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Research paper

Applications of Fixed Point Theory in Integral Equations and Homotopy Using Partially Ordered S_b Metric Spaces

[1]D. Ram Prasad, [2]GNV Kishore, [3]K.Priyanka

[1]Research scholar , Department of Mathematics, KL University,Vaddeshwaram,Guntur-522502
[2]Department of Mathematics , KL University, Vaddeshwaram, Guntur-522502.
[3]Department of Mathematics, Nalla Malla Reddy Engineering College, Divyanagar, Ghatkesar, Medchal-500088,
**Corresponding author E-mail:: kunda.priyanka40@gmail.com*

Abstract

In this paper we give some applications to integral equations as well as homotopy theory via Suzuki type fixed point theorems in partially ordered complete S_b - metric space by using generalized contractive conditions. We also furnish an example which supports our main result.

Keywords : *S_b-metric space · S_b–Cauchy sequence · S_b -completeness · fixed point · Suzuki type contraction*

1. Introduction

Banach contraction principle in metric spaces is one of the most important results in fixed theory and nonlinear analysis in general. Since 1922, when Stefan Banach formulated the concept of contraction and posted a famous theorem, scientists around the world publish new results related with generalization of metric space or with contractive mappings. Banach contraction principle is considered to be the initial result of the study of the fixed point theory in metric spaces. Recently Sedghi et al. defined Sb-metric spaces using the concept of S-metric spaces. The aim of this paper is to prove some Suzuki type unique fixed point theorems for generalized contractive conditions in partially ordered S_b-metric spaces, also provide an application of integral equations as well as an application of Homotopy theory. Throughout this paper R,R+ and N denote the set of all real numbers, non-negative real numbers and positive integers respectively. First we recall some definitions, lemmas and examples.

2. Definitions

Definition 1: Let X be a non-empty set. An S−metric on X is a function S : $x^3 \to [0,+\infty)$that satisfies the following conditions for each x,y,z,a ∈ X

(S1) : S(x, y, z) = 0 if and only if x = y = z,

(S2) : S(x, y, z) S(x, x, a) + S(y, y, a) + S(z, z, a) for all x, y, z, a ∈ X.

Then the pair (X, S) is called a S-metric space.

Definition 2: Let X be a non-empty set and b≥ 1 be given real number. Suppose that a mapping $S_b : X^3 \to [0,+\infty)$ be a function satisfying the following properties :

(S_b1) S_b(x, y, z) = 0 if and only if x = y = z,

(S_b2) S_b(x, y, z) ≤ b(S_b(x, x, a) + S_b (y, y, a) + S_b (z, z, a)) for all x, y, z, a ∈ X.

Then the function is called a S_b-metric on X and the pair (X, S) is called a S_b-metric space.

Remark: It should be noted that, the class of S_b -metric spaces is effectively larger than S-metric spaces. Indeed each S-metric space is a S_b metric space with b = 1.

Following example shows that a S_b -metric on X need not be a S-metric on X.

Let (X, S) be S-metric space and S_*(x, y, z) = $(S(x,y,z))^p$, where p > 1 is a real number. Note that S_* is a S_b -metric with b = $2^{2(p-1)}$. Also (X, S_*) is not necessarily a S-metric space.

Definition 3: Let (X, S) be S_b - metric space. Then, for xϵX, r > 0. We defined the open ball B_S(x, r) and closed ball B_S [x, r] with center x and radius r as follows respectively:

B_S(x, r) = {y ϵ X: S(y, y, x) < r}, B_S[x, r] = {y ϵ X : S(y, y, x) ≤ r}.

Lemma1: In a S_b-metric space, we have S(x, x, z) ≤2bS(x, x, y) + b^2S(y, y, z)

Definition 4: Let (X, S) be a S_b -metric space. A sequence $\{x_n\}$ in X is said to be:

(1) S_b -Cauchy sequence if, for each ϵ > 0, there exists n_0 ∈ N such that S(x_n, x_n, x_m) <ε for each m, n ≥ n_0.

(2) S_b -convergent to a point x ∈ X if, for each ε > 0, there exists a positive integer n_0 such that S(x_n, x_n, x) < ϵ or S(x, x, x_n)<ε for all n ≥ n_0 and we denote by $\lim_{n\to\infty} x_n = x$

Definition 5: A S_b -metric space (X, S) is called complete if every S_b - Cauchy sequence is S_b -convergent in X.

Lemma 2. Let (X,S) be a S_b-metric space with $b \geq 1$ and suppose that $\{x_n\}$ is a S_b-convergent to x, then we have

$\frac{1}{2b}S(y,y,x) \leq \lim_{n\to\infty} \inf S(y,y,x_n) \leq \lim_{n\to\infty} \sup S(y,y,x_n) \leq 2bS(y,y,x)$ and

$\frac{1}{b^2}S(x,x,y) \leq \lim_{n\to\infty} \inf S(x_n,x_n,y) \leq \lim_{n\to\infty} \sup S(x_n,x_n,y) \leq b^2 S(x,x,y)$ for all $y \in X$.

