Modeling and Pricing of Energy Derivative Market

Dr Sellamuthu Prabakaran,

School of Economics and Business Administration
Department of Accounting & Finance
Pontificia Universidad Javeriana Cali.
*Corresponding author E-mail: jopraba@gmail.com

Abstract

Energy derivatives are an energy instrument whose value be determined by on or derived from the values, more basic, a fundamental energy asset, such as crude oil, electricity, or natural gas. Energy derivatives are nonstandard products that have been generated by financial engineers (I. e exotic derivatives) and include exchange-traded contracts such as options and futures. In energy industries, the risk management and pricing model are important because the volatility of pricing in energy products. The price of the volatility can decrease the income of business strategies and its affects the consumer´s buying and selling decisions. For this reason, we have to manage the pricing risk and it became a pressure in the energy industries to continue the profitability and to avoid competitive disadvantages. The main goal of this study is to construct the option-pricing model for energy derivative markets.

Keywords: Energy Derivatives, Option Pricing, Energy Option Contract (Call and Put), Price Risk and Energy Spot Price.

1. Introduction

The usage of financial derivatives is traded on over-the-counter (OTC) and exchanges has established with low cost technique, which is used for hedging the pricing of risk. An extensive variety of derivative agreements exists, containing forwards, futures and options. Price risk is well defined as a condition in which a variable is expected to take on a value conflicting from that, which was projected and exposure to adverse price moves in the cash market and if the prices move lower in the cash market, producer or supplier will get loses. The consumer loses if prices change advanced. Basis risk can be representing, the difference between the price used as a benchmark in a transaction and the actual price of goods changing hands. If the difference between the benchmark price and the actual price does not remain constant, there will be a loss or a gain on one side of the deal. Encomiasts use the standard deviation to quantify the price risk and standard deviation performances the spread of possible conclude around the average, or predictable value of the variable in question.

2. Energy Derivatives

Energy Markets are everywhere undergoing fast deregulation and significant to additional competition, enlarged volatility in pricing of energy market, and revealing contributors to potentially higher risk. Deregulation influences (consumer and producers) and it consumes to an enlarged awareness, for monitoring the exposure in pricing of energy market, we need risk management and the use of derivative tools. Energy derivatives perform to be a new occurrence for many market participants; however, it is true that energy derivatives are comparatively novel concepts of energy markets.

Derivatives traded on the energy exchanges are liquid; however, OTC agreements normally are not. A party on either side of an exchange-traded agreement can cancel its position at any time by buying or selling a contract that is opposite its original agreement [2]. Energy derivatives are agreements related to a specific energy commodity (crude oil, Natural gas, heating oil, coal and electricity, etc.) and these energy instruments deliver an opportunity to manage the risk associated with the volatility in energy prices and permitting a party to lock the price of their energy in advance of the actual period of energy consumption.

Significant success of energy derivatives is the deregulation of the energy market place and energy commodity is free from any form of price regulation. The modest spot market can be established where pricing is liquid and reflective of the true cost of the energy commodity at any point of time. The typical energy portfolio in energy derivatives, like any other product sold in a market. Energy markets have their own choices on to quote futures. However, the most thought-provoking problem comes from the fixing the price of energy product, hedging and constructing of exotic tradable products linked to physical assets and it can call real derivatives.

The worth of a derivative is based on the price difference between the underlying energy product and can be used for both speculation and hedging purposes. Corporate, whether they sell or just use energy, can buy or sell of energy derivatives to hedge against variations in underlying energy prices. Speculators can use these derivatives to get the profit from the changes in the underlying price and can intensify those profits with advantage.

3. Standard Energy Options

Standard energy options, such as calls and puts, are some of the most frequently used risk management tools. The literature on options is quite extensive [1, 2 & 3]. We remind the reader that in energy markets, by definition, there is not much difference between calls, puts in energy markets and calls, and puts in all other markets. What sets them apart is an unusual diversity of trade energy options, a natural consequence of the diversity in the underlying commodity, especially power. Typically, energy option specifications include:

Place of delivery, time of maturity period, Delivery conditions - in the case of power, the type of delivered power (round-the-clock, on-peak and off-peak,), Price in the contract (i.e. Specific Price or exercise price) and Size of the contract.

