ \sigma = \overline{\sigma}/\overline{a},$$

$$Y_1 = \overline{Y}_1,\ Y_2 = \overline{Y}_2,\ d = \overline{d},\ \delta = \overline{\delta}, \quad (9)$$

Where $\overline{r}_p$ is plug core radius. The non-dimensional form of Eq. (1) is

$$\frac{\partial C}{\partial t} + u\,\frac{\partial C}{\partial z} = \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{Pe^2}\frac{\partial^2}{\partial z^2}\right)C. \quad (10)$$

The non-dimensional form of Eqs. (2), (5)-(8) are

$$C(r,z,0) = \begin{cases} Y_1(z)\,Y_2(r) & \text{if } |z| \le z_s/2, \\ 0 & \text{if } |z| > z_s/2, \end{cases} \quad (11)$$

$$Y_1(z) = \delta(z)/d^2 Pe, \quad (12)$$

$$Y_2(r) = \begin{cases} 1 & \text{if } 0 < r \le d, \\ 0 & \text{if } d < r \le 1, \end{cases} \quad (13)$$

$$-\frac{\partial C}{\partial r}(1,z,t) = \frac{\partial C_s(z,t)}{\partial t} = Da\left[\sigma C(1,z,t) - C_s(z,t)\right], \quad (14)$$

$$\partial C/\partial r = 0 \text{ at } r = 0, \quad (15)$$

$$C(r,\infty,t) = \frac{\partial C}{\partial z}(r,\infty,t) = 0, \quad (16)$$

$$C(0,z,t) = \text{finite}. \quad (17)$$

3. Method of solution

3.1. Dispersion functions and coefficients

The solution of (10) subject to the (11) and (14)-(17) is [15]

$$C(r,z_1,t) = \sum_{i=0}^{\infty} f_i(r,t)\frac{\partial^i C_m(z_1,t)}{\partial z_1^i}, \quad (18)$$

Where f_i is the dispersion function, z_1 is a new axial coordinate and C_m is the mean concentration of solute given by

$$C_m(z_1,t) = 2\int_0^1 C(r,z_1,t)\,r\,dr. \quad (19)$$

We expand the mean concentration as [6]

$$\frac{\partial C_m}{\partial t}(z_1,t) = \sum_{i=0}^{\infty} K_i(t)\frac{\partial^i C_m}{\partial z_1^i}(z_1,t), \quad (20)$$

Where K_i are the dispersion coefficients and are given by

$$K_i(t) = \frac{\delta_{i2}}{Pe^2} + 2\frac{\partial f_i}{\partial r}(1,t) - 2\int_0^1 f_{i-1}(r,t)\,u(r,t)\,r\,dr,\ i = 0,1,2,..., \quad (21)$$

With the Kronecker delta δ_{ij}. Substituting (18) and (20) into (10) and grouping the coefficients of $\partial^i C_m/\partial z_1^i$, we have

$$\frac{\partial f_i}{\partial t}(r,t) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)f_i(r,t) - u(r,t)f_{i-1}(r,t) + \frac{f_{i-2}}{Pe^2}(r,t)$$

$$-\sum_{j=0}^{i} K_j(t) f_{i-j}(r,t),\ i = 0,1,2,..., \quad (22)$$

Where $f_{-1} = f_{-2} = 0$. In(22), to find K_i, we need f_i . Use of Eq. (18) in Eqs. (11), (14), (17) and (19), (22) yields the respective initial, boundary and solvability conditions

$$f_i(r,0) = 0, \quad i = 1,2,... \quad (23)$$

$$\left[\partial f_i/\partial r\right](1,t) = -\beta f_i(1,t) - Da\sigma f_i(1,t), \quad i = 0,1,2,..., \quad (24)$$

$$f_i(0,t) = \text{finite}, \quad i = 0,1,2,... \quad (25)$$

$$\int_0^1 f_i(r,t)\,r\,dr = \delta_{i0}/2. \quad (26)$$

Setting $i = 0$ in Eq. (22), we get the differential equation for f_0 as

$$\frac{\partial f_0}{\partial t}(r,t) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f_0}{\partial r}(r,t)\right) - f_0(r,t)K_0(t). \quad (27)$$

With the initial and the solvability conditions

$$f_0(r,0) = Y_2(r)/2\int_0^1 Y_2(r)\,r\,dr \quad (28)$$

$$\int_0^1 f_0(r,t)\,r\,dr = 1/2. \quad (29)$$

Solving Eq. (27) using the variable separable method subject to Eqs. (24), (25), (28) and (29), the solution of $f_0(r,t)$ is found as

$$f_0(r,t) = \frac{\sum_{s=0}^{\infty} A_s J_0(\mu_s r)e^{-\mu_s^2 t}}{2\sum_{s=0}^{\infty}\left(\frac{A_s}{\mu_s}\right)J_1(\mu_s)e^{-\mu_s^2 t}}. \quad (30)$$