In particular if $x = y$ then we have $\lim_{n\to\infty} S(x_n,x_n,y) = 0$.

Now we prove our main results.

2.1 Results and Discussions

Definition 6: Let (X, S_b,≼) be a partially ordered complete S_b -metric space which is also regular, and f : X →X be mapping. We say that f is satisfy Suzuki type generalized φ- contraction if there exists φ : [0,∞) →[0,∞) such that

(6.1) f is non - decreasing and φ is lower semi continuous,

(6.2) φ(t) = 0 if and only if t = 0,

(6.3) $\frac{1}{4b^3}$ min {S_b(x, x, fx), S_b(y, y, fy)}≤ S_b (x, x, y) implies that

$4b^4$ S_b (fx, fx, fy)≤ M_f^i(x,y)- φ(M_f^i(x,y)) for all x,y ∈ X, x comparable to y, i=3 or 4 or. also

$M_f^5(x,y) = \max\{S_b(x,x,y), S_b(x,x,fx), S_b(y,y,fy), S_b(x,x,fy), S_b(y,y,fx)\}$

$M_f^4(x,y) = \max\{S_b(x,x,y), S_b(x,x,fx), S_b(y,y,fy), \frac{1}{4b^4}[S_b(x,x,fy), S_b(y,y,fx)]\}$

$M_f^3(x,y) = \max\{S_b(x,x,y), \frac{1}{4b^4}[S_b(x,x,fx)+S_b(y,y,fy)], \frac{1}{4b^4}[S_b(x,x,fy)+S_b(y,y,fx)]\}$

$M_f^4(x,y) = \max\{S_b(x,x,y), S_b(x,x,fx), S_b(y,y,fy), \frac{1}{4b^3}[S_b(x,x,fy), S_b(y,y,fx)]\}$

Theorem 1. Let $(X,S_b,\preccurlyeq)$ be an ordered complete S_b metric space, which is also regular and let $f : X \to X$ satisfies Suzuki type generalized φ - contraction with $i = 5$. If there exists $x_0 \in X$ with $x_0 \preccurlyeq f_{x_0}$,then f has a unique fixed point in X.

Proof: Since f is a mapping from X into X, there exists a sequence $\{x_n\}$ in X such that

$x_{n+1} = f_{x_n}$, n = 0, 1, 2, 3, · · ·

Case(i): If $x_n = x_{n+1}$, then x_n is a fixed point of f.

Case(ii): Suppose $x_n \neq x_{n+1}$ for all n.

Since $x_0 \preccurlyeq f_{x_0} = x_1$ and f is non-decreasing, it follows that $x_0 \preccurlyeq f_{x_0} \preccurlyeq f^2_{x_0} \preccurlyeq f^3_{x_0} \preccurlyeq \cdots \preccurlyeq f^n_{x_0} \preccurlyeq f^{n+1}_{x_0} \preccurlyeq \cdots\cdots\cdots$

Since $\frac{1}{4b^3}\min\{S_b(x_0,x_0,x_0), S_b(x_1,x_1,f_{x_1})\} \leq S_b(x_0,x_0,x_1)$.

Now from (21.3), we have that $4b^4 S_b(f_{x_0},f_{x_0},f^2_{x_0}) = 4b^4 S_b(f_{x_0},f_{x_0},f_{x_1}) \leq M_f^5(x_0,x_1) - \varphi\left(M_f^5(x_0,x_1)\right)$,

$$\leq \max\{S_b(x_0,x_0,f_{x_0}), S_b(f_{x_0},f_{x_0},f^2_{x_0}), S_b(x_0,x_0,f^2_{x_0})\}$$
$$- \varphi(\{S_b(x_0,x_0,f_{x_0}), S_b(f_{x_0},f_{x_0},f^2_{x_0}), S_b(x_0,x_0,f^2_{x_0})\})$$
$$\leq \{S_b(x_0,x_0,f_{x_0}), S_b(f_{x_0},f_{x_0},f^2_{x_0}), S_b(x_0,x_0,f^2_{x_0})\}.$$

Based on above, we have that

$$S_b(f_{x_0},f_{x_0},f^2_{x_0}) \leq \max,\left\{\frac{1}{4b^4}S_b(x_0,x_0,f_{x_0}), \frac{1}{4b^4}S_b(f_{x_0},f_{x_0},f^2_{x_0}), \frac{1}{4b^4}S_b(x_0,x_0,f^2_{x_0})\right\} \quad (2)$$

