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Research paper

The local multiset dimension of graphs

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Abstract

All graphs in this paper are nontrivial and connected graph. For k-ordered set $W = \{s_1, s_2, \dots, s_k\}$ of vertex set G, the multiset representation of a vertex v of G with respect to W is $r_m(v|W) = \{d(v, s_1), d(v, s_2), \dots, d(v, s_k)\}$ where $d(v, s_i)$ is a distance between of the vertex v and the vertices in W together with their multiplicities. The resolving set W is a local resolving set of G if$r_m(v|W) \neq r_m(u|W)$ for every pair u, v of adjacent vertices of G. The minimum local resolving set W is a local multiset basis of G. If G has a local multiset basis, then its cardinality is called local multiset dimension,denoted by $\mu_l(G)$. If G does not contain a local resolving set, then we write $\mu_l(G) = \infty$. In our paper, we will investigate the establish sharp bounds of the local multiset dimension of G and determine the exact value of some family graphs.

Keywords:*Local Resolving Set; Local Multiset Dimension; Distance; Some Family Graph.*

1. Introduction

In this paper, all graphs are nontrivial and connected graph, for detail definition of graph see [1,2,3]. The concept of metric dimension was independently introduced by Slater [4], Harrary and Melter [5]. In his paper, Slater said this concept as a locating set.
Chartrand, et al. in [9] define the distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest path between these two vertices. Suppose that $W = \{s_1, s_2, \dots, s_k\}$ is an ordered set of vertices of a nontrivial connected graph G. The metric representation of v with respect to W is the k-vector $r(v|W) = (d(v, s_1), d(v, s_2), \dots, d(v, s_k))$. Distance in graphs has also been used to distinguish all of the vertices of a graph. The set W is called a resolving set for G if distinct vertices of G have distinct representations with respect to W.The metric dimension of G, denoted by $\dim(G)$, is the minimum cardinality of resolving set W of G[5]. Furthermore, we consider those ordered sets W of vertices in G for which any two vertices of G having the same representation with respect to W are not adjacent in G. If $r(u|W) \neq r(v|W)$ for every pair u, v of adjacent vertices of G, then W is called a local resolving set of G. The minimum cardinality of local resolving set is local metric dimension of G, denoted by $\mathrm{ldim}(G)$[7]. In recent years, the local metric dimension has been studied by [6,7] and the related topic in resolving set [10,11,12,13].
Proposition 1.1: [7] Let Gbe a nontrivial connected graph of order n, $\mathrm{ldim}(G) = n - 1$ if and only if $G = K_n$ and $\mathrm{ldim}(G) = 1$ if and only if G is bipartite.
Proposition 1.1: [6] Let G be a nontrivial connected graph of order n. If G is paththen $\mathrm{ldim}(G) = 1$ andif G is cycle then $\mathrm{ldim}(G) = 1$ where n even and $\mathrm{ldim}(G) = 2$ where n odd.
Simanjuntaket. al.[8] started the definition of multiset dimension of G. Let G be a connected graph with vertex set V(G). Suppose $W = \{s_1, s_2, \dots, s_k\}$ of vertex set G, the multiset representation of a vertex v of G with respect to W is $r_m(v|W) = \{d(v, s_1), d(v, s_2), \dots, d(v, s_k)\}$ where $d(v, s_i)$ is a distance between of the vertex v and the vertices in W together with their multiplicities. The resolving set Wis aresolving set of G if $r_m(v|W) \neq r_m(u|W)$ for every pair of distances vertices u and v. The minimum resolving set W is a multiset basis of G. If G has a multiset basis, then its cardinality is called multiset dimension,denoted by $md(G)$. Until today, there is some results of multiset dimension as follows:

Theorem 1.3:*The multiset dimension of a graph G is one if and only if G is a path.*

Theorem 1.4:*Let G be a graph other than a path, we have $md(G) \geq 3$.*

Theorem 1.5:*If G is a graph of diameter at most 2 other than a path, then $md(G) = \infty$.*

Furthermore, we define the new notation of multiset dimension of G which is called local multiset dimension. We start definiton of local multiset dimension as follows:

