International Journal of Engineering & Technology, 7 (4) (2018) 3889-3896

International Journal of Engineering & Technology

Website: www.sciencepubco.com/index.php/IJET
doi: 10.14419/ijet. v7i4.20515
Research paper

Analytical and numerical approaches for calculating the static deflection of functionally graded beam under mechanical load

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Abstract

Functionally graded material is a new type of composite material and it is used in several smart applications. The static deflec-tion of the FG beam under mechanical and/or thermal load is a complex phenomenon because of the complexity of material properties. The power law model is used to described the material properties of FG beam. The analytical approach depending on the compound beam theory is derived in order to find the equivalent cross section area of FG beam and then calculate the static deflection for new beam. This approach can be achieved analytically when the power law index equals 1 and achieved by numerical integrals (Trapezoidal and Simpson Method) for any value of power law index. The numerical approach (Finite Element Method using ANSYS software) used in this work depends on the laminate theory , compound beam theory in order to build three different models. The validation of these methods were done by comparing the results with the results of Alexraj et. al. [9]. The comparison among the five methods were done and analysis to choose the suitable method.

Keywords: *FG Beam; Static Deflection; Finite Element Method; ANSYS Software; Trapezoidal Method; Simpson Method; Power Law Model; Cantilever Beam; Simply Supported Beam.*

1. Introduction

Composite material is a material made by combining two or more constituent materials with significantly different mechanical , physical or chemical properties in order to produce a material with characteristics different from the individual components. An ex-cellent combination of properties in additional to light weight will be produced when it compared with individual parent materials. With the increasing demands in modern technologies of the con-ventional homogeneous composite material, the new advanced materials with special mechanical characteristics should be fabri-cated. Functionally graded materials (FGM) are one of these ad-vanced materials which have unique properties such as thermal resistance, high toughness, and low density. Functionally graded materials can be defined as the material which the volume frac-tions of two or more individual components material are varied continuously as a function of position along certain dimensions of the structure to achieve a required function. The smooth variations of their mechanical properties along preferential directions lead to avoid the main drawbacks of the classical composites such as the stress discontinuities at the layer-interfaces and the low resistance to temperature shocks.
In 1984, a group of materials scientists in Japan introduced the concept of functionally graded materials (FGMs) by preparing thermal barrier materials. They investigated different models to produce a structure made of functionally graded materials resisting both of mechanical and thermal loads. FUH-Gwo YUAN et al [1] derived a new finite element model that can be used for long and short beams in 1989. There laminated finite element model in-cluded separate rotational degrees of freedom for each lamina . They compared there for both deflections and stresses results with other solutions and they get good agreement. Reddy et al [2] solved the governing equations for the bending of cross-ply lami-nated composite beams. They used the classical, first-order, sec-ond-order and third-order theories in their analysis. They devel-oped exact solutions for symmetric and anti-symmetric cross-ply beams with arbitrary boundary conditions under arbitrary load-ings. They presented numerical results and showed the deflection of the beam, the number of layers, the effect of shear deformation and the orthotropicity ratio on the static response of composite beams. In 2003, a new beam element Chakraborty et al [3] was developed to study the thermo-elastic behavior of functionally graded beam structures. They depend on the first-order shear de-formation theory and it accounts in order to vary elastic and ther-mal properties along its thickness. They examined different stress variations using both power-law and exponential variations of material property distribution. Their static models was showed that it is an effective way to smoothen stress jumps in bi-material beams. In 2009 Mseut Simsek [4] investigated the static analysis of a simply-supported functionally graded beam under a uniformly distributed load . He used Ritz method within the framework of Timoshenko and the higher order shear deformation beam theories in his work. He studied the displacements and the stresses of the beam for various material distributions. He found that in order to minimized stresses and displacements in a beam-type structure, the material properties of the FG beam can be designed by select-ing a suitable power-law exponent. In 2011 Almeida [5] used a Total Lagrangian formulation for presenting a geometric nonlinear analysis formulation for functionally graded beams. He studied the

influence of material gradation on the response. He compared between the behavior of functionally graded beam with homogeneous material beam.
A new finite element model was developed by El Shafei [6] in order to analyze the response of isotropic and orthotropic beams with different boundary conditions. He represented the field displacements equations of the beams by a first order shear deformation theory and the Timoshenko beam theory. He used Hamilton’s principle in order to derive equations of motion of the beams. He improved the obtained results by applying shear correction factor. He compared the results of the proposed model with the available results of other investigators and he get a good agreement. His proposed model decreased the error due to unaccurately modeling of the curvature present in the actual material under bending (i.e. shear Locking).
For an elasto-plastic FGM, A.R. Daneshmehr [7] studied simply supported Euler-Bernoulli beam using variation method. He loaded the rectangular section beam by uniformly distributed transverse loading. He used a power law model to describe material properties. He calculated analytically the required moment to have fully plastic beam and stress response of the beam. Also, a finite element model is used by Mehta [8] in order to study both static and dynamic behavior of functionally graded material beams. Mehta [8] used a power law in their study. Alexraj et al [9] used Finite Element Method (FEM) by ANSYS software in order to study a static analysis of Functionally Graded beam under uniformly mechanical and thermal load. They assumed power law model to represent the temperature dependent material properties. They used Timoshenko beam theory to calculate deflection and stress of cantilever and simply supported beam. Several methods and theories were used to discuss and analysis the static and/or dynamic response of beam made by functionally graded materials (FGMs) [10-20].
In this work, analytical solution, using the equivalent cross section area of the combined beam , was found in order to calculate the static deflection of cantilever and simply supported functionally graded beam under concentrated and distribution load. The analytical solution was compared with three finite element models.

2. Problem description

Consider a FG beam with rectangular cross-section W(width) * h (height) and length L as shown in Fig 1. This FG beam is constituted by a mixture of two constituents, typically ceramic and metal. The top surface of the beam is a pure ceramic (it is Al_2O_3 in this work) while the metal (it is Nickel in this work) located at the bottom surface of the beam. The following assumptions were considered in this work:

a) Linear elastic behavior.
b) Small deformations of materials.
c) The gravity is not taken into account.