But here

$$\frac{1}{4b^4}S_b(x_0,x_0,f^2_{x_0}) \leq \frac{1}{4b^4}[2bS_b(x_0,x_0,f_{x_0}) + b^2 S_b(f_{x_0},f_{x_0},f^2_{x_0})]$$
$$\leq \max\left\{\frac{1}{b^3}S_b(x_0,x_0,f_{x_0}), \frac{1}{2b^2}S_b(f_{x_0},f_{x_0},f^2_{x_0})\right\}.$$

From (2), we have that

$$S_b(f_{x_0},f_{x_0},f^2_{x_0}) \leq \max\left\{\frac{1}{b^3}S_b(x_0,x_0,f_{x_0}), \frac{1}{2b^2}S_b(f_{x_0},f_{x_0},f^2_{x_0})\right\}.$$

If $\frac{1}{2b^2}S_b(f_{x_0},f_{x_0},f^2_{x_0})$ is maximum, we get a contradiction. Hence

$$S_b(f_{x_0},f_{x_0},f^2_{x_0}) \leq \frac{1}{b^3}S_b(x_0,x_0,f_{x_0}) \quad (3)$$

Also since $\frac{1}{4b^3}\boldsymbol{min}\{S_b(x_1,x_1,f_{x_1}), S_b(x_2,x_2,f_{x_2})\} \leq S_b(x_1,x_1,x_2.)$.

From (6.3), it follows

$$4b^4 S_b(f^2_{x_0},f^2_{x_0},f^3_{x_0}) = 4b^4 S_b(f_{x_1},f_{x_1},f_{x_2})$$
$$\leq M_f^5(x_1,x_2) - \varphi\left(M_f^5(x_1,x_2)\right),$$
$$\leq \max\{S_b(f_{x_0},f_{x_0},f^2_{x_0}), S_b(f_{x_0},f_{x_0},f^2_{x_0}), S_b(f_{x_0},f_{x_0},f^3_{x_0})\}$$
$$- \varphi(\{S_b(f_{x_0},f_{x_0},f^2_{x_0}), S_b(f^2_{x_0},f^2_{x_0},f^3_{x_0}), S_b(f_{x_0},f_{x_0},f^3_{x_0})\})$$
$$\leq \max\{S_b(f_{x_0},f_{x_0},f^2_{x_0}), S_b(f^2_{x_0},f^2_{x_0},f^3_{x_0}), S_b(f_{x_0},f_{x_0},f^3_{x_0})\}.$$

Based on above , we have that

$$S_b(f^2_{x_0},f^2_{x_0},f^3_{x_0}) \leq \max\left\{\frac{1}{4b^4}S_b(f_{x_0},f_{x_0},f^2_{x_0}), \frac{1}{4b^4}S_b(f^2_{x_0},f^2_{x_0},f^3_{x_0}), \frac{1}{4b^4}S_b(f_{x_0},f_{x_0},f^3_{x_0})\right\}$$

$$\frac{1}{4b^4}S_b(f_{x_0},f_{x_0},f^3_{x_0})$$
$$\leq \frac{1}{4b^4}[2bS_b(f_{x_0},f_{x_0},f^2_{x_0}) + b^2 S_b(f^2_{x_0},f^2_{x_0},f^3_{x_0})]$$
$$\leq \max\left\{\frac{1}{b^3}S_b(x_0,x_0,f_{x_0}), \frac{1}{2b^2}S_b(f_{x_0},f_{x_0},f^2_{x_0})\right\}.$$

From (4), we have that $S_b(f^2_{x_0},f^2_{x_0},f^3_{x_0}) \leq \max\left\{\frac{1}{b^3}S_b(x_0,x_0,f_{x_0}), \frac{1}{2b^2}S_b(f_{x_0},f_{x_0},f^2_{x_0})\right\}$.

If $\frac{1}{2b^2}S_b(f^2_{x_0},f^2_{x_0},f^3_{x_0})$ is maximum, we get a contradiction. Hence

$$S_b(f_{x_0},f_{x_0},f^2_{x_0}) \leq \frac{1}{b^3}S_b(f_{x_0},f_{x_0},f^2_{x_0}) \leq \frac{1}{(b^3)^2}S_b(f_{x_0},f_{x_0},f^2_{x_0})$$

Continuing this process, we can conclude that

$$S_b(f^n_{x_0,}f^n_{x_0,}f^{n+1}_{x_0,}) \leq \frac{1}{b^3}\, S_b(f^{n-1}_{x_0,}f^{n-1}_{x_0,}f^n_{x_0,}) \quad (5)$$
$$\vdots$$
$$\leq \frac{1}{(b^3)^{n-1}}S_b(f_{x_0},f_{x_0},f^2_{x_0})$$
$$\leq \frac{1}{(b^3)^n}S_b(x_0,x_0,f_{x_0})$$
$$\to 0 \; as \; n \to \infty.$$

That is $\lim_{n\to\infty} S_b(f^n_{x_0,}f^n_{x_0,}f^{n+1}_{x_0,}) = 0$ (6)

Now we prove that $\{f^n_{x_0}\}$ is S_b-Cauchy sequence in $(X,S_b,\preccurlyeq)$. On contrary we suppose that $\{f^n_{x_0}\}$ is not a S_b−Cauchy. Then there exist $\epsilon > 0$ and monotonically increasing sequence of natural numbers $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k$.