Definition 1.1: *Let G be a connected graph with vertex set $V(G)$. Suppose $W = \{s_1, s_2, \dots, s_k\}$ of vertex set G, the multiset representation of a vertex v of G with respect to W is $r_m(v|W) = \{d(v, s_1), d(v, s_2), \dots, d(v, s_k)\}$ where $d(v, s_i)$ is a distance between of the vertex v and the vertices in W together with their multiplicities. The resolving set W is a local resolving set of G if $r_m(v|W) \neq r_m(u|W)$ for every pair u, v of adjacent vertices of G. The minimum local resolving set W is a local multiset basis of G. If G has a local multiset basis, then its cardinality is called local multiset dimension, denoted by $\mu_l(G)$.*

We will illustrate this concept in Figure 1. In this case, we have the resolving set $W = \{v_2, v_3, v_6\}$ which shown in Figure 1 (a) that $md(G) = 3$ and the representations of $v \in V(G)$ with respect to W are distinct. On other hand, we have $W = \{v_1\}$ which shown in Figure 1 (b) is a local resolving set. Hence, we first give the representation of the vertices of G with respect to W as follows

$$r_m(v_1|W) = \{0\},\ r_m(v_2|W) = \{1\}, r_m(v_3|W) = \{2\}$$

$$r_m(v_4|W) = \{1\}, r_m(v_5|W) = \{2\}, r_m(v_6|W) = \{1\}$$

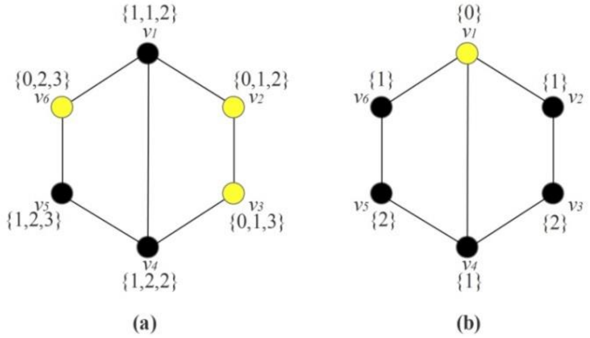

Fig. 1: A Graph with Multiset Dimension 3; (B) A Graph with Local Multiset Dimension 1 It Can Be Seen That $M_L(G) = 1$.

2. Main results

In this paper, we introduce the new concept of multiset dimension namely local multiset dimension. We found the lower bound of local multiset dimension and also determine the exact values of local multiset dimension of some graph families in the following theorems.

Lemma 2.1: *For every nontrivial connected graph G of order n, we have $\mu_l(G) \geq ldim(G)$.*

Proof: Let W be a local resolving set of G. If we have the vertices u adjacent to v which have representation $r(u|W) = (a, b, c)$ and $r(v|W) = (b, a, c)$ for a, b, c represented of distance $d(u, w)$ for $w \in W$, then $r(u|W) \neq r(v|W)$. It satisfies the properties of local metric dimension. But, If we focus to multiset of distance which causes $\{a, b, c\} = \{b, a, c\}$, then we have same multiset representation $r_m(u|W) = r_m(v|W) = \{a, b, c\}$. It does not satisfy the properties of local multiset dimension. Other hand, If the vertices u adjacent to v have $r(u|W) = (a, b, c)$ and $r(v|W) = (b, a, d)$ for a, b, c, d represent of distance $d(u, w)$ for $w \in W$, then $r(u|W) \neq r(v|W)$. It satisfies the properties of local metric dimension and also $\{a, b, c\} \neq \{b, a, d\}$ such that we have distinct multiset representation $r_m(u|W) \neq r_m(v|W)$. It satisfies the properties of local multiset dimension. Thus, we concludes that $\mu_l(G) \geq ldim(G)$.

Lemma 2.2: *For every nontrivial connected graph G of order n, we have $\mu_l(G) \leq md(G)$.*

Proof: Let W be a resolving set of G, the vertices in G have distinct multiset representation. Such that, every resolving set is also a local resolving set. Hence, we have $\mu_l(G) \leq md(G)$.

Lemma 2.3: *Let T be a tree graph of order n, we have $\mu_l(T) \geq 1$.*

Proof: Let T be a tree graph with order n. For two vertices $x, v \in V(T)$ which have at most one path between the vertices x and v. Suppose $W = \{u\}$ with u of T, there is representation local multiset as follows.