Where μ_s is the root of the transcendental equation

$$\mu_s J_1(\mu_s) = Da\sigma J_0(\mu_s), \quad s = 0,1,2,..., \quad (31)$$

$$A_s = \frac{\mu_s^2 \int_0^1 J_0(\mu_s r) Y_2(r)\, r\, dr}{J_0^2(\mu_s)\left\{\mu_s^2 + (Da\sigma)^2\right\} \int_0^1 Y_2(r)\, r\, dr}. \tag{32}$$

As $f_0(r, t)$ is known, $K_0(t)$ is obtained from Eq. (21) as below:

$$K_0(t) = 2\frac{\partial f_0}{\partial r}(1, t) = -\frac{\sum_{s=0}^{\infty} A_s \mu_s J_1(\mu_s) e^{-\mu_s^2 t}}{\sum_{s=0}^{\infty}\left(\frac{A_s}{\mu_s}\right) J_1(\mu_s) e^{-\mu_s^2 t}}. \tag{33}$$

Since the computations of the higher order $f_1, f_2, \ldots$ and $K_1, K_2, \ldots$ have difficult integral equations, the asymptotic computation has been used to give beneficial physical insight into the nature of the problem [6]. At large time, expanding Eq. (30) and ignoring the negligibly small terms in Eq. (30) yields

$$f_0(r) = \mu_0 J_0(\mu_0 r) / 2J_1(\mu_0). \tag{34}$$

Applying Eq. (32) in Eq. (33), K_0 at large time is obtained as

$$K_0 = -\mu_0^2, \tag{35}$$

Where μ_0 is the first root of Eq. (31). Using Eq. (35) in Eq. (22), we obtain the following differential equation at large time for $f_i(r)$:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{\partial f_i}{\partial r}(r)\right) + \mu_0^2 f_i(r) = u(r) f_{i-1}(r)$$

$$-\frac{f_{i-2}}{Pe^2}(r) + \sum_{j=1}^{i} K_j f_{i-j}(r) \tag{36}$$

And using Eq. (35) in Eqs. (24), (25) and (26), we obtain the following conditions:

$$[df/dr](1) = -Da\sigma f_i(1),\ \ i = 1, 2, 3, \ldots, \tag{37}$$

$$f_i(0) = \text{finite} \tag{38}$$

$$\int_0^1 f_i(r)\, r\, dr = 0, \quad i = 1, 2, 3, \ldots \tag{39}$$

Note that K_j's in Eq. (36) are the constant steady-state ($t \rightarrow \infty$) values. Multiplying both sides of Eq. (21) by $rJ_0(\mu_0 r)$ and then integrating it with respect to r and applying the orthogonal property of Bessel function and using Eqs. (34) and (39), we get K_i as below:

$$K_i = \frac{\int_0^1 r J_0(\mu_0 r)\left[\frac{1}{Pe^2} f_{i-2}(r) - u(r) f_{i-1}(r) - \sum_{j=1}^{i-1} K_j f_{i-j}(r)\right] dr}{\int_0^1 f_0(r)\, r\, J_0(\mu_0 r)\, dr}. \tag{40}$$

In order to get the solution of longitudinal convection coefficient K_1, we apply i = 1 in (34) and using Eq. (40), we get

$$K_1 = -\frac{-2\mu_0^2 \int_0^1 u(r)\, r\, J_0^2(\mu_0 r)\, dr}{\left\{\mu_0^2 + (Da\sigma)^2\right\} J_0^2(\mu_0)}. \tag{41}$$

Thus, Eq. (36) is reduced to

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{\partial f_1}{\partial r}(r)\right) + \mu_0^2 f_1(r) = u(r) f_0(r) + K_1 f_0(r) \tag{42}$$

And the respective boundary and solvability conditions reduce to

$$[df_1/dr](1) = -Da\sigma f_1(1), \tag{43}$$

$$f_1(0) = \text{finite}, \tag{44}$$

$$\int_0^1 f_1(r)\, r\, dr = 0. \tag{45}$$

Using Eqs. (41), (43)-(45), the solution of Eq. (42) is obtained as

$$f_1(r) = \sum_{s=1}^{\infty} B_s\left[-\frac{\mu_0 J_1(\mu_s) J_0(\mu_0 r)}{\mu_s J_1(\mu_0)} + J_0(\mu_s r)\right]. \tag{46}$$

Using Bessel properties, we get B_s, as

$$B_s = \frac{2\mu_s^2 \int_0^1 \left[u(r) + K_1\right] f_0(r) J_0(\mu_s r)\, r dr}{\left(\mu_0^2 - \mu_s^2\right)\left\{\mu_s^2 + (\beta + \mathrm{Da}\sigma)^2\right\} J_0^2(\mu_s)}. \tag{47}$$