$S_b\left(f^{m_k}_{x_0,}f^{m_k}_{x_0,}f^{n_k}_{x_0,}\right) \geq \epsilon$ (7) and $S_b\left(f^{m_k}_{x_0,}f^{m_k}_{x_0,}f^{n_k-1}_{x_0,}\right) < \epsilon$ (8)

First we claim that

$\frac{1}{4b^3}\min\left\{S_b\left(x_{m_k},x_{m_k},f_{x_{m_k}}\right), S_b\left(x_{n_k-1},x_{n_k-1},f_{x_{n_k-1}}\right)\right\} \leq S_b(x_{m_k},x_{m_k},x_{n_k-1})$. (9)

On contrary suppose that

$\frac{1}{4b^3}\min\left\{S_b\left(x_{m_k},x_{m_k},f_{x_{m_k}}\right), S_b\left(x_{n_k-1},x_{n_k-1},f_{x_{n_k-1}}\right)\right\} > S_b(x_{m_k},x_{m_k},x_{n_k-1})$

Now consider

$$\epsilon \le S_b\left(f_{x_0,}^{m_k} f_{x_0,}^{m_k} f_{x_0,}^{n_k}\right)$$
$$\le 2bS_b\left(f_{x_0,}^{m_k} f_{x_0,}^{m_k} f_{x_0,}^{n_k-1}\right) + b^2 S_b\left(f_{x_0,}^{n_k-1} f_{x_0,}^{n_k-1} f_{x_0,}^{n_k}\right)$$
$$< \frac{1}{2b^2} min\left\{S_b\left(f_{x_0,}^{m_k} f_{x_0,}^{m_k} f_{x_0,}^{m_k+1}\right), S_b\left(x_{n_k-1}, x_{n_k-1}, f_{x_{n_k-1}}\right)\right.$$
$$\left. + b^2 S_b\left(f_{x_0,}^{n_k-1} f_{x_0,}^{n_k-1} f_{x_0,}^{n_k}\right)\right\}.$$

Letting $k \to \infty$, it follows that$\epsilon \le 0$. It is a contradition. Hence our claim (9) is holds. From (7) and (8), we have

$$\epsilon \le S_b\left(f_{x_0,}^{m_k} f_{x_0,}^{m_k} f_{x_0,}^{n_k}\right)$$
$$\le 2bS_b\left(f_{x_0,}^{m_k} f_{x_0,}^{m_k} f_{x_0,}^{m_k+1}\right)$$
$$+ b^2 S_b\left(f_{x_0,}^{m_k+1} f_{x_0,}^{m_k+1} f_{x_0,}^{n_k}\right)$$

Letting $k \to \infty$, we have that

$$4b^2\epsilon \le \lim_{k\to\infty} 4b^4 S_b\left(f_{x_0,}^{m_k+1} f_{x_0,}^{m_k+1} f_{x_0,}^{n_k}\right)$$
$$= \lim_{k\to\infty} 4b^4 S_b\left(x_{m_k+1}, x_{m_k+1}, x_{n_k}\right)$$
$$= \lim_{k\to\infty} 4b^4 S_b\left(f_{x_{m_k}}, f_{x_{m_k}}, f_{x_{n_k-1}}\right)$$
$$\le \lim_{k\to\infty} M_f^5\left(x_{m_k}, x_{n_k-1}\right)$$
$$- \lim_{k\to\infty} \varphi\left(M_f^5\left(x_{m_k}, x_{n_k-1}\right)\right)$$
$$\le \lim_{k\to\infty} M_f^5\left(x_{m_k}, x_{n_k-1}\right)$$
$$= \lim_{k\to\infty} max\left\{S_b\left(f_{x_0,}^{m_k} f_{x_0,}^{m_k} f_{x_0,}^{n_k-1}\right), S_b\left(f_{x_0,}^{m_k} f_{x_0,}^{m_k} f_{x_0,}^{m_k+1}\right), S_b\left(f_{x_0,}^{n_k-1k} f_{x_0,}^{n_k-1} f_{x_0,}^{n_k}\right), S_b\left(f_{x_0,}^{m_k} f_{x_0,}^{m_k} f_{x_0,}^{n_k}\right), S_b\left(f_{x_0,}^{n_k-1} f_{x_0,}^{n_k-1} f_{x_0,}^{m_k+1}\right)\right\}$$
$$< \lim_{k\to\infty} max\left\{\epsilon, 0, 0, S_b\left(f_{x_0,}^{m_k} f_{x_0,}^{m_k} f_{x_0,}^{n_k}\right), S_b\left(f_{x_0,}^{n_k-1} f_{x_0,}^{n_k-1} f_{x_0,}^{m_k+1}\right)\right\}$$