Options contract, the buyer must make the payment at the time beginning of the contracts; this is option price or premium (Initial Investment). At the time of expiration date, if the spot price of the asset less than the agreed price (K), the holder of the option contracts are worthless (less premium), and the buyer, buys the assets in the markets at the market price. Else, if spot asset price is more than K, the option holder exercise the option, buying the assets at K and the capability to immediately make a profit - difference between the spot and the exercise prices (less premium).
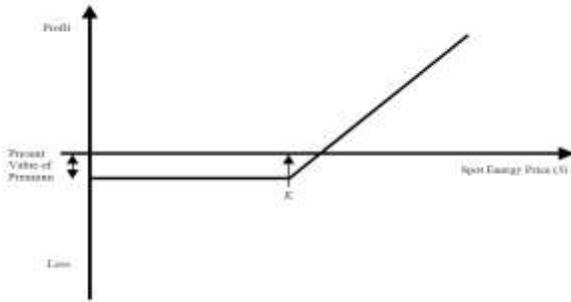

Fig. 1: Payoff to Call Option

A call option gives right to the owner to buy the fixed number of underlying assets with the specific price and specific date.

The mathematical expression for call option (payoff)

$$Max\left[\left(S_T - K\right)^+, 0\right] \tag{1}$$

Another type of option, a put, gives the holder the right to sell an asset, but not the obligation with the specific date and specific price;
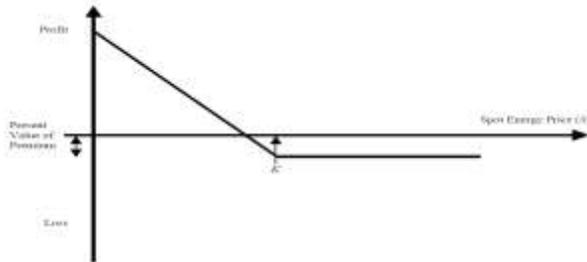

Fig. 2: Payoff to Put Option

The mathematical expression for the put option (Payoff)

$$Min\left[0, \left(S_T - K\right)^-\right] \tag{2}$$

Every day the daily option is exercised during the exercise month. It allows its owner to the take the daily decisions during the exercise month about buying (call option) or selling (put option) spot power or gas with agreed strike price. In the index or cash option, the option is exercised every day during the exercise month with a specified monthly index at an agreed strike price defined at the beginning of the month as a settled value of the monthly index.

Options on the spot commodity are very common among energy derivatives, because they answer the real need to manage price risks on a daily basis. They are typically structured as a strip of options exercised daily during a certain time period (month, quarter, season, and so on). Therefore, their payoffs can be represented as follows for the call options.

$$C_{Daily} = \sum_{i=1}^{n} \max\left(0, S_T - K\right) = \sum_{i=1}^{n}\left(S_T - K\right)^+ \tag{3}$$

$$P_{Daily} = \sum_{i=1}^{n} \max\left(0, K - S_T\right) = \sum_{i=1}^{n}\left(K - S_T\right)^+ \tag{4}$$

Where n is the no. of the days in agreement and T is the time of the expired date in the contract.

4. Construct of the Option Pricing Model

The contemporary concept of option pricing model is one of most important contribution in modern finance. To calculate a unique risk, the neutral price for energy derivatives, we have to assume that there is no arbitrage is used in quantitative finance. The arbitrage profits can be made from the price changes between the actual and the theoretic futures price.

Consider the price of the assets S_t and divide the time [t, T] into small intervals with equal size Δ. For each time $t + i\Delta$ with, $i = 1, ..., n$ and the assets price move to $S_{t+i\Delta}$.

We can get the change in asset price at time t - $S_{t+\Delta} - S_t$.