Once f_1 is known, the longitudinal diffusion coefficient K_2 can be obtained as below by setting i = 2 in Eq. (40):

$$K_2 = \frac{1}{Pe^2} - \frac{4\mu_0 J_1(\mu_0)}{\left\{\mu_0^2 + (\beta + \mathrm{Da}\sigma)^2\right\} J_0^2(\mu_0)} \times$$

$$\int_0^1 \left[u(r) + K_1\right] f_1(r) J_0(\mu_0 r)\, r dr. \tag{48}$$

3.2. Mean concentration

Expanding Eq. (20) and then neglecting the terms $K_3(t)$, $K_4(t)$,… (since the magnitude of these terms are negligibly small), one can obtain

$$\frac{\partial C_m}{\partial t} = K_0 C_m + K_1 \frac{\partial C_m}{\partial z_1} + K_2 \frac{\partial^2 C_m}{\partial z_1^{\,2}} \tag{49}$$

Substituting Eqs. (11) and (16) into (19), we get the following initial and boundary conditions:

$$C_m(z_1, 0) = 2Y_1(z_1) \int_0^1 Y_2(r)\, r\, dr, \tag{50}$$

$$C_m(\infty, t) = \left[\partial C_m / \partial z_1\right](\infty, t) = 0. \tag{51}$$

The solution for $C_m(z, t)$ is obtained as below by solving Eq. (49) subject to the conditions (50) and (51) and using Fourier Transform (FT) method:

$$C_m(z_1, t) = \frac{1}{2Pe\sqrt{\pi\phi}} e^{\left(\zeta - \frac{z_1^2}{4\phi}\right)}, \tag{52}$$

Where

$$\zeta(t) = \int_0^t K_0(t)\, dt, \tag{53}$$

$$z_1(z, t) = z + \int_0^t K_1(t)\, dt, \tag{53}$$

$$\phi(t) = \int_0^t K_2(t)\, dt. \tag{54}$$

4. Results and discussions

The objective of this paper is to discuss the effects of various parameters on the negative asymptotic phase exchange, negative asymptotic convection, longitudinal diffusion coefficients and mean concentration.

4.1. Negative exchange coefficient

The variation of negative asymptotic exchange coefficient ($-K_0$) with Damköhler number (*Da*) for different values of partition coefficient σ is shown in Fig. 2. It is noted that ($-K_0$) increases very rapidly with the increase of *Da* from 0 to 10 and then it increases very slowly (almost constant) from 10 to 100. It is also observed that ($-K_0$) increases when σ increases due to the increasing of the molecules reaction at the wall, thus, the solute exchange becomes faster. At quite large value of *Da* and σ, ($-K_0$) becomes effective when a high amount of the solute has been exchanged to the wall tissues of blood vessels and less amount of solute at the central region.

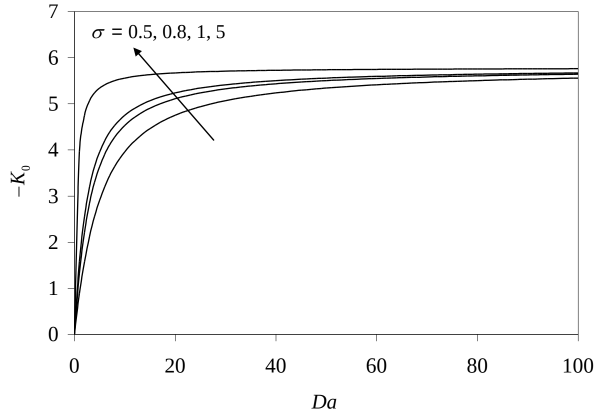

Fig. 2: Variation of ($-K_0$) with Da for different values of σ.

4.2. Negative asymptotic convection coefficient

The variation of negative asymptotic convection coefficient ($-K_1$) with Damköhler number *Da* for different values of partition coefficient σ and yield stress r_p is depicted in Fig. 3. It is noticed that ($-K_1$) increases when *Da* and σ increase and it tends to be a constant when the value of *Da* is large ($Da > 10$). It is also clear that ($-K_1$) decreases when r_p increases because when r_p increases, the accumulation of red blood cells at the centre increase and thus, the velocity tends to decrease and it slows down the convection process. The plot of ($-K_1$) for Newtonian fluid ($r_p = 0$) without the phase exchange ($\sigma = 0$) is in good agreement with the corresponding plot given in Gill and Sankarasubramanain [5].