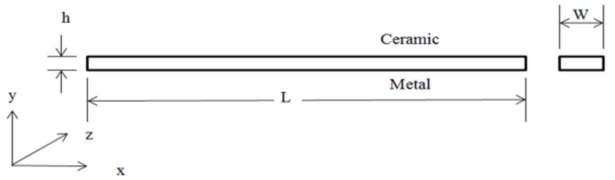


Fig. 1: Geometry of Functionally Graded Beam.

3. Effective material properties of FG beam

In the composite materials, the rule of mixture is the main rule used to calculate the mechanical and physical properties of composite materials that constituted two or more separated materials. In other words, there is discontinuity in mechanical and physical properties. In FGM, the continuity in mechanical physical properties is the main purpose of this materials. The mechanical and thermal properties of FG are described by three mathematical models as follow [21]:

a) Power Law Model.
b) Sigmoid Law Model.
c) Exponential Law Model.

In this work, the Power Law Model was considered and can be write as [21-22]:

$$E(y) = (E_c - E_m)\left[\frac{y}{h} + \frac{1}{2}\right]^K + E_m \quad (1)$$

Where:

E_c= Modulus of Elasticity of ceramic.

E_m= Modulus of Elasticity of metal.

h= thickness of beam

K= Power Law Index.

4. Analytical model

According to the theory of compound beam [23-24], the analytical solution can be derived by the following procedure:

a) The height of beam (h) is divided into (N) layers as shown in Fig. 2.

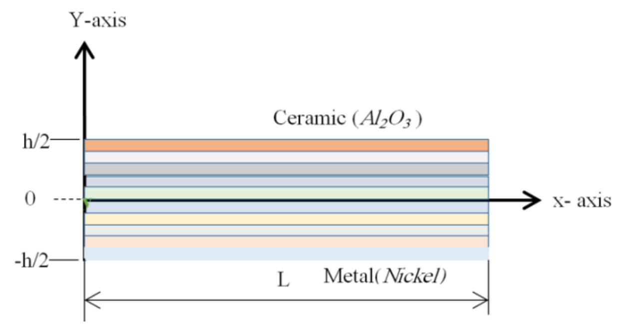


Fig. 2: The Model of Ten Layered FGM Beam.

(1) Rewrite equation (1):

$$\frac{E(y)}{E_m} = \left(\frac{E_c}{E_m} - 1\right)\left[\frac{y}{h} + \frac{1}{2}\right]^K + \quad (2)$$

b) According to the theory of compound beam the width of each layer can be calculated as follow:

$$(A_m)_{eq}E_m = (A_{layer})_{real}E_{layer}$$

But $A_* = W_*(\Delta y)$; *=m, layer W_{layer}= W=W_m

$$(W_m)_{eq} = (W_{layer})_{real}\frac{E_{layer}}{E_m} = W\frac{E(y)}{E_m} \quad (3)$$

Substituting equation (2) into equation (3):

$$W(y) = (W_m)_{eq} = W\left[\left(\frac{E_c}{E_m} - 1\right)\left[\frac{y}{h} + \frac{1}{2}\right]^K + 1\right] \quad (4)$$

c) Calculate the area of cross section using equation (5):

$$Area = \int_{-\frac{h}{2}}^{\frac{h}{2}} W(y)dy = Wh\left[\frac{1}{(K+1)}\left(\frac{E_c}{E_m} - 1\right) + 1\right] \quad (5)$$

d) Calculate the centroid of cross section area using equation (6):

$$\bar{Y}A = \int \bar{y}dA = \int_{-\frac{h}{2}}^{\frac{h}{2}} yW\left[\left(\frac{E_c}{E_m}-1\right)\left[\frac{y}{h}+\frac{1}{2}\right]^K+1\right]dy$$

Where: $\bar{y} = y$ and $A = W\left[\left(\frac{E_c}{E_m}-1\right)\left[\frac{y}{h}+\frac{1}{2}\right]^K+1\right]$

$$\bar{Y} = \frac{\left[W\int_{-\frac{h}{2}}^{\frac{h}{2}} y\left[\left(\frac{E_c}{E_m}-1\right)\left[\frac{y}{h}+\frac{1}{2}\right]^K+1\right]dy\right]}{W\left[\left(\frac{E_c}{E_m}-1\right)\left[\frac{y}{h}+\frac{1}{2}\right]^K+1\right]} \quad (6)$$

If K=1

$$(\bar{Y})_{K=1} = \frac{\left(\frac{E_c}{E_m}-1\right)h}{6\left[\left(\frac{E_c}{E_m}-1\right)+2\right]} \quad (7)$$

e) Calculate the second moment of area using equation (8):

$$I = \int y^2 dA = \int_{-h/2}^{h/2} y^2 W\left[\left(\frac{E_c}{E_m}-1\right)\left[\frac{y}{h}+\frac{1}{2}\right]^K+1\right]dy \quad (8)$$

If K=1 then:

$$I = \frac{Wh^3}{24}\left[\left(\frac{E_c}{E_m}-1\right)+2\right] \quad (9)$$

Now, the maximum deflection of cantilever and simply supported beam under concentrated load (F (N)) can be calculated by:

$$\delta = \frac{FL^3}{3IE} \quad (10)$$

$$\delta = \frac{FL^3}{48IE} \quad (11)$$

Now, the maximum deflection of cantilever and simply supported beam under distribution load (P (N/m)) can be calculated by:

$$\delta = \frac{PL^4}{8IE} \quad (12)$$

$$\delta = \frac{5PL^3}{384IE} \quad (13)$$

If (K=1), the maximum deflection can be calculated directly using equations (5, 7, 9, 10, 11, 12 and 13). But if (k≠1), The numerical integral must be used in equations (6 and 8). In this work, Trapezoidal and Simpson's integral are used.