To find out the expected value of the change under the working probability (denote by P^*)

If $P^* = \tilde{P}$, we find the risk neutral expected net [4] return by

$$E_t^{P^*}\left[S_{t+\Delta} - S_t\right] = r_t S_t \Delta \tag{5}$$

Now, we have the risk – neutral probability by using of probability switching method and Martingale property [5],

$$\left[S_{t+\Delta} - S_t\right] = r_t S_t \Delta + \varepsilon_t, \tag{6}$$

The equivalent form $\varepsilon_t = \sigma\left(S_t\right)S_t \Delta W_t$ $\qquad$ (7)

Where ΔW_t is a Wiener process increment with variance equal to Δ. Thus, the arbitrage – free dynamics under the $\tilde{P}$ measure can be written as

$$\frac{\left(S_{t+\Delta} - S_t\right)}{S_t} = r_r\Delta + \sigma\left(S_t\right)\Delta W_t \tag{8}$$

Let $\Delta \to 0$, this equation (8) becomes a SDE during the short period dt, then the SDE can be written as

$$\frac{dS_t}{S_t} = r_t dt + \sigma_t dW_t \qquad (9)$$

The above equation (9) developed from the Black – Scholes – Merton (BSM) model. The assumptions used to derive the BSM – PDE, where proportional changes in the asset price (S), drift (r), and volatility, (σ).

Where,

dS_t - The increments in the asset price processing during the small interval of time (dt),

dW_t - An increment in a Weiner process during dt (Uncertainty)

According to the risk – neutral assumption, it gives $\mu = r$.

Let V denotes the value of any derivatives security. The arguments allow for the derivatives of the following PDE elucidating the evolution of the derivatives price through time, [6]

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S} r S_t + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma^2 S_t^2 = rV \qquad (10)$$

With $t \in [0, \infty]$ and $S \in [0, S_{max}]$. The BSM equation can be derived from several approaches [7 & 8].

The BSM differential equation must be fulfilled by the price of any derivatives with non - dividend - paying stock [2]. These are similar to the no arbitrage arguments we used to value stock options for the situation where stock price movement are binomial and contain setting up a riskless portfolio involving of a position in the energy derivatives and a position in the stock. In the absences of arbitrage opportunities, the return from the portfolio must be the risk free interest rate, r.

Assume asset price evolve according to the stochastic process called GBM.

$$dS_t = \mu dt + \sigma dW_t, \quad t \in [0, \infty] \qquad (11)$$

Where μ (drift) and σ (volatility) are constants. The lognormal evolution follows

$$S(t) = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma dW_t\right)$$

$$= S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma \varepsilon \sqrt{T}\right) \qquad (12)$$

Replacing the risk – neutral lognormal for the pricing of asset path into the expectation, then the integrated expectation as follows

$$c = \frac{1}{e^{rT}} E^T\left[\max(S_T - K)^+, 0\right] \qquad (13)$$

$$c = \frac{1}{e^{rT}} \int_{-\infty}^{\infty} \max\left(0, S_0 e^{\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\varepsilon\sqrt{T}\right)} - K\right)^+ \frac{1}{\sqrt{2\pi}} e^{-\varepsilon^2/2} d\varepsilon \qquad (14)$$

ε - Normal distribution.

To maximize this above payoff term and eliminate the integration part, then we obtain with positive value

$$S_0 e^{\left(r - \sigma^2/2\right)T + \sigma\varepsilon\sqrt{T}} - K \geq 0 \qquad (15)$$

Which occurs when [9]

$$\varepsilon_1 \geq \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{K}{S_0}\right) - \left(r - \frac{\sigma^2}{2}\right)T\right] \qquad (16)$$

Chang the appropriate limits of integration and it gives

$$c = \frac{1}{e^{rT}} \frac{1}{\sqrt{2\pi}} \int_{\varepsilon_1}^{\infty} \left(S_0 e^{\left(r - \sigma^2/2\right)T + \sigma\varepsilon\sqrt{T}} - K\right) e^{-\varepsilon^2/2} d\varepsilon \qquad (17)$$

Eliminate the r from the first term

$$c = \frac{S_0}{\sqrt{2\pi}} \int_{\varepsilon_1}^{\infty}\left[\left(e^{-\varepsilon^2/2 + \sigma\varepsilon\sqrt{T} - \left(\sigma^2/2\right)T} - Ke^{-rT}\frac{1}{\sqrt{2\pi}} e^{-\varepsilon^2/2}\right)\right] d\varepsilon \qquad (18)$$

