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Research paper

Monophonic Wirelength in Graph Embedding

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Abstract

In this paper, we define the monophonic embedding of graph G into another graph H and this paper presents a monophonic algorithm to find the monophonic wirelength of circulant networks G[n, ±S], where S ⊆ {1,2,3,⋯,n/2} into the family of Cycle C_n, n≥ 4. The mono-phonic embedding of a graph G into a graph H is an embedding denoted by $f_m$is a bijective map from the vertex set of G into the vertex set of H and f_m is a one-one mapping from the edge set (x, y) of G into $P_m(H)$ where $P_m(H)$ is the set of monophonic paths be-tween $f_m(x)$ and $f_m(y)$ for every $f_m(x)$, $f_m(y)$ ∈ H. The monophonic wirelength of f_m of G into H is the sum of distances of monophonic paths between two vertices $f_m(x)$ and $f_m(y)$ in H such that (x, y) ∈ E(G). In addition, the eccentricity, radius and diameter of an embed-ding of G into H are defined. The average wirelength of an embedding is defined and the bounds of average wirelength of some embed-dings have been found.

Keywords:*Circulant Networks; Congestion; Cycles; Embedding; Wirelength.*

1. Introduction

For vertices u and v in a connected graph G, The distance d(u, v) is the length of the shortest u-v path in G. A chord of a path u_0, u_1, u_h is an edge u_iu_j, with j ≥ i + [2]. A u - v path is called a mono-phonic path if it is a chordless path. For two vertices u and v in a connected graph G, the monophonic distance $d_m(u, v)$ is the length of the longest u - v monophonic path in G. An u - v monophonic path of length $d_m(u, v)$ is called an u - v monophonic as stated in [1, 2].
By an embedding f : G → H and a monophonic embedding f_m: GH, it is meant that the graphs G(V, E) and H(V, E) are finite, simple and connectedwith n vertices. Given a host graph H, which repre-sents the network into which other networks are to be embedded, and a guest graph G, which represents the network to be embed-ded, the problem is to find a mapping from V(G) to V(H) such that each edge of G can be mapped to a path in Has given in [3-7].
An embedding f of G into H is defined as follows:

1) f is a bijective map from V(G) to V(H).
2) f is an one-to-one mapping from E(G) to $P_f(f(u), f(v))$ where $P_f(f(u), f(v))$ is a path in H between f(u) and f(v) for (u,v) ∈ E(G) as defined in [8-10].

An embedding f_m: G → H is called a monophonic embedding if f_m maps each vertex of G into a vertex of H and each edge (x, y) of G is mapped to a monophonic path between $f_m(x)$ and $f_m(y)$ in H. The edge congestion of an embedding f of G into H is the maxi-mum number of edges of the graph G that are embedded on any single edge of H. The wirelength of an embedding f of G into H is given by,

$$WL_f(G,H) = \sum_{(u,v)\in E(G)} d_H(f(u),\ f(v)) = \sum_{e\in E(H)} EC_f(G,\ H(e))$$

The wirelength problem of a graph G into H is to find an embed-ding of G into H that induces the minimum wirelength WL(G, H) as defined in [6],[11], [12].

2. Preliminaries

In this section, we have given definition, example, Lemma and Theorem which are needed in the sequel.

Definition 2.1:*f_m: G→H be an monophonic embedding. The mon-ophonic wirelength MWL(G, H) of f_m is given by,* $MWL_{f_m}(G,H) = \sum_{(x,y)\in E(G)} d_m(f_m(x),\ f_m(y))$

Definition 2.2: *Let f_m :G → H be a monophonic embedding. The monophonic edge congestion off_m of G into H is the maximum number of edges of the graph G that are embedded on an edge e ∈ H and is given by, $MEC_{f_m}(G,\ H) = \max MEC_{f_m}(G,\ H(e))$. The monophonic wirelength problem of a graph G into H is the problem of finding a monophonic embedding f_m :G → H that produces the monophonic wirelength MWL(G, H).*