5. Finite element models

In order to simulate the FG beam by ANSYS software, three models are suggested and these models are:

a) Equivalent Cross Section Area:

In this model, the cross section area of the beam changes due to the value of power law index (K) .From equation (4), the cross section of the beam can be drawn. Due to this change in cross section area, the material properties of the beam are the properties of metal only. Three dimensional beam is drawn in ANSYS software and element (SOILID187) is used in this model. This model is called " ANSYS – Case 1" (see Fig. 3 – a).

b) Equivalent Properties of Layer :

In this model, the height of beam is divided into (N) layer and the properties of each layer can be calculated using equation (1). Two dimensional beam with (N) layers is drawn in ANSYS software and element (SHELL181) is used in this model. This model is called " ANSYS – Case 2"(see Fig. 3 – b). Also, three dimensional beam with (N) layers is drawn in ANSYS software and element (SOILID187) is used in this model. This model is called " ANSYS – Case 3"(see Fig. 3 – c).

The convergent criteria of numerical solution is considered for all the above ANSYS models.

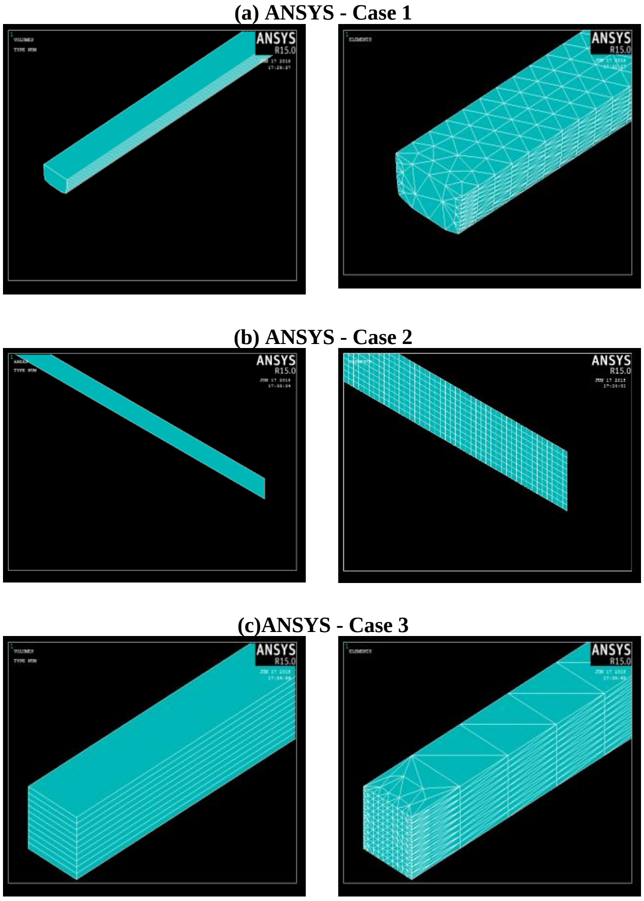


Fig. 3: The Three Ansys Models Used I This Work.

6. Validation

In order to check the models described in this work, the comparison with the static deflection results of Alexraj et. al. [9] were done. with Dimensions and material properties of the FG beam and applied load used by Alexraj et. al. [9] are summarized in Table (1):

Table 1: Dimension, Materiales Properties and Applied Load Used Be Alexraih Et.Al. [9]

1- Material Properties:		
Property	Metal (Al)	Ceramic(Zirconia (ZrO_2))
Modulus of Elasticity, E (GPa.)	70	200
Poisson Ratio, ν.	0.3	0.3
Thermal Expansion, α	23.1e-6	10.3e-6
2- Dimensions:		
3- Length of the FG Beam =0.4m.		
4- Width of the FG Beam =0. 1 m.		
5- Height of the FG Beam =0. 1 m.		
6- Applied Load:		
7- Distribution Load and its Value = 5000 N/m^2.		

The comparison between the static deflection results of Alexraj et. al. [9] and the results of the calculation methods used in this work are made as shown in Table 2. When the number of layers increases, the statics deflection converges to exact value. when

K=1,the exact value, that calculated analytically, is (1.61e-6) and equals to the value of Trapezoidal and Simpson method when the number of layers is (50) layer. Generally, a very good agreement can be seen between Alexraj ET. al. [9] and Trapezoidal and Simpson method at any value of power law index (K).

Table.2: Comparession between the Result Calculating by the Method of This Work and Ref [9]

Method of Calculation			No.	1	2	3	4	5	6	7
			K	0	0.2	0.5	1	2	5	100
Ref.[9]			Deflection (m)	2.88E-06	2.18E-06	1.83E-06	1.61E-06	1.43E-06	1.25E-06	1.01E-06
Trapezoidal Meth.		N=10.		2.96E-06	1.96E-06	1.75E-06	1.58E-06	1.42E-06	1.25E-06	1.09E-06
		N=50.		2.86E-06	2.13E-06	1.81E-06	1.61E-06	1.44E-06	1.26E-06	1.02E-06
Simpson Meth.		N=10.		2.96E-06	2.05E-06	1.8E-06	1.61E-06	1.44E-06	1.26E-06	1.07E-06
		N=50.		2.86E-06	2.14E-06	1.81E-06	1.61E-06	1.44E-06	1.26E-06	1.02E-06
ANSYS	Case 2	N=10.		3.01E-06	2.00E-06	1.74E-06	1.56E-06	1.39E-06	1.21E-06	1.02E-06
		N=50.		3.05E-06	2.10E-06	1.75E-06	1.56E-06	1.4E-06	1.22E-06	1.03E-06
	Case 3	N=10.		3.05E-06	2.1E-06	1.88E-06	1.70E-06	1.5E-06	1.31E-06	1.09-06
		N=50.		2.90E-06	2.27E-06	1.95E-06	1.76E-06	1.55E-06	1.36E-06	1.06E-06

7. Results and discussion

The material properties , dimensions and applied loads used in this work are summarized in Table (3).

Table 3: Dimenision, Materials Propperties and Applied Load Used in This Work

1- Material Properties:		
Property	Metal (Nickel)	Ceramic(Alumina (Al_2O_3))
Modulus of Elasticity, E (GPa.)	*393*	*199.5*
Density, ρ (Kg/m^3)	*3970*	*8900*
Poisson Ratio, v	*0.3*	*0.3*
Thermal Expansion, α (1/k)	*8.80E-06*	*13.3E-6*
Thermal Conductivity, k *(W/(k.m))*	*30.1*	*90.7*
2- Dimensions: (a) Length of the FG Beam =1 m. (b) Width of the FG Beam =0.01 m. (c) Height of the FG Beam =0.01 m.		
3- Applied Load: (a) Concentrated Load and its Value = 100 N for cantilever and simply supported beam. (b)Distribution Load and its Value = 100 N/m for cantilever and simply supported beam.		