- If the vertices x and v are pendant vertices and its vertices aren't adjacent, then it is satisfies the condition local multiset dimension.
- If $d(x) = 1$ and $d(v) \neq 1$ in one path, then $r_m(x|W) \neq r_m(v|W)$. It satisfies its condition.
- If $d(x) \neq 1$ and $d(v) \neq 1$ in one path, then $r_m(x|W) \neq r_m(v|W)$. It satisfies its condition.
- The vertices x and v in one path and $r_m(x|W) \neq r_m(v|W)$$. If there is at least one vertex w' adjacent to vertex x and $d(w', x) = d(v, x)$, then $r_m(w'|W) = r_m(v|W)$ with v and w' aren't adjacent. It satisfies its condition.

Based on all possible that we obtain the local multiset dimension of T is $\mu_l(T) \geq 1$.

Theorem 2.1: *Let P_n be a path graph with $n \geq 3$, the local multiset dimension of P_n is $\mu_l(P_n) = 1$.*

Proof : The path P_n is a tree graph with n vertices. The vertex set $V(P_n) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(P_n) = \{v_i v_{i+1};\ 1 \leq i \leq n - 1\}$. The cardinality of vertex set and edge set, respectively are $|V(P\_n)| = n$ and $|E(P\_n)| = n - 1$.
Based on Lemma 2.3 that the lower bound of local multiset dimension of tree graph T is $\mu_l(T) \geq 1$. We know that P_n is tree graph such that $\mu_l(P_n) \geq 1$. However, we attain the sharpest lower bound. Furthermore, The upper bound of local multiset dimension of path P_n is $\mu_l(P_n) \leq 1$. Suppose $W = \{v_1\}$, the representation of vertices $v \in V(P_n)$ respect to W is $r_m(v_1|W) = \{0\}$ and $r_m(v_i|W) = \{i - 1\}$; $2 \leq i \leq n$. It can be seen that $r_m(v_i|W) \neq r_m(v_{i+1}|W)$. Thus, we obtain the upper bound of local multiset dimension of P_n is $\mu_l(P_n) \leq 1$. We conclude that $\mu_l(P_n) = 1$.

Theorem 2.2: *Let S_n be a star graph with $n \geq 3$, the local multiset dimension of S_n is $\mu_l(S_n) = 1$.*

Proof : The star S_n is a tree graph with $n + 1$ vertices. The vertex set $V(S_n) = \{v, v_1, v_2, \dots, v_n\}$ and edge set $E(S_n) = \{vv_i;\ 1 \leq i \leq n\}$. The vertex v is a central vertex and the vertices v_i is pendant vertex with degree 1. The cardinality of vertex set and edge set, respectively are $|V(S_n)| = n + 1$ and $|E(S_n)| = n$.
Based on Lemma 2.3 that the lower bound of local multiset dimension of tree graph T is $\mu_l(T) \geq 1$. We know that S_n is tree graph such that $\mu_l(S_n) \geq 1$. However, we attain the sharpest lower bound. Furthermore, The upper bound of local multiset dimension of star S_n is $\mu_l(S_n) \leq 1$. Suppose $W = \{v\}$, the representation of vertices $v \in V(S_n)$ respect to W is $r_m(v|W) = \{0\}$ and $r_m(v_i|W) = \{1\};\ 1 \leq i \leq n$. It can be seen that$r_m(v_i|W) = r_m(v_j|W)$ with v_i and v_j aren't adjacent for $1 \leq i, j \leq n$. Thus, we obtain the upper bound of local multiset dimension of S_n is $\mu_l(S_n) \leq 1$. We conclude that $\mu_l(S_n) = 1$.