Simplified the above (18) equation and we can get

$$c = S_0 \frac{1}{\sqrt{2\pi}}\left\{\int_{\varepsilon_1}^{\infty} e^{-\frac{1}{2}\left(\varepsilon - \sigma\sqrt{T}\right)^2} d\varepsilon - Ke^{-rT}\int_{\varepsilon_1}^{\infty} e^{-\varepsilon^2/2} d\varepsilon\right\} \qquad (19)$$

Let $N(x)$ - Cumulative normal distribution function, i.e. probability that a variable with a standard normal distribution $\phi(0,1) < 1$,

$$N(x_1) = \int_{-\infty}^{x_1} \frac{dx}{e^{x^2/2}} = 1 - \int_{x_1}^{\infty} \frac{dx}{e^{x^2/2}} = 1 - N(-x_1) \qquad (20)$$

Now, simplify the valuation equation to

$$c = S_0\left[1 - N\left(\varepsilon - \sigma\sqrt{T}\right)\right] - \frac{K}{e^{rT}}\left[1 - N(\varepsilon)\right] \qquad (21)$$

The distribution can be rewritten by the property $N(x_1) = 1 - N(-x_1)$ as

$$1 - N\left(\varepsilon - \sigma\sqrt{T}\right)$$

$$= N\left\{\frac{\left[\ln\left(S_0/K\right) - \left(r + \sigma^2/2\right)T\right]}{\sigma\sqrt{T}}\right\} \tag{22}$$

$$= N\left(-\varepsilon + \sigma\sqrt{T}\right)$$

This is standard B S equation, and we can write this equation for a European Call option

$$c = \frac{S_0}{e^{qT}}N(d_1) - \frac{K}{e^{rT}}N(-d_2) \tag{23}$$

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S_0}{K}\right) - \left(r - q + \frac{\sigma^2}{2}\right)T\right] \tag{24}$$

$$d_2 = d_1 - \sigma\sqrt{T} \tag{25}$$

Similarly, we can write the European Put option

$$p = -S_0 e^{-qT}N(-d_1) + Ke^{-rT}N(-d_2) \tag{26}$$

Where q- Dividend yield.

5. Model for Energy Spot Price

Here, we construct a stochastic model for energy spot price by using of Ordinary Least Square Regression Model (OLSR) [10]. On the contract, if the returns were independents and it is assumed by the BSM, the correlation coefficient and close to zero and indicating inadequate sign for an original seasonality.
In particular,

$$DR_t = r_t - \bar{r}_d \tag{27}$$

Where

DR_t - Deseasonalised return at time t,

$\bar{r}_d$ - Corresponding mean of every month and

r_t - The return at time, t.

That the parameter evaluate from OLSR method should be ´rationally´ close.

$$dP(t) = \alpha\left[\mu - \ln P(t)P(t)dt + \sigma P(t)dW(t)\right] \tag{28}$$

Where,
$\mu =$ Long-term mean level for the logarithm of Electricity spot prices,
$\alpha =$ Mean Reversion rate,
$W =$ Standard Wiener Process, and
$\sigma =$ The variance in electricity spot prices.
To solve this SDE with the following equation, substituting with the help of Stochastic Calculus.

$$z = \ln(P) \tag{29}$$

$$dz = d\left(\ln(P)\right) \tag{30}$$

Using Ito´s Lemma as given in [2], we get

$$d(InP) = \left[\frac{\left[\alpha P(\mu - \ln P)\right]}{P} - \frac{1}{2}\left(\frac{\sigma}{P}\right)^2 (dP)^2\right]dt + \frac{\sigma P}{P}dW \tag{31}$$

$$dz = (\mu - \ln P_t)\alpha + \sigma dW_t - \frac{\sigma^2}{2}dt \tag{32}$$

$$\Rightarrow dz_t = \alpha\left(\mu - \frac{\sigma^2}{2\alpha} - z_t\right)dt + \sigma dW_t \tag{33}$$