The results of this work can be divided into three parts as follow:

1) Material Properties of FG Beam:

The material properties (especially modulus of elasticity) depend generally on two factors (material properties of parents (i.e. metal and ceramic) and power law index (K)). The difference between the material properties of parents is the range of the material properties of each point in the FG beam. While the power law index (K) is responsible on the variation in each point. Figure 4 shows the change in volume fraction of each point in the height of FG beam depending on the value of Power law index (K). The effect of material properties of parents is shown in Figure 5. The change in material properties , when the power law index equals (1), is linear. While the change in material properties , when the power law index is less than (1), is nonlinear and depending on the value of power law index. The same behavior can be seen but with different nonlinearity curve when the power law index is larger than (1) (see Figure 6 and 7). In Figures (8,9 and 10), the difference in change rate of material properties due to increase the power law index appears sharply when the material properties ratio ($E_k(y)/E_{k=1}(y)$) is considered. Finally, the effect of power law index on the material properties appears in Figures (11 and 12) at different position of height of FG beam.

2) Equivalent Cross Section Area:

The analysis of compound beam described in this work assumed that the cross section area of the beam changes due to the variation in the material properties along the height of the FG beam. In order to explain the effect of power law index on the shape of cross section area , the cross section area for different value of power law index is drawn in Figure 13 when the number of layers is 10. When the power law index equals 1, the linear variation in width of FG beam appears. While different nonlinear variations appear when power law index less and larger than 1.

The number of dividing layers is an important factor in order to converge the calculated parameter to the exact value. The effect of layer 's number is not appear in the shape of cross section area when the power law index equals 1 because of the linear variation in material properties (see Figure 14). While the effect of layer 's number appears in the shape of cross section area when the power law index less or larger than 1(see Figures 15 and 16).

According to the above factors, the maximum static deflection in this work are calculated when the power law index equals or less than 1only.

3) Maximum Static Deflection:

As maintained in Table 3, two types of support are used and for each type of support, two kinds of applied load are used. Therefore, this section is divided into two parts according to the support type.

a) Cantilever FG Beam:

Figure 17 shows the comparison among the maximum deflection of cantilever FG beam calculating by different methods when the applied load is concentrated Force (100 N) for the power law Index value (K) between 0.1 and 1. While the same comparison , when the applied load is distribution load, is shown in Figure 18. The following points can be noted from these figures:

1) In all methods except ANSYS – Case 1, when the number of layers increases, the maximum static deflection converges to the exact value in the two applied loads.
2) In ANSYS – Case 1, the number of layers is not effect on the maximum deflection at any value of power law index. But the value of maximum deflection converges to exact solution when the power law index increases to 1.

b) Simply supported FG Beam:

Figures 19 and 20 show the comparison among the maximum deflection results calculated by the five methods used in this work when the FG beam supports as cantilever and simply supported beam respectively. Generally, the results of method ANSYS – Case 1 is constant when the number of layers increase and its results converge to exact value when the power law index converges to 1 for the cantilever FG beam. But for simply supported beam, the results of the five method are constant when the number of layers and power law index increase. The results of method ANSYS – Case 1 is divergent when its results compare to the results of the other four methods.

8. Conclusions

From the results, the following points can concluded:

1) The analytical solution (i.e. Trapezoidal and Simpson method) and he numerical approach (ANSYS – Case 2 and Case 3) are a very good approach to calculate the static deflection of cantilever and simply supported FG beam under concentrated and distribution load.
2) The numerical approach (ANSYS – Case 1) fails in calculation the static deflection of cantilever and simply supported FG beam under concentrated and distribution load especially when the power law index smaller than 1.

For the next work, the normal stress and shear stress will calculated using the same methods described in this work. Also, the static deflection, normal stress and shear stress due to thermal load will studied too.

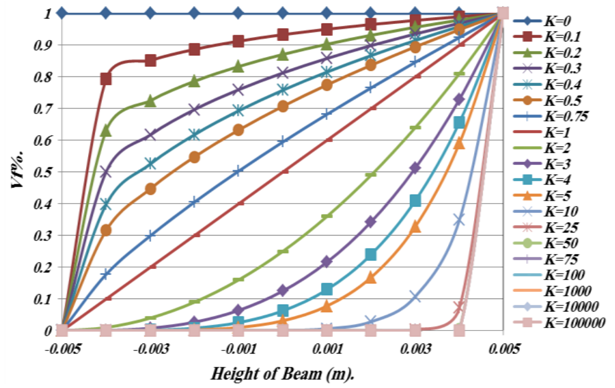


Fig. 4: Distribution of Volume Fraction along the Height of Beam for Different Power Law Index When the Number of Layers is 10.

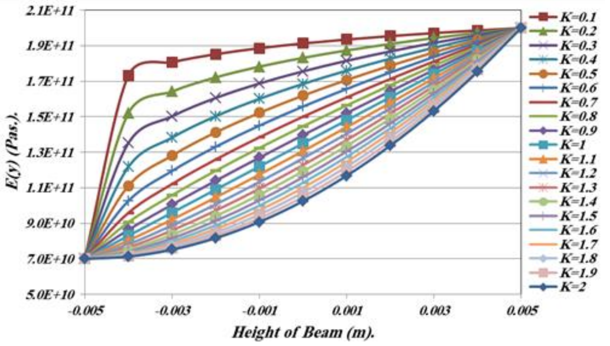


Fig. 5: Distribution of Modulus of Elasticity (E) Along the Height of Beam for Different Power Law Index when the Number of Layers is 10.