But

$$\lim_{k\to\infty} S_b\left(f_{x_0,}^{m_k} f_{x_0,}^{m_k} f_{x_0,}^{n_k}\right) \le \lim_{k\to\infty}\left[2bS_b\left(f_{x_0,}^{m_k} f_{x_0,}^{m_k} f_{x_0,}^{n_k-1}\right) + b^2 S_b\left(f_{x_0,}^{n_k-1} f_{x_0,}^{n_k-1} f_{x_0,}^{n_k}\right)\right] < 2b\epsilon.$$

Also,

$$\lim_{k\to\infty} S_b\left(f_{x_0,}^{n_k-1} f_{x_0,}^{n_k-1} f_{x_0,}^{m_k+1}\right) \le \lim_{k\to\infty}\left[2bS_b\left(f_{x_0,}^{n_k-1} f_{x_0,}^{n_k-1} f_{x_0,}^{m_k}\right) + b^2 S_b\left(f_{x_0,}^{m_k} f_{x_0,}^{m_k} f_{x_0,}^{m_k+1}\right)\right] < 2b^2\epsilon.$$

Therefore from (10), we have that, $4b^2\epsilon \le max\{\epsilon, 2b\epsilon, 2b^2\epsilon\} = 2b^2\epsilon$.
It is a contradiction. Hence $\{f_{x_0}^n\}$ is a S_b −Cauchy sequence in complete regular Sb - metric space $(X, S_b, \preccurlyeq)$.

By completeness of (X, S_b), it follows that the sequence $\{f_{x_0}^n\}$ is converges to α in (X, S_b). Thus $\lim_{k\to\infty} f_{x_0}^n = \alpha = \lim_{k\to\infty} f_{x_0}^{n+1}$.

First we claim that for each $n \ge 1$, at least one of the following assertion is holds.

$\frac{1}{4b^3} S_b(x_{n+1}, x_{n+1}, x_n) \le S_b(\alpha, \alpha, x_n)$ or
$\frac{1}{4b^3} S_b(x_n, x_n, x_{n-1}) \le S_b(\alpha, \alpha, x_{n-1})$.

On contrary suppose that
$\frac{1}{4b^3} S_b(x_{n+1}, x_{n+1}, x_n) > S_b(\alpha, \alpha, x_n)$
and $\frac{1}{4b^3} S_b(x_n, x_n, x_{n-1}) > S_b(\alpha, \alpha, x_{n-1})$.

Now consider

$$S_b(x_{n-1}, x_{n-1}, x_n) \le 2bS_b(x_{n-1}, x_{n-1}, \alpha) + b^2 S_b(\alpha, \alpha, x_n)$$
$$< 2b^2 S_b(\alpha, \alpha, x_{n-1})$$
$$+ b^2 \frac{1}{4b^3} S_b(x_{n+1}, x_{n+1}, x_n)$$
$$< 2b^2 \frac{1}{4b^3} S_b(x_n, x_n, x_{n-1})$$
$$+ \frac{1}{4b} S_b(x_{n+1}, x_{n+1}, x_n)$$
$$= \frac{1}{2b} bS_b(x_{n-1}, x_{n-1}, x_n)$$
$$+ \frac{1}{4b} bS_b(x_{n+1}, x_{n+1}, x_n)$$
$$\le \frac{1}{2} S_b(x_{n-1}, x_{n-1}, x_n)$$
$$+ \frac{1}{4b^3} S_b(x_{n-1}, x_{n-1}, x_n)$$
$$= \frac{2b^3+1}{4b^3} S_b(x_{n-1}, x_{n-1}, x_n)$$
$$\le \frac{3}{4} S_b(x_{n-1}, x_{n-1}, x_n).$$

It is a contradiction. Hence our claim is holds.
Now we have to prove that α is fixed point of f.

Since x_n, $\alpha \in X$ and X is regular, it follows that either $x_n \preccurlyeq \alpha$ or $\alpha \preccurlyeq x_n$.
Suppose $f\alpha \ne \alpha$,
From (6.3) and by known result, we have that

$$4b^4\left(\frac{1}{2b} S_b(f\alpha, f\alpha, \alpha)\right)$$
$$\le \lim_{n\to\infty} \inf 4b^4\left(S_b\left(f\alpha, f\alpha, f_{x_0}^{n+1}\right)\right)$$
$$\le \lim_{n\to\infty} \inf M_f^5(\alpha, x_n)$$
$$- \lim_{n\to\infty} \inf \varphi\left(M_f^5(\alpha, x_n)\right). \quad (11)$$