This model is very similar to Vasicek Model for interest rates, we use this methodology, and to solve this above model, we have to consider.

$$Y = e^{\alpha t}z \tag{34}$$

Then, we have to use Ito´s Lemma,

$$d(Y) = d\left(e^{\alpha t}z\right) = e^{\alpha t}dz_t + \alpha e^{\alpha t}zdt + 0$$

From the equation (33)

$$d\left(e^{\alpha t}z\right) = e^{\alpha t}\left[\alpha\left(\mu - \frac{\sigma^2}{2\alpha} - z\right)dt + \sigma dW_t\right] + \alpha e^{\alpha t}zdt \tag{35}$$

$$d\left(e^{\alpha t}z\right) = \alpha\left(\mu - \frac{\sigma^2}{2\alpha}\right)e^{\alpha t}dt + \sigma e^{\alpha t}dW_t \tag{36}$$

Integrating equally sides w.r.t to t from t_i to t_{i+1} yields,

$$e^{\alpha t_{i+1}}z_{i+1} - e^{\alpha t_i}z_i$$
$$= \alpha\left(\mu - \frac{\sigma^2}{2\alpha}\right)\int_{t_i}^{t_{i+1}}e^{\alpha s}ds + \sigma\int_{t_i}^{t_{i+1}}e^{\alpha s}dW_s \tag{37}$$

$$z_{i+1} = z_i e^{-\alpha\Delta t} + \left(\mu - \frac{\sigma^2}{2\alpha}\right)\left(1 - e^{-\alpha\Delta t}\right)$$
$$+ \sigma e^{-\alpha t_{i+1}}\int_{t_i}^{t_{i+1}}e^{\alpha s}dW_s \tag{38}$$

Where $\Delta = t_{i+1} - t_i$

Now, OLSR model is overdue and to deduct the z_i term from equally sides of the equation, then we get the following equation.

$$z(t_{i+1}) - z(t_i) = \left(\frac{1}{e^{\alpha \Delta t}} - 1\right) z(t_i)$$
$$+ \left(1 - \frac{1}{e^{\alpha \Delta t}}\right)\left(\mu - \frac{\sigma^2}{2\alpha}\right) + \frac{\sigma}{e^{\alpha t_{i+1}}} \int_{t_i}^{t_{i+1}} e^{\alpha s} dW_s \tag{39}$$

Now, we analyze this equation and get an algebra equation,

$$Y = mX + c + \varepsilon \tag{40}$$

Where,

$$m = \left(\frac{1}{e^{\alpha \Delta t}} - 1\right), c = \left(\mu - \frac{\sigma^2}{2\alpha}\right)\left(1 - \frac{1}{e^{\alpha \Delta t}}\right)$$
$$and \quad \varepsilon = \frac{\sigma}{e^{\alpha \Delta t}} \int_{t_i}^{t_{i+1}} e^{\alpha s} dW_s \tag{41}$$

The above equation as a system of Linear equation and we consider

$$Y = XA + \varepsilon \tag{42}$$

Where,
ε = Residual term.

In specific,

$$Y = \begin{pmatrix} z_1 - z_0 \\ z_2 - z_1 \\ ... \\ ... \\ z_n - z_{n-1} \end{pmatrix}, X = \begin{pmatrix} 1 & z_0 \\ 1 & z_1 \\ . & ... \\ . & ... \\ 1 & z_n \end{pmatrix}, A = \begin{pmatrix} c \\ m \end{pmatrix} \tag{43}$$

From the Equation (41) we got the three indicators
1. The slope of the regression line is equal to coefficient of the log prices.

$$m = \left(e^{-\alpha \Delta t} - 1\right) \tag{44}$$

Taking the natural logarithm of both sides gives,

$$-\alpha \Delta t = \ln(m+1) \tag{45}$$

$$\Rightarrow \alpha = \frac{-\ln(m+1)}{\Delta t} \tag{46}$$

2. The deterministic part of the equation (40) is equal to the intercept of the regression line,

3. $\quad c = \left(\mu - \frac{\sigma^2}{2\alpha}\right)\left(1 - e^{-\alpha \Delta t}\right) \tag{47}$