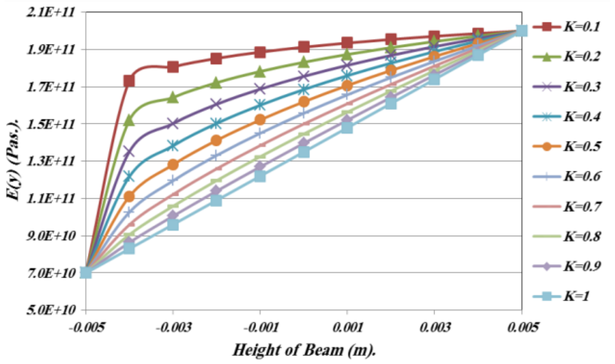


Fig. 6: Distribution of Modulus of Elasticity (E) along the Height of Beam for Power Law Index Is (0.1 – 1) when the Number of Layers is 10.

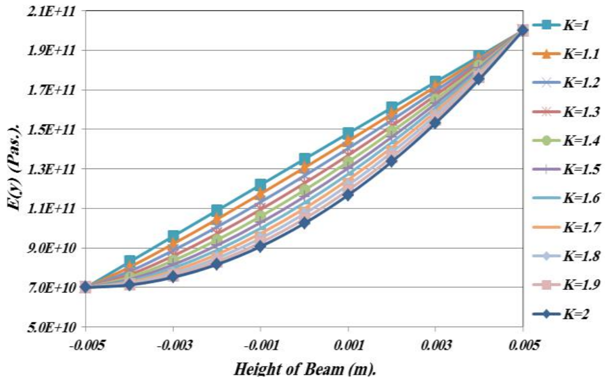


Fig. 7: Distribution of Modulus of Elasticity (E) Along the Height of Beam for Power Law Index is (1 – 2) When the Number of Layers is 10.

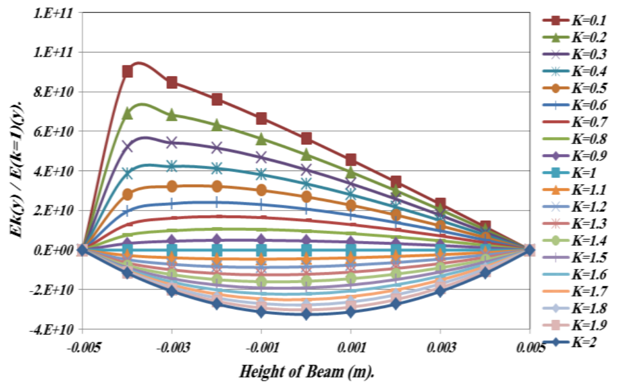


Fig. 8: Distribution of Modulus of Elasticity Ratio (Ek(y)/Ek=1(y)) Along the Height of Beam for Different Power Law Index When the Number of Layers is 10.

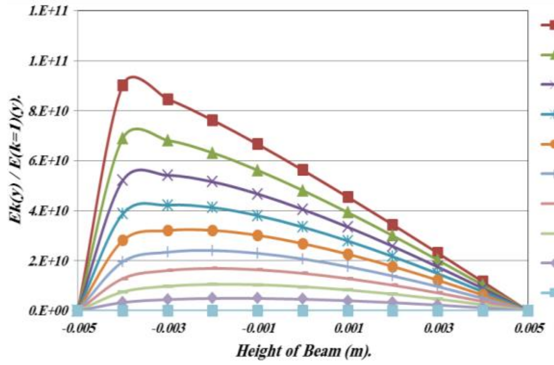


Fig. 9: Distribution of Modulus of Elasticity Ratio (Ek(y)/Ek=1(y)) Along the Height of Beam for Power Law Index is (0.1 – 1) When the Number of Layers is 10.

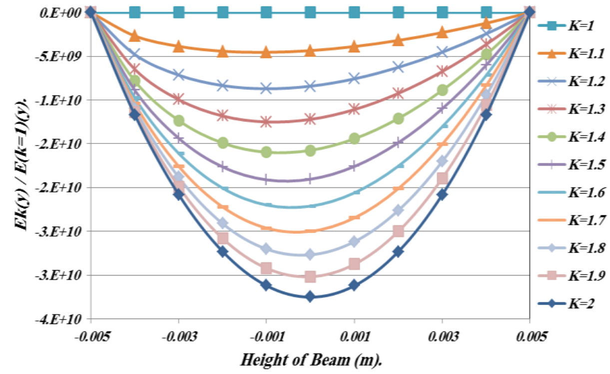


Fig. 10: Distribution of Modulus of Elasticity Ratio (Ek(y)/Ek=1(y)) Along the Height of Beam for Power Law Index is (1 – 2) When the Number of Layers is 10.

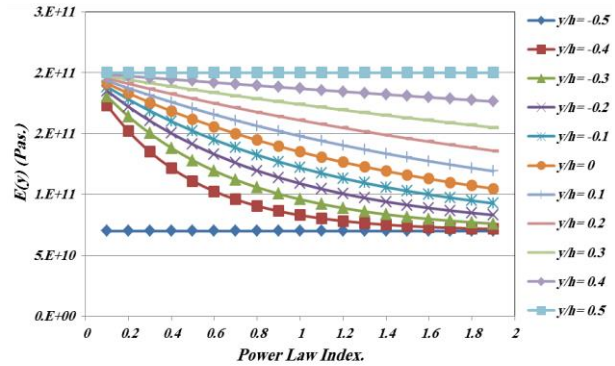


Fig. 11: Distribution of Modulus of Elasticity Ek(y) Due to Change in Power Law Index at Different Depth of Beam When the Number of Layers is 10.

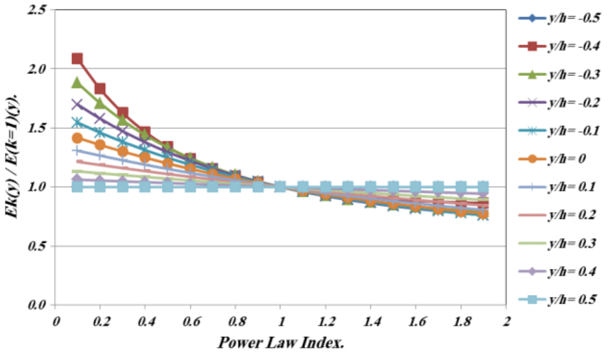


Fig. 12: Distribution of Modulus of Elasticity Ratio (Ek(y)/Ek=1(y)) Due to Change in Power Law Index at Different Depth of Beam When the Number of Layers is 10.