Theorem 2.3: *Let T be a complete k-ary tree of height h, the local multiset dimension of T is $\mu_l(T) = 1$.*

Proof : A complete k-ary tree is a k-ary tree which is maximally space efficient. It must be completely filled on every level except for the last level. However, if the last level is not complete, then all nodes of the tree must be "as far left as possible.
Based on Lemma 2.3 that the lower bound of local multiset dimension of tree graph T is $\mu_l(T) \geq 1$. We know that T is complete k-ary tree graph such that $\mu_l(T) \geq 1$. However, we attain the sharpest lower bound. Furthermore, The upper bound of local multiset dimension of complete k-ary tree graph is $\mu_l(T) \leq 1$. Suppose $W = \{v\}$, the representation of vertices $v \in V(T)$ respect to W is $r_m(v|W) = \{0\}$ and $r_m(v_j^i|W) = \{j\};\ 1 \leq i \leq k, 1 \leq j \leq h$. It can be seen that $r_m(v_j^i|W) = r_m(v_j^l|W)$ with v_j^i and v_j^l aren't adjacent for $1 \leq i, l \leq k$. Thus, we obtain the upper bound of local multiset dimension of complete k-ary tree graph is $\mu_l(T) \leq 1$. We conclude that $\mu_l(T) = 1$.

Theorem 2.4: *Let $C_{n,m}$ be a caterpillar graph with $n \geq 3$ and $m \geq 1$, the local multiset dimension of $C_{n,m}$ is $\mu_l(C_{n,m}) = 1$.*

Proof : The path $C_{n,m}$ is a tree graph with $nm + n$ vertices. The vertex set $V(C_{n,m}) = \{v_1, v_2, \dots, v_n\} \cup \{v_{i,j};\ 1 \leq i \leq n, 1 \leq j \leq m\backslash\}$ and edge set $E(C_{n,m}) = \{v_i v_{i+1};\ 1 \leq i \leq n-1\} \cup \{v_i v_{i,j};\ 1 \leq i \leq n, 1 \leq j \leq m\}$. The cardinality of vertex set and edge set, respectively are $|V(C_{n,m})| = n + nm$ and $|E(C_{n,m})| = nm + n - 1$.
Based on Lemma 2.3 that the lower bound of local multiset dimension of tree graph T is$\mu_l(T) \geq 1$. We know that $C_{n,m}$ is tree graph such that $\mu_l(C_{n,m}) \geq 1$. However, we attain the sharpest lower bound. Furthermore, The upper bound of local multiset dimension of path $C_{n,m}$ is $\mu_l(C_{n,m}) \leq 1$. Suppose $W = \{v_1\}$, the representation of vertices $v \in V(C_{n,m})$ respect to W is $r_m(v_1|W) = \{0\}, r_m(v_i|W) = \{i-1\};\ 2 \leq i \leq n$ and$r_m(v_{i,j}|W) = \{i\};\ 1 \leq i \leq n, 1 \leq j \leq m$. Thus, we obtain the upper bound of local multiset dimension of $C_{n,m}$ is $\mu_l(C_{n,m}) \leq 1$. We conclude that $\mu_l(C_{n,m}) = 1$.

Theorem 2.5: *Let $K_{n_1,n_2,\dots,n_k}$ be a k-partite graph with $1 \leq l \leq k$ and $n_l \geq k-1$, the local multiset dimension of $K_{n_1,n_2,\dots,n_k}$ is $\mu_l(K_{n_1,n_2,\dots,n_k}) = \frac{k(k-1)}{2}$.*

Proof : The k-partite graph $K_{n_1,n_2,\dots,n_k}$be a connected graphwith $n_l \geq 2$ and $1 \leq l \leq k$. The vertex set $V(K_{n_1,n_2,\dots,n_k}) = \{v_{l,i};\ 1 \leq i \leq n_l, 1 \leq l \leq k\}$ and edge set $E(K_{n_1,n_2,\dots,n_k}) = \{v_{l,i} v_{l+r,i+r};\ 1 \leq i \leq n_l, 1 \leq l \leq k, 1 \leq r \leq k-l\}$.Firstly, we prove that the lower bound of local multiset dimension of k-partite graph is $\mu_l(K_{n_1,n_2,\dots,n_k}) \geq \frac{k(k-1)}{2}$. We assume that $\mu_l(K_{n_1,n_2,\dots,n_k}) < \frac{k(k-1)}{2}$, suppose $W = W_1 \cup W_2 \cup \dots \cup W_{k-1}$with$W_l = \{v_{l,i}; 1 \leq i \leq k-l, 1 \leq l \leq k-2\}$ such that $(k-1)$th partite and kth partite do not have at least one vertex as resolving set. Hence, the representation of vertices $v \in V$ at least two adjacent vertices which have some representation include $r_m(v_{k-1,i}|W) = r_m(v_{k,i}|W) = \{1^{\frac{k(k-1)-2}{2}}\backslash\}$. It is a contradiction. Thus, we have the lower bound of local multiset dimension of k-partite graph is $\mu_l(K_{n_1,n_2,\dots,n_k}) \geq \frac{k(k-1)}{2}$.
Furthermore, we show that the upper bound of local multiset dimension of k-partite graph is $\mu_l(K_{n_1,n_2,\dots,n_k}) \leq \frac{k(k-1)}{2}$. Suppose $W = W_1 \cup W_2 \cup \dots \cup W_{k-1}$ with $W_l = \{v_{l,i}; 1 \leq i \leq k-l, 1 \leq l \leq k-1\}$, the representation of vertices $v \in V(K_{n_1,n_2,\dots,n_k})$ respect to W as follows.