4.

$$\Rightarrow \mu = \frac{c}{(1 - e^{\alpha \Delta t})} + \frac{\sigma^2}{2\alpha} \tag{48}$$

5. Form the equation according to volatility parameter, σ and reshuffle the equation (40) for the ε and we can get the following equation,

$$\varepsilon = Y - mX - c \tag{49}$$

Now, we have to minimize the variance (residual error ε), which would be calculated later by taking the second moment of the equation (38).

$$Var(\varepsilon) = E(\varepsilon^2) - [E(\varepsilon)]^2 \tag{50}$$

We, entitlement to minimum variance $E(\varepsilon) = E(Y - mX - c) = 0$.

$$Var(\varepsilon) = E[((Y - mX) - c)^2]$$
$$= E[(Y - mX)^2 + c^2 - 2c(Y - mX)]$$
$$= E[Y^2 - 2mXY + (mX)^2 - 2cY + 2mcX + c^2] \tag{51}$$

Now, by linearly of the expectation operator

$$Var(\varepsilon) = G(m,c) = E(Y^2)$$
$$- 2mE(XY) + m^2 E(X^2) \tag{52}$$
$$- 2cE(Y) + 2mcE(X) + c^2$$

Notice that in the above equation the *X*'s and *Y*'s are known entities that come from the data. Hence, the variance is a function of *m* and *c* alone. As discussed earlier, with the use of one variable calculus for optimization, we get,

$$\frac{\partial G}{\partial m} = -2E(XY) + 2mE(X^2) + 2cE(E) = 0 \tag{53}$$

$$\frac{\partial G}{\partial c} = -2E(Y) + 2mE(X) + 2c = 0 \tag{54}$$

Now, we got the proof for the claim from second equation, again rearrange the equation.

$$-2E(Y) + 2mE(X) + 2c = 0$$
$$\Rightarrow E(Y) - mE(X) - c = 0 \tag{55}$$
$$\Rightarrow E(Y - mX - c) = E(\varepsilon) = 0$$

For finding the optimum values for *m* and *c*.
From the above, two equations of linear system and set up with following equations.

$$E(X^2)m + E(X)c = E(XY)$$
$$[E(X)m + 1c = E(Y)E(X)] \times E(X) \tag{56}$$

Now, the system is solved immediately from multiplying the second equation $E(X)$ and deducting from the initial yielding the equation

$$m\left[E(X^2)-\underbrace{(E(X))^2}\right]=\left[\underbrace{E(XY)-E(X)E(Y)}\right] \qquad (67)$$
$$\underset{Variance(X)}{} \qquad \underset{Covariance(X,Y)}{}$$

$$\Rightarrow m = \frac{Cov(X,Y)}{\sigma_X^2} \qquad (68)$$

$$\Rightarrow c = E(Y)-\frac{Cov(X,Y)E(X)}{\sigma_X^2} \qquad (69)$$

Now, substituting equation (68) and (69) into equation (67) and we can get the minimum variance, and After combining the term who have been marked with the same symbol above them, the equation is reduced to,

$$=\underbrace{\left[E(Y^2)\right]-\left[E(Y)\right]^2}_{\sigma_Y^2}-2\frac{Cov(X,Y)}{\sigma_X^2}\underbrace{\left[E(XY)-E(Y)E(X)\right]}_{Cov(X,Y)}$$

$$+\frac{Cov(X,Y)^2}{\left(\sigma_X^2\right)^2}\left[\underbrace{E\left(X^2-\left[E(X)\right]^2\right)}_{\sigma_X^2}\right]$$

$$\Rightarrow Var_{\min}(\varepsilon)=\sigma_Y^2-\frac{\left[Cov(X,Y)\right]^2}{\sigma_X^2} \qquad (70)$$

The second part of the equation (RHS) (38) and calculated the set equal from the above equation (70) get the following equation for σ

$$\varepsilon_{model}=\frac{\sigma}{e^{\alpha t_{i+1}}}\int_{t_i}^{t_{i+1}}e^{\alpha s}dW_s$$

$$Var(\varepsilon_{model})=E\left(\varepsilon_{model}^2\right)$$

$$\Rightarrow Var(\varepsilon_{model})=\sigma^2 e^{-2\alpha t_{i+1}}\left(\int_{t_i}^{t_{i+1}}e^{\alpha s}dW_s\right)^2 \qquad (71)$$

For estimate the integral above, have to use Ito´s isometry theorem, [5], [6].