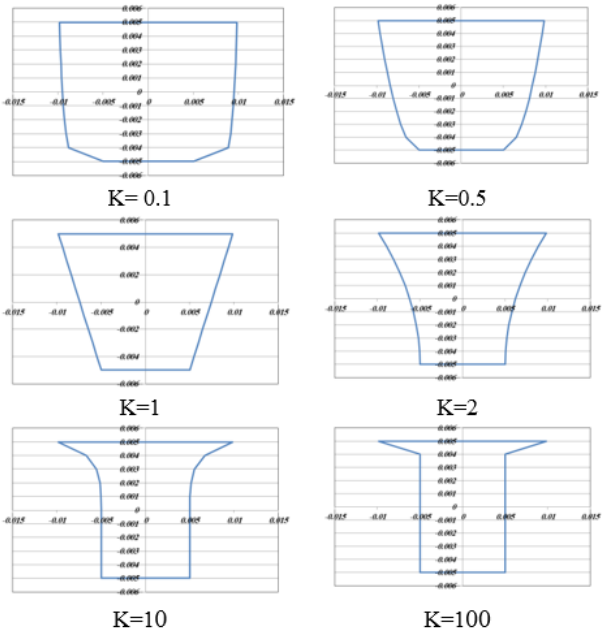


Fig. 13: Equivalent Cross Section Area when the Number of Layers is 10.

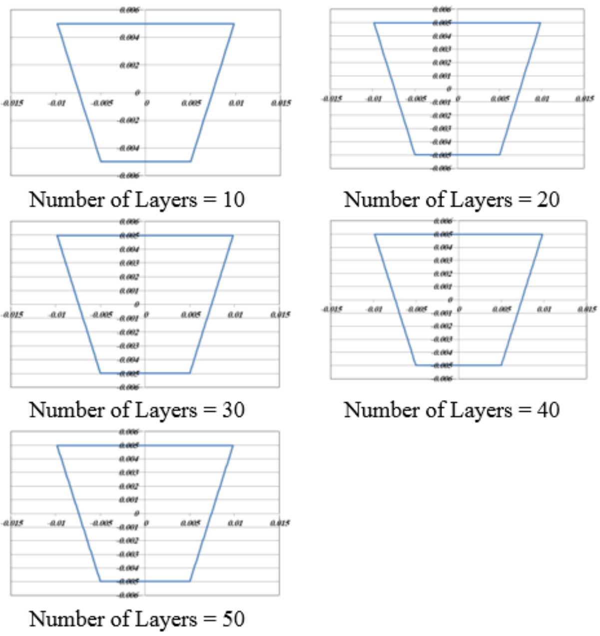


Fig. 14: Equivalent Cross Section Area when K=1.

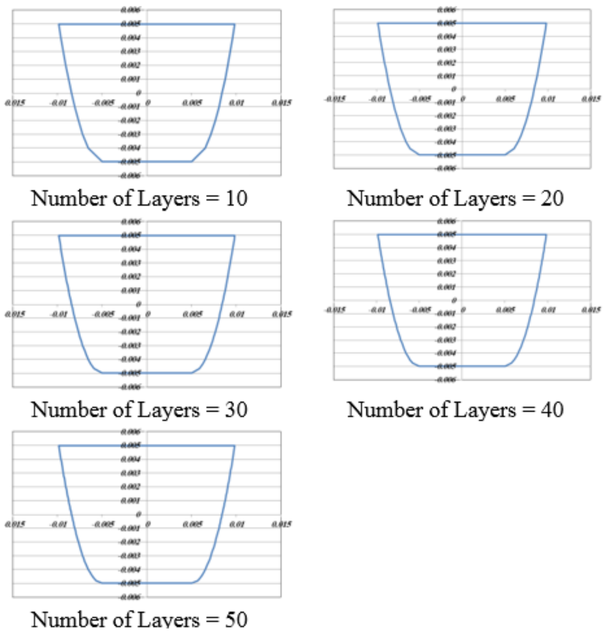


Fig. 15: Equivalent Cross Section Area when K=0.5.

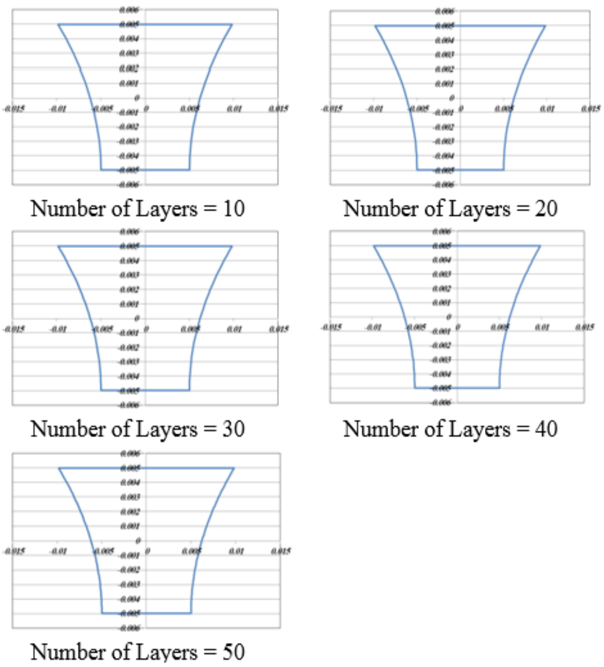
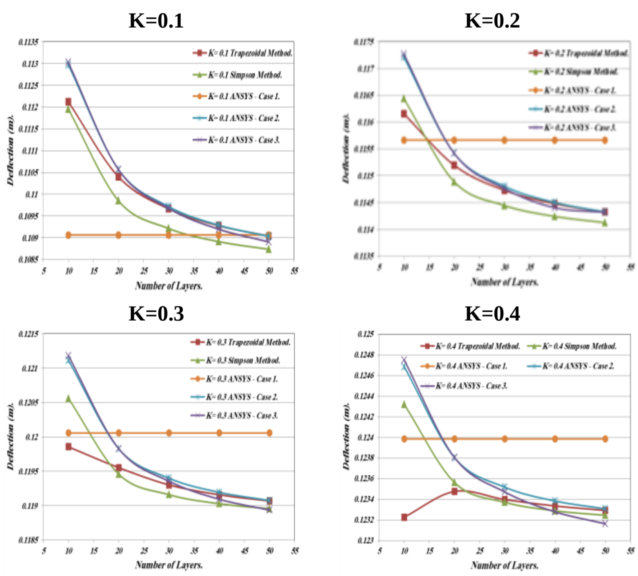


Fig. 16: Equivalent Cross Section Area when K= [2].