Definition 2.3:*The eccentricity of a vertex of graph G into H of an embedding f is given by $e_f(v)$ = max{d(f(u), f(v)) / (u, v) ∈ E(G) and $P_f(f(u), f(v))$ is a path in H for every (u, v) ∈ E(G) and for any vertex u in G}.*

Definition 2.4:*The radius of f : G→H is given by $r_f(G,H)$ = min{$e_f(v)$/v ∈ V(G)} and the diameter of f : G→H is given by $dm_f(G, H)$ = max{$e_f(v)$/v ∈V(G)}.*

Definition 2.5:*The monophonic eccentricity of a vertex of graph G into H of an embedding f_m is given by* $e_{f_m}(v)$ =

$\max\{d_m(f_m(u), f_m(v)) / (u,v) \in E(G)$ *and* $P_{f_m}(f_m(u), f_m(v))$ *is a path in H for every (u, v)* $\in E(G)$ *and for any vertex u in G}.*

Definition 2.6: *The monophonic radius of* f_m*: G→H is given by* $r_{f_m}(G,H) = min\{e_{f_m}(v) / v \in V(G)\}$ *by and the diameter of* f_m*: G→H is given by* $dm_{f_m}(G,H) = max\{e_{f_m}(v) / v \in V(G)\}$.

Lemma 2.7: (*Congestion lemma): Let G be an r-regular graph and let f: G → H be an embedding. Let the graph H \ E has the components* H_i*, i =1, 2 and* $G_i = f^{-1}(H_i)$ *then the edge cut E of H has the following properties:*

1) The path $P_f(f(x), f(y))$ has no edges in E for every edge (x, y) $\in G_i$, i = 1, 2.
2) The path $P_f(f(x), f(y))$ has exactly one edge in E for every edge (x, y) in G with x ∈ G_1 and y ∈ G_2.
3) G_1 is a maximum subgraph of G.

Then $EC_f(E)$, is minimum and $EC_f(E) = r|V(G_1)|-2|E(G_1)|$ as proved in [6].

Lemma 2.8: **(***Partition Lemma): Let f: G→H be an embedding. Let* $\{E_1,E_2,…,E_p\}$ *be a partition of E (H) such that each* E_i *is an edge cut of H. Then* $WL_f(G,H) = \sum_{i=i}^{p} EC_f(E_i)$ *as proved in [6].*

Lemma 2.9: **(***k-partition Lemma): Let f be an embedding of G into H. Let* $\{E_1,E_2,…,E_p\}$ *be a partition of k[E(H)] such that each* E_i *is an edge cut of H. Then* $WL_f(G,H) = \frac{1}{k}\sum_{i=i}^{p} EC_f(E_i)$ *as proved in [13].*

Lemma 2.10: **(***Generalized partition Lemma): Let f be an embedding of G into H. For* $1 \le i \le k$*, suppose* $S_i = \{S^i_1, S^i_2, .., S^i_{pi}\}$ *partitions* $E(H)\backslash F_i$ *for mutually disjoint* F_i*'s such that* S^i_j*,* $1 \le j \le p_i$*,* $1 \le i \le k$ *and* $S = \bigcup_{i=1}^{k} F_i$ *are all edge cuts of H. Then* $WL_f(G,H) = \frac{1}{k}\left[\sum_{i=1}^{k}\sum_{j=1}^{p_i} EC_f(S^i_j) + EC_f(S)\right]$ *as proved in [5].*

Lemma 2.11: *(Monophonic congestion Lemma): Let G be an r-regular graph with n vertices. Let H be a finite graph with n vertices. Let* f_m*: G → H be a monophonic embedding of G into H. Let the graph H * E_j*, j = 1,2,…,p; 0 < p <|E(G)|, have the components* H_i*, i =1, 2 and* $G_i = f_m^{-1}(H_i)$*, where* E_j*'s are the edge cuts of H, form a partition in H and have the following properties:*

1) For m ≥ 0, there are m edges (x, y) $\in G_i$, i = 1,2, such that the monophonic path $P_{f_m}(f_m(x), f_m(y))$ has exactly two edges in E_j.
2) The monophonic path $P_{f_m}(f_m(x), f_m(y))$ has exactly one edge in E_j for every (x, y) ∈ G with x ∈ G_1 & y ∈ G_2.