7. Mathematical Model for Pricing of Energy Derivative [11]

To manage the energy derivatives (price, hedge, and risk), it is understood that the dynamics of volatility in energy derivatives markets, for pricing of energy derivatives in the occurrence of unspanned stochastic volatility (USV). This chapter highlighting on the risk – neutral dynamics of the model and efficient pricing of derivatives [12 - 21]. We assume that interest rates are deterministic, which is innocuous when pricing energy derivatives with short and intermediate maturities. Let $S(t)$ denote the time - t spot price of the commodity and let $F(t,T)$ denote the time - t price of a futures contract expiring at time T. In the case of a constant continuously compounded cost of carry rate δ, the relation between spot and future prices is simply

$$F(t,T)=S(t)e^{\delta(T-t)} \qquad (72)$$

In the absence of arbitrage opportunities, futures prices are martingales [22] under the risk- neutral measure from which it follows that $\frac{1}{dt}E_t\left[\frac{dS(t)}{S(t)}\right]=\delta$. More generally, the cost of carry varies stochastically, reflecting stochastic variation in convenience yields. Let $\delta(t)$ denote the time t instantaneous spot rate of carrying cost. Furthermore, let $y(t,T)$ denote the time-t instantaneous forward cost of carry rate at time T, defined such that futures prices are followed

$$F(t,T)=S(t)\exp\left\{\int_t^T y(t,u)du\right\} \qquad (73)$$

In the limit as $T\rightarrow t$, $y(t,t)=\delta(t)$. It follows that the term structure of forward cost of carry rates can be inferred from the term structure of future prices.

One strand of the commodity derivatives literature specifies the dynamics of $S(t)$, $\delta(t)$ and derives futures prices endogenously. Another strand takes futures prices as given and specifies the dynamics of the entire futures curve, which is equivalent to the dynamics of $S(t)$ and the entire forward cost of carry curve.

Here, I am introducing the basic HJM model [13] where $S(t)$ and $y(t,T)$ have the following dynamics:

$$\frac{dS(t)}{S(t)}=\delta(t)dt+\sigma_S dW_1(t) \qquad (74)$$

$$dy(t,T)=\mu_y(t,T)dt+\sigma_y(t,T)dW_2(t) \qquad (75)$$

Where $W_1(t)$ and $W_2(t)$ denote the Wiener process under the risk – neutral measure with correlation ρ.

For convenience, introduce the process

$$Y(t,T)=\int_t^T y(t,u)du, \qquad (76)$$

The dynamics of which are given by

$$dY(t,T)=\left(-\delta(t)+\int_t^T\mu_y(t,u)du+\int_t^T\sigma_y(t,u)du dW_2(t)\right) \qquad (77)$$

Then, from the equation (73), we can write, $F(t,T)$

$$F(r,T)=S(t)e^{Y(t,T)} \qquad (78)$$

With the following dynamics

$$\frac{dF(t,T)}{F(t,T)} = \left(\int_t^T \mu_y(t,u)du + \frac{1}{2}\left(\int_t^T \sigma_y(t,u)du\right)^2 + \rho\sigma_S\int_t^T \sigma_y(t,u)du \right)dt \qquad (79)$$

$$+ \sigma_S dW_1(t) + \int_t^T \sigma_y(t,u)du dW_2(t)$$

Setting the drift (79) to zero (futures prices are martingale under the risk – neutral measure and differentiating w.r.t yields

$$\mu_y(t,T) = -\sigma_y(t,T)\left(\rho\sigma_S + \int_t^T \sigma_y(t,u)du\right) \qquad (80)$$