$$r_m(v_{l,i}|W) = \left\{0, 1^{\frac{k^2-3k+2l}{2}}, 2^{k-l-1}\right\};\ 1 \leq i \leq k-l, 1 \leq l \leq k-1$$

$$r_m(v_{l,i}|W) = \left\{1^{\frac{k^2-3k+2l}{2}}, 2^{k-l}\right\};\ k-l+1 \leq i \leq n_l, 1 \leq l \leq k-1.$$

$$r_m(v_{k,i}|W) = \left\{1^{\frac{k^2-k}{2}}\right\};\ 1 \leq i \leq n_k.$$

It can be seen that $r_m(v_{l,i}|W) = r_m(v_{l,j}|W)$ with $v_{l,i}$ and $v_{l,j}$ aren't adjacent for $1 \leq i, j \leq n_l$. Thus, we obtain the upper bound of local multiset dimension of k-partite graph is $\mu_l(K_{n_1,n_2,\dots,n_k}) \leq \frac{k(k-1)}{2}$. We conclude that $\mu_l(K_{n_1,n_2,\dots,n_k}) = \frac{k(k-1)}{2}$.

Theorem 2.6: *Let C_n be a cycle graph with $n \geq 3$, the local multiset dimension of C_n is*

$$\mu_l(C_n) = \begin{cases} 1, \text{if } n is even \\ 3, if n is odd \end{cases}$$

Proof : The cycle C_n is a cyclic graph with n vertices. The vertex set $V(C_n) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(C_n) = \{v_1 v_n, v_i v_{i+1};\ 1 \leq i \leq n-1\}$. The cardinality of vertex set and edge set, respectively are $|V(C_n)| = n$ and $|E(C_n)| = n$. The proof divided into two cases as follows.

Case 1: For n is even, Based on Proposition 1.2 and Lemma 2.1 that the lower bound of local multiset dimension of cycle C_n is $\mu_l(G) \geq ldim(G)$. We know that $ldim\ (C_n) = 1$ such that $\mu_l(C_n) \geq\ ldim(C_n) = 1$. However, we attain the sharpest lower bound. Furthermore, The upper bound of local multiset dimension of cycle is $\mu_l(C_n) \leq 1$. Suppose $W = \{v_1\}$, the representation of vertices $v \in V(C_n)$ respect to W as follows.

$$r(v_1|W) = \{0\}$$

$$r(v_i|W) = \{i-1\};\ 2 \leq i \leq \frac{n}{2} + 1$$

$$r(v_i|W) = \{n-i+1\}; \frac{n}{2} + 2 \leq i \leq n$$

It can be seen that $r_m(v_i|W) \neq r_m(v_{i+1}|W)$ with v_i and v_{i+1} are adjacent for $1 \leq i \leq n-1$. Thus, we obtain the upper bound of local multiset dimension of cycle C_n is $\mu_l(C_n) \leq 1$. We conclude that $\mu_l(C_n) = 1$ for n is even.