Here,

$$\lim_{n\to\infty} \inf M_f^5(\alpha, x_n)$$
$$= \lim_{n\to\infty} \inf \max\{S_b(\alpha, \alpha, x_n), S_b(\alpha, \alpha, f_\alpha), S_b(x_n, x_n, f_{x_n}), S_b(\alpha, \alpha, f_{x_n}),$$
$$\le \lim_{n\to\infty} \sup \max\{0, S_b(\alpha, \alpha, f_\alpha), 0, 0, S_b(x_n, x_n, f_\alpha)\}$$
$$\le \max\{bS_b(f_\alpha, \alpha, \alpha), b^3 S_b(f_\alpha, f_\alpha, \alpha)\} = b^3 S_b(f_\alpha, f_\alpha, \alpha).$$

Hence from (11), we have that

$$2b^3 S_b(f_\alpha, f_\alpha, \alpha) \le b^3 S_b(\alpha, \alpha, f_\alpha)$$
$$- \lim_{n\to\infty} \inf \varphi\left(M_f^5(\alpha, x_n)\right)$$
$$\le b^3 S_b(f_\alpha, f_\alpha, \alpha).$$

It is a contradiction. So that α is fixed point of f.
Suppose α^* is another fixed point of f such that $\alpha \ne \alpha^*$

Since $\frac{1}{4b^3} \min\{S_b(\alpha, \alpha, f_\alpha), S_b(\alpha^*, \alpha^*, f_{\alpha^*})\} \le S_b(\alpha, \alpha, \alpha^*)$.

Since $\alpha, \alpha^* \in X$ and X is regular we have that α and α^* are comparable.
From (6.3), we have

$$4b^4 S_b(\alpha, \alpha, \alpha^*) \le M_f^4(\alpha, \alpha^*) - \varphi\left(M_f^4(\alpha, \alpha^*)\right)$$
$$= \max\{S_b(\alpha, \alpha, \alpha^*), S_b(\alpha^*, \alpha^*\alpha)\}$$
$$- \varphi(\max\{S_b(\alpha, \alpha, \alpha^*), S_b(\alpha^*, \alpha^*\alpha)\})$$
$$\le bS_b(\alpha, \alpha, \alpha^*).$$

It is a contradiction. Hence α is unique fixed point of f in (X, S_b).

Example 1. let $X = [0,1]$ and $S: X \times X \times X \to \mathbb{R}^+$ by $S_b(x,y,z) = (|y+z-2x|) + |y-z|^2$ and $\preccurlyeq$ by $a \preccurlyeq b \iff a \leq b$, then $(X, S_b, \preccurlyeq)$ is complete ordered S_b- metric space with $b = 4$.
Define $f: X \to X$ by $f(x) = \frac{x}{32\sqrt{2}}$. also definr $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\varphi(t) = \frac{1}{2}$.
Clearly for all $x, y \in X, \frac{1}{4b^3}\min\{S_b(x,x,f_x), S_b(y,y,f_y)\} \leq S_b(x,x,y)$. And

$$4b^4 S_b(f_x, f_x, f_y) = 4b^4(|y+z-2x|) + |y-z|)^2$$
$$= 4b^4\left(2\left|\frac{x}{32\sqrt{2}} - \frac{y}{32\sqrt{2}}\right|\right)^2 = \frac{1}{2}S_b(x,x,y)$$
$$\leq \frac{1}{2}M_f^5(x,y) = M_f^5(x,y) - \varphi\left(M_f^5(x,y)\right),$$

Where
$M_f^5(x,y) =$
$\max\{S_b(x,x,y), S_b(x,x,f_x)S_b(y,y,f_y)S_b(x,x,f_y)S_b(y,y,f_x).\}$
Hence from Theorem 1, 0 is Unique fixed point of f.

Theorem 1.2: Let $(X, S_b, \preccurlyeq)$ be an ordered complete S_b- metric space and let $f: X \to X$ be satisfies Suzuki type generalized $\varphi - contraction$ with $i = 3$ or 4. If there exists $x_0 \in X\ with\ x_0 \preccurlyeq f_{x_0}$. Then f has unique fixed point in X.

Theorem 1.3: Let $(X, S_b, \preccurlyeq)$ be an ordered complete S_b- metric space and let $f: X \to X$ be satisfies

$$\frac{1}{4b^3}\min\{S_b(x,x,f_x), S_b(y,y,f_y)\} \leq S_b(x,x,y) \Rightarrow S_b(f_x, f_x, f_y)$$
$$\leq \lambda M_f^i(x,y),$$

Where $\lambda \in \left[0, \frac{1}{4b^3}\right)$ and $i = 3\ or\ 4\ or\ 5$. If there exists $x_0 \in X\ with\ x_0 \preccurlyeq f_{x_0}$.Then f has unique fixed point in X.

3. APPLICATIONS

3.1 Application to integral equations

In this section, we study the existence of a unique solution to an initial value problem, as an application to Theorem 1.