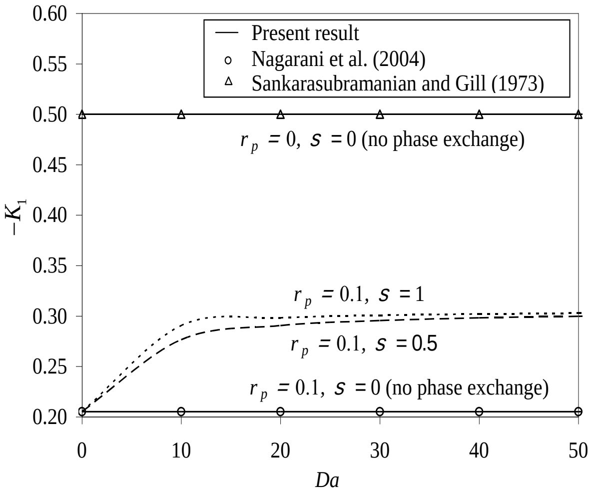

Fig. 3: Variation Of ($-K_1$) With Da for Various Values of σ and R_p.

4.3 Longitudinal diffusion coefficient

Fig. 4 shows the variation of the longitudinal diffusion coefficient (K_2-1/Pe^2) x103 with Damköhler number *Da* for different values of partition coefficient σ. It is seen that the (K_2-1/Pe^2) x103 decreases very rapidly with the increase of *Da* from 0 to 20 and then it decreases slowly from when *Da* increases from 20 to 100. It is also found that the (K_2-1/Pe^2) x103 decreases with the increase of σ. It is of interest to note that the plot of (K_2-1/Pe^2) x10^3 for Newtonian with $\sigma = 0$ is in good agreement with the corresponding plot in Gill and Sankarasubramanian [5].

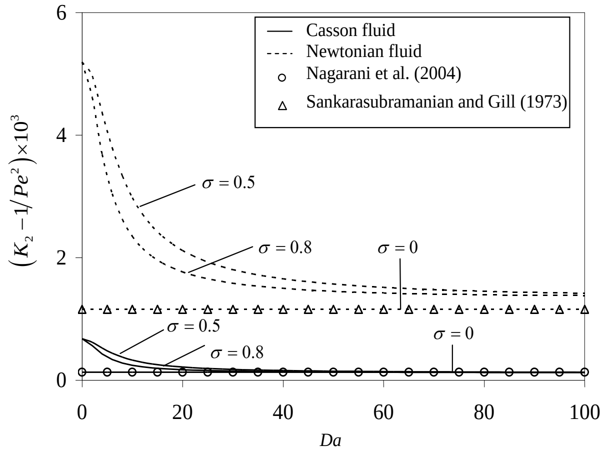

Fig. 4: Variation of (K_2-1/Pe^2) 10^3 with Da for Different Values of σ and R_p.

4.4. Mean concentration of solute

Fig. 5 depicts the variation of $C_m \times Pe$ with time t at $z = 0.5$ for different values of *Da*, σ and r_p. In an early period of time, $C_m \times Pe$ increases rapidly and reaches the maximum value at the middle of the period and then it falls suddenly. The peaks in the graph show an effective $C_m \times Pe$ and it helps us to predict the time to achieve effective $C_m \times Pe$ in Casson fluid as 0.3 and Newtonian fluid as 0.1.

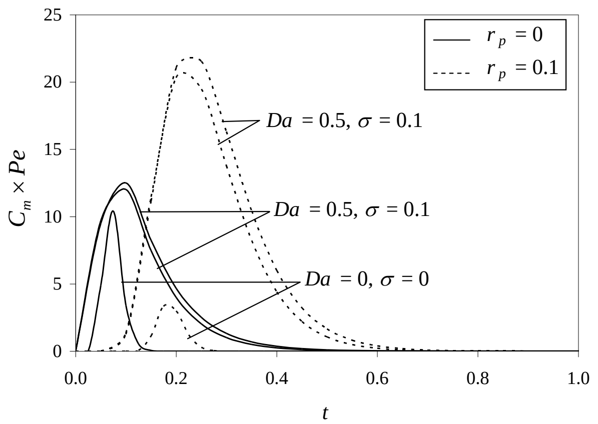

Fig. 5: Variation of $C_m \times Pe$ with Da for Different Values of σ And R_p.

5. Conclusions

This study investigates the effects of yield stress and reversible reactions at the wall on the phase exchange, convection, longitudinal diffusion coefficients and mean concentration of the solute. The major findings of this study are summarised below:

- The negative exchange coefficient, the negative convective coefficient increase and the longitudinal diffusion coefficient decreases with the increase of the Damköhler number and partition coefficient.
- The negative exchange coefficient, the negative convective coefficient and the longitudinal diffusion coefficient decrease with the increase of yield stress.

In view of the results obtained, this study may be considered as an improvement in the studies of solutes dispersion in blood flow through arteries with reversible reaction at the wall of the artery.

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