Case 2 : For n is odd, we will show that lower bound of the local multiset dimension of C_n is $\mu_l(C_n) \geq 3$. Assume that $\mu_l(C_n) < 3$, suppose the resolving set $W = \{u, v\}$ so that there is some condition as follows

- If $u, v \in W$ are adjacent, then $r_m(u|W) = r_m(v|W) = \{0,1\}$, it is a contradiction.
- If $u, v \in W$ aren't adjacent then there is at most two path P_1 and P_2 between two vertices u and v. If $|V(P_1)| = k_1$ with k_1 is odd, then $|V(P_2)| = k_2$with k_2 is even.
- We take the cardinality $|V(P_2)| = k_2$ with k_2 is even and the vertices in P_2 includes path graph.
- Let the vertices in P_2 be $v_1, \dots, v_{2l} \in V(P_2)$ for $l \in Z^+$\$ such that $d(v_l, v_1) = d(v_{l+1}, v_{2l})$.
- We obtain that $d(v_1, u) = 1$ and $d(v_{2l}, v) = 1$, based on point d) that $r_m(v_l|W) = \{d(v_l, v_1) + d(v_1, u), d(v_{l++\text{ d \$ain that \$inality \$ path \$1}}, v_1) + d(v_1, u) + 1\} = \{d(v_l, v_1) + 1, d(v_l, v_1) + 2\}$ and $r_m(v_{l+!}\}|W) = \{d(v_{l+1}\}, v_{2l}) + d(v_{2l}, v), d(v_{l+1}, v_{2l}) + d(v_{2l}, v) + 1\} = \{d(v_{l+1}, v_{2l}) + 1, d(v_{l+1}, v_{2l}) + 2\backslash\}\$$.
- Based onpoint d), e) that $r_m(v_l|W) = \{d(v_l, v_1) + 1, d(v_l, v_1) + 2\} = \{d(v_{l+1}, v_{2l}) + 1, d(v_{l+1}, v_{2l}) + 2\} = r_m(v_{l+1}|W)$ and we know that v_l is adjacent to v_{l+1}, it is a contradiction.

Based on point a),b),c),d),e), f) that the lower bound of local multiset dimension of C_n is $\mu_l(C_n) \geq 3$. Furthermore, the upper bound of the local multiset dimension of C_n is $\mu_l(C_n) \leq 3$.Suppose the resolving set $W = \{v_1, v_3, v_4\}$, we can obtain the representation v respect to W as follows

$$r_m(v_1|W) = \{0,2,3\}$$

$$r_m(v_2|W) = \{1,1,2\}$$

$$r_m(v_3|W) = \{0,1,2\}$$

$$r_m(v_4|W) = \{0,1,3\}$$

$$r_m(v_i|W) = \{i-4, i-3, i-1\};\ 5 \leq i \leq \frac{n+1}{2}$$

$$r_m(v_i|W) = \{i-4, i-3, i-2\};\ i = \frac{n+3}{2}$$

$$r_m(v_i|W) = \{i-4, i-3, i-4\};\ i = \frac{n+5}{2}$$

$$r_m(v_i|W) = \{n-i+1, n-i+3, n-i+3\};\ i = \frac{n+7}{2}$$

$$r_m(v_i|W) = \{n-i+1, n-i+3, n-i+4\};\ \frac{n+9}{2} \leq i \leq n$$

The representation of the vertices v_i which is adjacent are distinct such that W is a local resolving set of C_n. Thus, we obtain the upper bound of the local multiset dimension of C_n is $\mu_l(C_n) \leq 3$. It concludes that $\mu_l(C_n) = 3$ for $n$is odd.

Theorem 2.7:*Let K_n be a complete graph with $n \geq 3$, the local multiset dimension of K_n is $\mu_l(K_n) = \infty$.*

Proof : The complete K_n is a $n-1$-regular graph with n vertices. The vertex set $V(K_n) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(K_n) = \{v_i v_{i+k};\ 1 \leq i \leq n, 1 \leq k \leq n-i\}$. The cardinality of vertex set and edge set, respectively are $|V(K_n)| = n$ and $|E(K_n)| = \frac{n(n-1)}{2}$. Diameter of K_n is 1 and all verteices are adjacents.

We prove this theorem by contradiction. Assume that all vertices in W is distance 1 and W is a local resolving set of complete graph K_n. There is some condition as follows.