Where

G_1 is a maximum subgraph of G. Then *$MEC_{f_m}(E_j)$* is monophonic and the monophonic wirelength of f_m of G into H is given by $MWL_{f_m}(G,H) = \sum_{j=1}^{p} MEC_{f_m}(E_j)$, where the monophonic edge congestion, $MEC\{f_m\}(E_j) = r|V(G_1)|-2|E(G_1)|+2m, m \ge 0$.

Theorem 2.12:*Let f : G→ H be an embedding, where G be the circulant graph G[2n, ±S], S ⊆ {1,2,3,….,n} and H be the grid M[n × 2]. If d is the diameter of the embedding of G into H, then,*

1) WL(G[2n,{1,2,…,n - 1}], M[n × 2]) = 2n|S| + 1/3 [d(d-1)(2d-1)].
2) WL(G[2n,{1,2,…,n}], M[n × 2]) = (2n|S|1) - n + 1/3 [d(d-1)(2d-1)].
3) WL(G[2n,{1,2,..,n-2}], M[n × 2]) = 2(|S|-1)(n-1) + 1/3[d(d-1)(2d-1)].

3. Average wirelength

The average wirelength of an embedding f: G → H denoted by $\mu W_f(G, H)$, is the expected edge congestion of G into H between a randomly chosen pair of distinct vertices f(x), f(y) ∈ H such that (x, y) ∈ E(G). Therefore $\mu W_f(G,H) = \frac{1}{|E(G)|}\sum_{(x,y)\in E(G)} d_H(f(x), f(y)) = \frac{1}{|E(G)|}\sum_{e\in E(H)} EC_f(G, H(e))$

In the same manner, the average monophonic wirelength of f_m: G → H is given by,

$\mu MW_{f_m}(G,H) = \frac{1}{|E(G)|}\sum_{e\in E(H)} MEC_{f_m}(G, H(e))$.

3.1. Bounds on average wirelength and average monophonic wire length

For n ≥ 2,

$2 \le \mu W_f(G[2n, \{1,2,3,…,n-1\}], M[n \times 2]) \le \lceil n/2 \rceil$ *and*

$$2 \le \mu MW_{f_m}(G[2n,\{1,2,3,….,n-1\}], M[n \times 2]\} \le \left\lceil \frac{n+1}{2} \right\rceil$$

For n ≥ 3,

$2 \le \mu W_f(G[2n, \{1,2,3,…,n-2\}], M[n \times 2]) \le \lceil n/4 \rceil$ *and*

$2 \le \mu MW_{f_m}(G[2n,\{1,2,3,….,n-2\}], M[n \times 2]\} \le \left\lceil \frac{n}{3} \right\rceil$.

For n ≥ 4,

$2 \le \mu W_f(G[2n, \{1,2,3,…,n-3\}], M[n \times 2]) \le \left\lceil \frac{n-1}{4} \right\rceil$ *and*

$2 \le \mu MW_{f_m}(G[2n,\{1,2,3,….,n-3\}], M[n \times 2]\} \le \left\lceil \frac{n-1}{3} \right\rceil$.

4. Monophonic wirelength of circulant networks into cycles

Definition 4.1:*A connected undirected graph represented by G [n, ±S] where S ⊆ {1,2,3,...,[n/2]}, n ≥ 3 is said to be a circulant graph if it consists of the vertex set.*
V = {0,1,2…,n-1} and the edge set
E= {(x, y): |x-y| ≅s (mod n), s ∈ S} as defined in [5].