Theorem 2. Consider the initial value problem $x'(t) = T(t, x(t))$, $t \in I = [0,1]$, $x(0) = x_0$ (1)
Where $T: I \times \left[\frac{x_0}{4}, \infty\right) \to \left[\frac{x_0}{4}, \infty\right)$ with $\int_0^t T(x(s), y(s))ds = \min\left\{\int_0^t T(s, x(s))ds, \int_0^t T(s, y(s))ds,\right\}$ and $x_0 \in \mathbb{R}$.
Then there exists unique solution in $C(I, \left[\frac{x_0}{4}, \infty\right))$ for the initial value problem (1).

Proof: The integral equation corresponding to initial value problem (1) is $x(t) = x_0 + \int_0^t T(s, x(s))ds$,
Let $X = C\left(I, \left[\frac{x_0}{4}, \infty\right)\right)$ and $S_b(x,y,z) = (|y+z-2x|) + |y-z|)^2$ for $x, y \in X$.
Define $\varphi: [0,\infty) \to [0,\infty)$ by $\varphi(t) = \frac{3t}{4}$
Define $f: X \to X$ by
$f(x)(t) = \frac{x_0}{4b^2} + \int_0^t T(x(s), y(s))ds$ (2)
Clearly for all $x, y \in X, we\ have\ \frac{1}{4b^3}\min\{S_b(x,x,f_x), S_b(y,y,f_y)\} \leq S_b(x,x,y)$
Now

$$4b^4 S_b(f\,x(t), f\,x(t), f\,y(t))$$
$$= 4b^4\{(|fx(t) + fy(t) - 2fx(t)|) + |fx(t) - fy(t)|^2\}$$
$$= 16b^4|fx(t) - fy(t)|^2 = \frac{16b^4}{16b^4}|x_0 - y_0|^2$$
$$\leq |x(t) - y(t)|^2 = \frac{1}{4}S_b(x,x,y)$$
$$\leq M_f^5(x,y) - \varphi\left(M_f^5(x,y)\right),$$

Where
$M_f^5(x,y) =$
$\max\{S_b(x,x,y), S_b(x,x,f_x)S_b(y,y,f_y)S_b(x,x,f_y)S_b(y,y,f_x).\}$

It follows from Theorem 1, we conclude that f has a unique fixed point in X.

3.2 Applications to homotopy

Theorem 3 : Let (X, S_b) be a complete S_b - metric space, U be an open subset of X and $\overline{U}$ be closed subset of X such that $U \subseteq \overline{U}$. Suppose $H: \overline{U} \times [0,1] \to X$ be an operator such that the following conditions are satisfying,
(3.1)$x \neq H(x, \lambda)$ for each $x \in \partial U$ and $\lambda \in [0,1]$, (here ∂U denotes the boundary of b U in X),
(3.2) $\frac{1}{4b^3}\min\{S_b(x,x,H(x,\lambda)), S_b(y,y,H(y,\lambda))\} \leq S_b(x,x,y)$
implies that

$$4b^4 S_b(H(x,\lambda), H(x,\lambda), H(y,\lambda)) \leq S_b(x,x,y) - \varphi((x,x,y))$$

For all $x, y \in \overline{U}$ and $\lambda \in [0,1]$ where$f: [0,\infty) \to [0,\infty)$ is continuous, non decreasing and $\varphi : [0,\infty) \to [0,\infty)$ is lower semi continuous with $\varphi(t) > 0$ for $t > 0$,
(3.3) there exists $M \geq 0$ such that

$$S_b(H(x,\lambda), H(x,\lambda), H(y,\lambda)) \leq M|\lambda - \mu|$$

for every x ∈ U and λ, μ ∈ [0, 1]. Then H(., 0) has a fixed point if and only if H(., 1) has a fixed point.

Proof : Consider the set

$$A = \{\lambda \in [0,1]: x = H(x,\lambda) \text{ for some } x \in U\}.$$

Since H (., 0) has a fixed point in U, we have that $0 \in A$.
So that A is non-empty set.

We will show that A is both open and closed in [0, 1] and so by the connectedness we have that $A = [0,1]$.
As a result, H (., 1) has a fixed point in U. First we show that A is closed in [0, 1].

To see this let $\{\lambda_n\}_{n=1}^{\infty} \subseteq A$ with $\lambda_n \to \lambda \in [0,1]$ as $n \to \infty$.

Since $\lambda_n \in A$ for n = 1, 2, 3, · · ·, there exists $x_n \in U$ with $x_n = H(x_n, \lambda_n)$Consider
$S_b(x_n, x_n, x_{n+1}) = S_b(H(x_n,\lambda_n), H(x_n,\lambda_n), H(x_{n+1},\lambda_{n+1})) \leq$
$2bS_b(H(x_n,\lambda_n), H(x_n,\lambda_n), H(x_{n+1},\lambda_{n+1})) +$
$b^2 S_b(H(x_{n+1},\lambda_n), H(x_{n+1},\lambda_n), H(x_{n+1},\lambda_{n+1})) \leq$
$2bS_b(H(x_n,\lambda_n), H(x_n,\lambda_n), H(x_{n+1},\lambda_{n+1})) + b^2 M|\lambda_n - \lambda_{n+1}|$.