- If we take $W = \{v_1\}$, then $r_m(v_1|W) = \{0\}$ and $r_m(v_2|W) = r_m(v_3|W) = \dots = r_m(v_{n-1}\}|W) = r_m(v_n|W) = \{1\}$, we know that $v_2, v_3, \dots, v_n$ are adjacent such that it is a contradiction.

- If we take $W = \{v_1, v_2\}$, then $r_m(v_1|W) = r_m(v_2|W) = \{0,1\}$ and $r_m(v_3|W) = \cdots = r_m(v_{n-1}\}|W) = r_m(v_n|W) = \{1^2\}$, we know that$v_3, \dots, v_n$ are adjacent such that it is a contradiction.
- If we take $W = \{v_1, v_2, \dots, v_k\}$ for $2 \leq k \leq n-1$, then $r_m(v_1|W) = \cdots = r_m(v_k|W) = \{0,1^{k-1}\}$ and $r_m(v_{k+1}|W) = \cdots = r_m(v_n|W) = \{1^k\}$, we know that $v_{k+1}, \dots, v_n$ are adjacent such that it is a contradiction.

Hence, W is not a local resolving set of complete graph K_n. It is conclude that $\mu\_l\ (K\_n) = \infty$.

3. Conclusion

In this paper we have given an result the lower bound of local multiset dimension and determine the exact values of some special graphs. Hence the following problem aries naturally.

3.1. Open problem

Determine the local multiset dimension of family graph namely family tree, unicyclic, regular graphs, and others.

3.2. Open problem

Determine the local multiset dimension of operation graph namely corona product, cartesian product, joint, comb product, and others.

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References

[1] J. L. Gross, J. Yellen and P. Zhang. {\it Handbook of graph Theory} Second Edition CRC Press Taylor and Francis Group, (2014).
[2] G. Chartrand and L. Lesniak. {\it Graphs and digraphs} {\it 3rd} ed (London: Chapman and Hall), (2000)
[3] N. Hartsfield dan G. Ringel. \textit{Pearls in Graph Theory} Academic Press. United Kingdom, (1994).
[4] P.J. Slater, Leaves of trees, in:\textit{Proc. 6th Southeast Conf. Comb., Graph Theory, Comput. Boca Rotan}, {\bf 14} (1975), 549-559.
[5] F. Harary and R.A. Melter, on the metric dimension of a graph, \textit{Ars Combin}, {\bf 2} (1976), 191-195.
[6] E. Ulfianita, N. Estuningsih, L. Susilowati. DimensiMetrikLokal pada Graf Hasil Kali Comb dari Graf Siklus dan Graf Lintasan. Journal of Mathematics, Vol. 1 (2014) No. 3, 24 - 33.
[7] F. Okamoto, B. Phinezy, P. Zhang. The Local Metric Dimension of A Graph. Mathematica Bohemica, Vol. 135 (2010) No. 3, 239 - 255
[8] R. Simanjuntak, T. Vetrik, and P. B. Mulia. The multiset dimension of graphs. arXiv preprint arXiv:1711.00225, (2017)
[9] Chartrand, G., Eroh, L., Johnson, M.A., amdOellermann, O.R., 2000, Resolvability in Graphs and the Metric Dimension of a Graph, Discrete Appl. Math., 105: 99-113.https://doi.org/10.1016/S0166-218X(00)00198-0.
[10] Dafik, Agustin I H, Surahmat, Syafrizal Sy and Alfarisi R 2017 On non-isolated resolving number of some graph operations {\it Far East Journal of Mathematical Sciences} {\bf 102} (2) 2473 – 2492.https://doi.org/10.17654/MS102102473.
[11] Darmaji, and Alfarisi, R. "On the partition dimension of comb product of path and complete graph." AIP Conference Proceedings. Vol. 1867. No. 1. AIP Publishing, 2017.https://doi.org/10.1063/1.4994441.
[12] Alfarisi, R., Darmaji, and Dafik. "On the star partition dimension of comb product of cycle and complete graph." Journal of Physics Conference Series. Vol. 855. No. 1. 2017.https://doi.org/10.1088/1742-6596/855/1/012005.
[13] Alfarisi, Ridho, and Darmaji. "On the star partition dimension of comb product of cycle and path." AIP Conference Proceedings. Vol. 1867. No. 1. AIP Publishing, 2017.https://doi.org/10.1063/1.4994419.