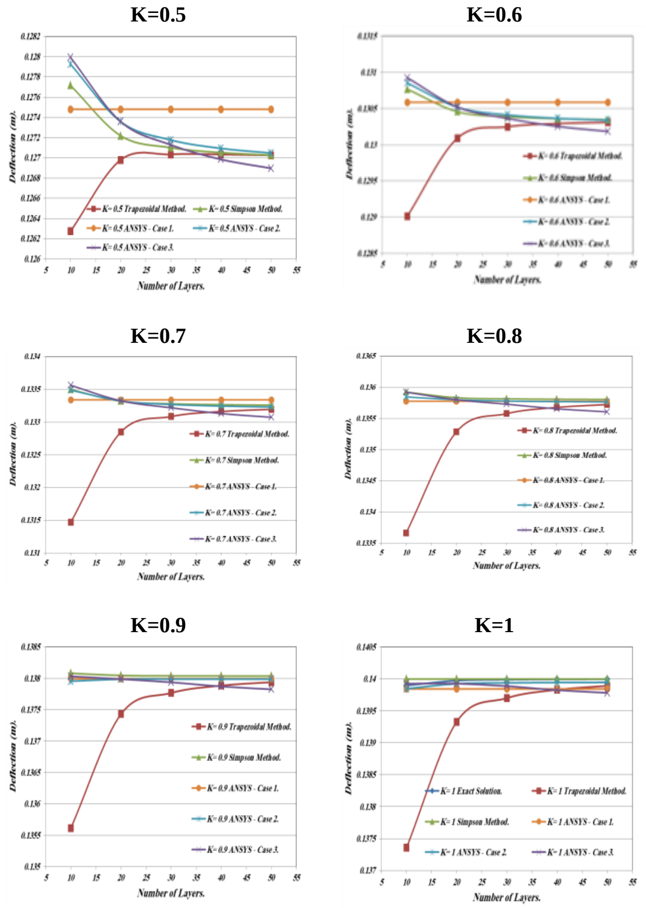
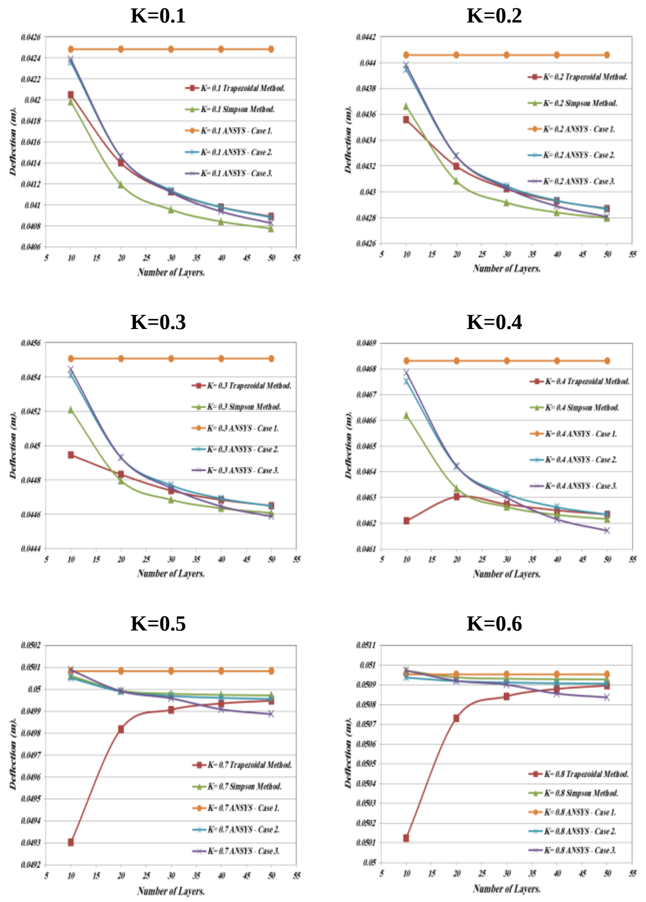


Fig. 17: Comparison Among the Maximum Deflection of Cantilever Beam Calculating by Different Methods Due to Change in Number of Layers When the Load is a 100 N concentrated Force and Different Value of Power Law Index (K).



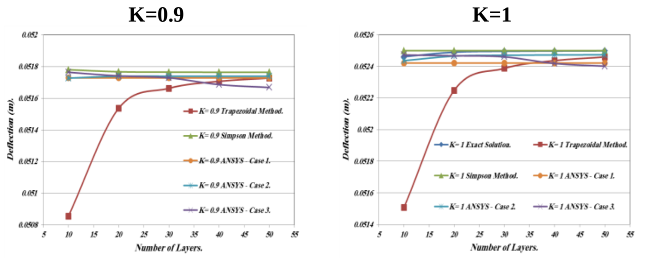


Fig. 18: Comparison Among the Maximum Deflection of Cantilever Beam Calculating by Different Methods Due to Change in Number of Layers When the Load is a 100 N/m distributed load and Different Value of Power Law Index (K).