Definition 4.2:*A graph with a closed walk consisting of n points is called a cycle, denoted by* C_n *as defined in [14].*

Theorem 4.3:*A set of k consecutive vertices of G[n, ±1];* $1 \le k \le n$ *induces a maximum subgraph of G[n,±S]*
Where S ⊆{1,2,3,…,j}, 1 ≤ j ≤ [n/2], n ≥ 3 as defined in [5].

In this section, G denotes the circulant graph G[n, ±S] where S ⊆{1,2,3,…,[n/2]} of n vertices and H denotes the cycle graph C_n with n vertices and f_m, the monophonic embedding from G to H.

Theorem 4.4: If *f_m: G → H is a monophonic embedding, then the wirelength induced by* f_m *from the Circulant graph to Cycle graph is monophonic.*

Proof:

We prove this for two cases.

Case (i) (n is odd)

Let $P_1 = \{P^1_1, P^1_2, ….., P^1_{\frac{n-1}{2}}\}$ and $P_2 = \left\{P^2_1, P^2_2, ….., P^2_{\frac{n-1}{2}}\right\}$ where $P^1_s = \left\{(s-1, s), \left(\frac{(n-3)}{2}+s, \frac{(n-1)}{2}+s\right)\right\}$ and $P^2_s = \left\{(s-1,s), \left(\frac{(n-1)}{2}+s, \frac{(n+1)}{2}+s\right)\right\}$, $1 \le s \le \frac{(n-1)}{2}$, taken modulo n.

Consider the edges, $R_1 = \{(n-1, 0)\}$ and $R_2 = \left\{\left(\frac{(n-1)}{2}, \frac{n+1}{2}\right)\right\}$, clearly P_t partitions $E(H)\backslash R_t$, t=1,2 and the sets R_1 and R_2 are disjoint and their union is an edge cut of H. For each s, $E(H)\backslash P^t_s$ has two components $H^t_{s_1}$ and $H^t_{s_2}$ induced by consecutive vertices on H with $|H^t_{s_1}| = \left\lceil\frac{n}{2}\right\rceil = |H^t_{s_2}|$.Let $G^t_{s_1} = f_m^{-1}(H^t_{s_1})$ and $G^t_{s_2} = f_m^{-1}(H^t_{s_2})$.Then $G^t_{s_1}$ is on $\left\lceil\frac{n}{2}\right\rceil$ consecutive vertices of G[n, ±1] and these vertices induce a maximal subgraph of G[n, ±S] by theorem 4.3. Hence P^t_s satisfies the monophonic congestion lemma and therefore $MEC_{f_m}(P^t_s)$ is monophonic and thus the wire length induced by f_m is monophonic.

Case (ii) (n is even)

Let P_t, t = 1,2,…,n/2 be the edge cuts of H, form a partition in H. For each t, let $P_t = \left\{(t-1,t),\left(\frac{n}{2}+t-1\right),(\frac{n}{2}+t)\right\}$ $1 \le t \le n/2$ where the vertices are taken modulo n. That is P_t has two direct opposite edges of the cycle graph H. Let H_{t_1} and H_{t_2} are the components of H\$P_t$ for each t. Let $G_{t_1} = f_m^{-1}(H_{t_1})$ and $G_{t_2} = f_m^{-1}(H_{t_2})$. Then the graphs G_{t_1} and G_{t_2} are on $\left[\frac{n}{2}\right]$ consecutive points of G[n, ±1] and these points induce a maximal subgraph of G[n, ±S] by theorem 4.3. Hence, P_t satisfies the monophonic congestion lemma and therefore $MEC_{f_m}(P_t)$ is monophonic and thus the wirelength induced by f_m is monophonic.

5. Monophonic Embedding Algorithm

Aim: To find a monophonic embedding $f_m : G \rightarrow H$ that produces the monophonic wirelength $MEC_{f_m}(G,H)$ where G is the family of circulant graph with n vertices of r-regular and H is the cycle graph C_n, $n \ge 4$.