Letting $n \to \infty$, we get $\lim_{n\to\infty} S_b(x_n, x_n, x_{n+1}) \leq$
$\lim_{n\to\infty} 2bS_b(H(x_n,\lambda_n), H(x_n,\lambda_n), H(x_{n+1},\lambda_{n+1})) + 0.$
Since
$\frac{1}{4b^3}\min\left(S_b(x_n, x_n, H(x_n,\lambda)), S_b(x_{n+1}, x_{n+1}, H(x_{n+1},\lambda))\right) \leq S_b(x_n, x_n, x_{n+1})$.

Therefore from 3.2, we have that

$$\lim_{n\to\infty} S_b(x_n, x_n, x_{n+1})$$
$$\leq \lim_{n\to\infty} 4b^3 S_b(H(x_n,\lambda_n), H(x_n,\lambda_n), H(x_{n+1},\lambda_n))$$
$$\leq \lim_{n\to\infty} [S_b(x_n, x_n, x_{n+1}) - \varphi(S_b(x_n, x_n, x_{n+1}))].$$

It follows that $\lim_{n\to\infty} S_b(x_n, x_n, x_{n+1}) = 0$ (3)

Now we prove that $\{x_n\}$ is a S_b−Cauchy sequence in (X, S_b).
On contrary suppose that $\{x_n\}$ is not a S_b−Cauchy.

There exists an $\epsilon > 0$ and monotone increasing sequence of natural numbers $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k$,

$S_b(x_{m_k}, x_{m_k}, x_{n_{k-1}}) < \epsilon$ (5)

From (4) and (5), we get $\epsilon \le S_b(x_{m_k}, x_{m_k}, x_{n_k}) \le 2bS_b(x_{m_k}, x_{m_k}, x_{m_k+1}) + b^2 S_b(x_{m_k+1}, x_{m_k+1}, x_{n_k})$.
Letting $k \to \infty$, we have, $\frac{\epsilon}{b^2} \le \lim_{n\to\infty} S_b(x_{m_k+1}, x_{m_k+1}, x_{n_k})$.
But

$\lim_{n\to\infty} S_b(x_{m_k+1}, x_{m_k+1}, x_{n_k}) \le 4b^4 \lim_{n\to\infty} S_b\left(H(x_{m_k+1}, \lambda_{m_k+1}), H(x_{m_k+1}, \lambda_{m_k+1}), H(x_{n_k}, \lambda_{n_k})\right) \le \lim_{n\to\infty} \left[S_b(x_{m_k+1}, x_{m_k+1}, x_{n_k}) - \varphi\left(S_b(x_{m_k+1}, x_{m_k+1}, x_{n_k})\right)\right]$.

It follows that $\lim_{n\to\infty} S_b(x_{m_k+1}, x_{m_k+1}, x_{n_k}) = 0$ such that $\epsilon = 0$,
It is a contradiction.
Hence $\{x_n\}$ is a S_b−Cauchy sequence in (X, S_b). and by the completeness of (X, S_b), there exists $\alpha \in U$ with

$\lim_{n\to\infty} x_n = \alpha = \lim_{n\to\infty} x_{n+1}$ (6)

Since $\frac{1}{4b^3} \min\left(S_b(\alpha, \alpha, H(\alpha, \lambda)), S_b(x_n, x_n, H(x_n, \lambda))\right) \le S_b(\alpha, \alpha, x_n)$.

$$\begin{aligned}
&\frac{1}{2b} S_b(H(\alpha,\lambda), H(\alpha,\lambda), \alpha) \\
&\le \lim_{n\to\infty} \inf \frac{1}{2b} S_b\left(H(\alpha,\lambda), H(\alpha,\lambda), H(x_n, \lambda)\right) \\
&\le \lim_{n\to\infty} \inf 4b^4 S_b\left(H(\alpha,\lambda), H(\alpha,\lambda), H(x_n, \lambda)\right) \\
&\le \lim_{n\to\infty} \inf \left[S_b(\alpha, \alpha, x_n) - \varphi\left(S_b(\alpha, \alpha, x_n)\right)\right] = 0.
\end{aligned}$$

4. Conclusions

In this paper we conclude some applications on Homotopy theory and integral equations by using fixed point theorems in partially ordered S_b - metric spaces.

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Author Details:

[1] Research Scholar ,Department of Mathematics, KL University, Vaddeshwaram, Guntur-522502, Andhrapradesh , ,India
[2]Department of Mathematics, K L University, Vaddeswaram, Guntur -522 502, Andhra Pradesh, India
[3]Department of Mathematics, Nalla Malla Reddy Engineering college, DivyaNagar, Narapally, Hyderabad- 500088, Telangana, India

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