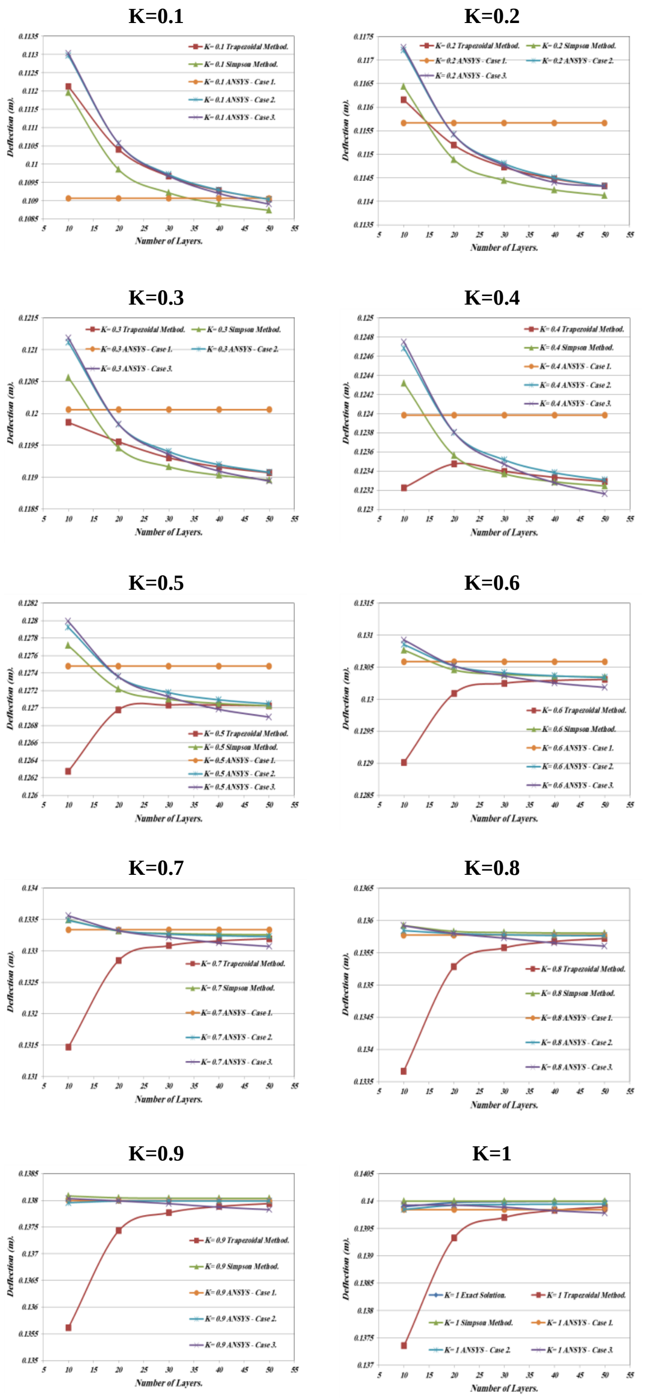


Fig. 19: Comparison Among the Maximum Deflection of Simply Supported Beam Calculating by Different Methods Due to Change in Number of Layers When the Load is a 100 N concentrated Force and Different Value of Power Law Index (K).

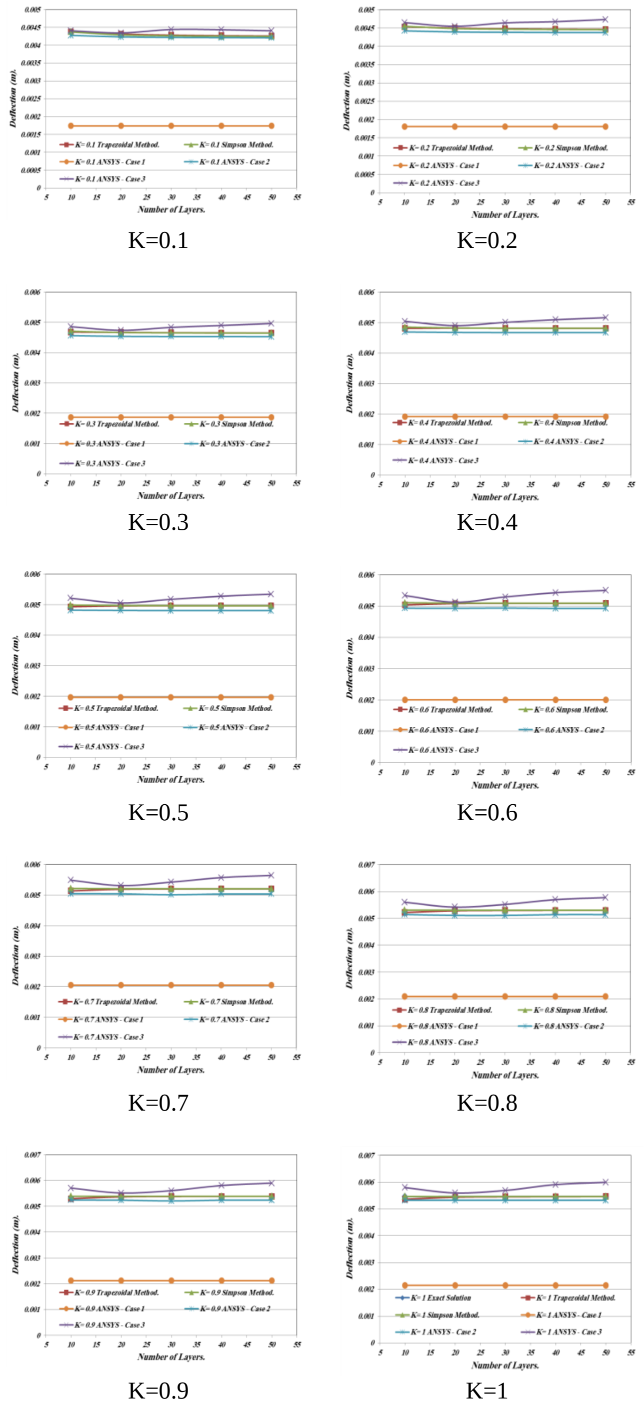


Fig. 20: Comparison Among the Maximum Deflection of Simply Supported Beam Calculating by Different Methods Due to Change in Number of Layers when the Load Is A 100 N/M Distributed Load and Different Value of Power Law Index (K).

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