5.1. Monophonic Algorithm

1) Name the vertices of G[n, ±S], S ⊆ {1,2,3,…,n/2} as the vertices of Cycle from 0,1,2,…,n-1.
2) Name the vertices of C_n as 0,1,2,…,n-1 in clockwise.

Input: A family of circulant graphs G[n, ±S], S ⊆ {1,2,3,…,n/2} and the cycle graph C_n.

Output: A monophonic embedding f_m from G[n, ±S], into C_n given by $f_m(x) = x$ with monophonic wirelength MWL(G[n, ±S], C_n) = $\left[\frac{n}{2}\right][MEC_{f_m}(P)]$.

Proof: In theorem 4.4, it is proved that the wirelength induced by f_m from G[n, ±S] to C_n is monophonic. We prove two cases.

Case (i) (n is odd)

Using the notations used in case (i) of theorem 4.4, we have by generalized partition lemma,

$$MWL(G[n,\pm S],C_n) = \frac{1}{2}\{\sum_{t=1}^{2}\sum_{s=1}^{\frac{(n-1)}{2}} MEC_{f_m}(P_s^t) + MEC_{f_m}(P)\}$$

As H\P is isomorphic to H\P_s^t,

$$MWL(G[n,\pm S],C_n) = \frac{1}{2}\{(n-1)MEC_{f_m}(P) + MEC_{f_m}(P)\}$$
$$= \frac{n}{2}MEC_{f_m}(P)$$

Case (ii) (n is even)

Using the notations used in case (ii) of theorem 4.4, we have by generalized partition lemma,

$$MWL(G[n,\pm S],C_n) = \frac{1}{2}\{\sum_{t=1}^{\frac{n}{2}} MEC_{f_m}(P_t)\}$$
$$= \frac{1}{2}\{nMEC_{f_m}(P)\} = \frac{n}{2}MEC_{f_m}(P)$$

5.1.1. Monophonic algorithm (i)

Input: A family of circulant graph $G[n,\left\{1,2,\dots,\left[\frac{n}{2}-1\right]\right\}]$ and the cycle graph C_n, $n \ge 4$.

Output: A monophonic embedding f_m from $G[n,\left\{1,2,\dots,\left[\frac{n}{2}-1\right]\right\}]$ into C_n given by $f_m(x) = x$ with monophonic wirelength,

$$MWL(G\left[n,\left\{1,2,\dots,\left[\frac{n}{2}-1\right]\right\},C_n\right) = \begin{cases} \frac{n}{2}(3|s|^2-|s|)\ ifniseven \\ \frac{n}{2}(3|s|^2-3|s|+2)\ ifnisodd \end{cases}$$

5.1.2. Monophonic algorithm (ii)

Input: A family of circulant graph $G[n,\left\{1,2,\dots,\left[\frac{n}{2}-2\right]\right\}]$, and the cycle graph C_n, $n \ge 5$.

Output: A monophonic embedding f_m from $G[n,\left\{1,2,\dots,\left[\frac{n}{2}-2\right]\right\}]$ into C_n given by $f_m(x) = x$ with monophonic wirelength,

$$MWL(G\left[n,\left\{1,2,\dots,\left[\frac{n}{2}-2\right]\right\},C_n\right) = \begin{cases} \frac{n}{2}(3|s|^2-3|s|-4\ ifniseven \\ \frac{n}{2}(3|s|^2-|s|-2ifnisodd \end{cases}$$

5.1.3. Monophonic algorithm (iii)

Input: A family of circulant graph $G[n,\left\{1,2,\dots,\left[\frac{n}{2}-3\right]\right\}]$, and the cycle graph C_n, $n > 6$.

Output: A monophonic embedding f_m from $G[n,\left\{1,2,\dots,\left[\frac{n}{2}-3\right]\right\}]$ into C_n given by $f_m(x) = x$ with monophonic wirelength,

$$MWL(G\left[n,\left\{1,2,\dots,\left[\frac{n}{2}-3\right]\right\},C_n\right) = \begin{cases} \frac{n}{2}(3|s|^2+7|s|-8)\ ifniseven \\ \frac{n}{2}(3|s|^2+5|s|-6)\ ifnisodd \end{cases}$$

6. Conclusion

In this paper we have applied the monophonic idea on graph embedding of two graphs and also we have given a monophonic algorithm for finding the monophonic wirelength of circulant networks into cycles.

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References

[1] F. Buckley and F. Harary, Distance in Graph, *Addison- Wesley*, California, 1990.

[2] A.P. Santhakumaran and P. Titus, "Monophonic Distance in Graphs", *Discrete Mathematics Algorithms Application*, Vol.3, No.2, (2011), pp.159-169, https://doi.org/10.1142/S1793830911001176.

[3] P. Maya and T. Nicholas, "Some Results on Integer Edge Cordial Graph", *DJ Journal of Engineering and Applied Mathematics*, Vol.3, No.1, (2017), pp. 1-12, https://doi.org/10.18831/djmaths.org/2017011001.

[4] P. Manuel, "Minimum Average Congestion of Enhanced and Augmented Byper-Cube into Complete Binary Tree", *Discrete Applied Mathematics*, Vol.159, No.5, (2010), pp.360-366, https://doi.org/10.1016/j.dam.2010.12.001.

[5] IndraRajasingh, Paul Manuel, M. Arockiaraj and BharatiRajan, "Embeddings of Circulant Networks", *Journal of Combinatorial Optimization*, Vol.26, No.1, (2013), pp.135-151, https://doi.org/10.1007/s10878-011-9443-x.

[6] P. Manuel, I. Rajasingh, B. Rajan and H. Mercy, "Exact Wirelength of Hypercube on a Grid", *Discrete Applied Mathematics*, Vol.157,

No.7, (2009), pp.1486-1495.https://doi.org/10.1016/j.dam.2008.09.013.

[7] Jess Mathew, "Subdivided Stars Super (b, d) - Edge-Antimagic Total Graph Labelling", *DJ Journal of Engineering and Applied Mathematics*, Vol.1, No.1, (2015), pp.30-34, https://doi.org/10.18831/djmaths.org/2015011005.

[8] S.L. Bezrukov, J.D. Chavez, L. H. Harper, M. Rottger and U. P.Schroeder, "Embedding of Hypercubes into Grids", *International Symposium on Mathematical Foundations of Computer Science*, Vol.1450, (1998), pp.693-701, https://doi.org/10.1007/BFb0055820.

[9] L. Bezrukov, "Embedding Complete Trees into the Hypercube", Discrete Applied Mathematics, Vol.110, No.2-3, (2001), pp.101-119,https://doi.org/10.1016/S0166-218X(00)00256-0.

[10] JasinthaQuadras, S. Sarah Surya, "Linear Wirelength of Circulant Networks", *International Journal of Pure and Applied Mathematics*, Vol.86, No.6, (2013), pp.883-891,

[11] P. Manuel, M. Arockiaraj, I. Rajasingh and B. Rajan, "Embedding Hypercubes into Cylinders, Snakes and Caterpillars for Minimizing Wirelength", *Discrete Applied Mathmetics*, Vol.159, No.17, (2011), pp.2109-2116, https://doi.org/10.1016/j.dam.2011.07.003.

[12] I. Rajasingh, J. Quadras, P.Manuel and A.William, "Embedding of Cycles and Wheels into Arbitrary Trees", *Networks*, Vol.44, (2004), pp.173-178, https://doi.org/10.1002/net.20027.

[13] Arokiaraj, Paul Manuel, IndraRajasingh and BharatiRajan, "Wirelength of 1-Fault Hamiltonian Graphs into Wheels and Fans", *Information Processing Letters*, Vol.111, No.18, (2011), pp.921-925,https://doi.org/10.1016/j.ipl.2011.06.011.

[14] F. Harary, Graph Theory, *Addison- Wesley*, 1969.https://doi.org/10.21236/AD0705364.