This condition on the drift of $y(t,T)$ is similar to the famous HJM drift conditions in term structure modeling. The particular model depends on the choice of $\sigma_y(t,T)$ and consider the following time – homogeneous specification

$$\sigma_y(t,T) = \alpha e^{-\gamma(T-t)} \qquad (81)$$

From (80) and follows the drift $\mu_y(t,T)$ is given by

$$\mu_y(t,T) = -\frac{1}{e^{\gamma(T-t)}}\left(\rho\alpha\sigma_S + \frac{\alpha^2}{\gamma}\right) + \frac{\alpha^2}{\gamma}\frac{1}{e^{2\gamma(T-t)}} \qquad (82)$$

Integrating (75) and using that $\frac{1}{e^{\gamma(T-t)}} = e^{-\gamma(T-t)}e^{-\gamma(t-u)}$, one obtains

$$y(t,T) = y(0,T) + e^{-\gamma(T-t)}x(t) + \frac{\alpha^2}{2\gamma^2}\frac{(e^{2\gamma t}-1)}{e^{2\gamma T}} \qquad (83)$$

Where

$$x(t) = -\int_t^T\left(\rho\alpha\sigma_S + \frac{\alpha^2}{\gamma}\right)e^{-\gamma(t-u)}du + \int_0^t \alpha e^{-\gamma(t-u)}dW_2(u) \qquad (84)$$

It follows that $x(t)$ has the mean – reverting dynamics

$$dx(t) = \gamma(\theta - x(t))dt + \alpha dW_2(t), \quad x(0) = 0 \qquad (85)$$

Where $\theta = -\left(\frac{\rho\alpha\sigma_S}{\gamma} + \frac{\alpha^2}{\gamma^2}\right)$, finally, from (73) and using that

$$\frac{F(0,T)}{F(0,t)} = \exp\left\{\int_t^T y(0,u)du\right\}, \text{ future prices are given by}$$

$$F(t,T) = S(t)\frac{F(0,T)}{F(0,t)}e^{\{B(T-t)x(t)+A(t,T)\}} \qquad (86)$$

Where

$$B(T-t) = \frac{1}{\gamma}\left(1 - e^{-\gamma(T-t)}\right) \qquad (87)$$

$$A(t,T) = \frac{\alpha^2}{4\gamma^3}\left(1 - e^{2\gamma t}\right)\left(e^{-2\gamma T} - e^{-2\gamma t}\right) \qquad (88)$$

This model is the HJM equivalent of the two-factor [13, 18] model, the dynamics of $S(t)$ is given by (3) and $\delta(t)$ (or, alternatively, the convenience yield) follows a mean-reverting Gaussian process. To see the equivalence, note that $\delta(t)$ is obtained by setting $T = t$ in (83). It is straightforward to the dynamics of $\delta(t)$ are given by

$$d\delta(t) = \gamma(\theta_\delta(t) - \delta(t)dt + \alpha dW_2)(t) \qquad (89)$$

With

$$\theta_\delta(t) = \frac{1}{\gamma}\frac{dy(0,t)}{dt} + y(0,t) - \frac{\rho\alpha\sigma_S}{\gamma} - \frac{\alpha^2}{2\gamma^2}\left(1 - e^{-2\gamma t}\right) \qquad (90)$$

Therefore, the present model implies dynamics of $S(t)$ and $\delta(t)$ that are similar to [18], with the exception that the mean-reversion level is time-dependent, due to the model matching the initial futures curve. Here we extend the framework with stochastic volatility and resulting model is equivalent to the stochastic volatility model in [11].

$S(t)$ and $y(t,T)$ have the following dynamics:

$$\frac{dS(t)}{S(t)} = \delta(t)dt + \sigma_S[\upsilon(t)]^{0.5}dW_1(t) \qquad (91)$$

$$dy(t,T) = \mu_y(t,T)dt + \sigma_y(t,T)[\upsilon(t)]^{0.5}dW_2(t) \qquad (92)$$

$$d\upsilon(t) = k(\theta - \upsilon(t))dt + \sigma_\upsilon\sqrt{\upsilon(t)}dW_3(t) \qquad (93)$$

Where $W_1(t), W_2(t)$ *and* $W_3(t)$, denote correlated Wiener processes under the risk – neutral [4] measure, with ρ_{12}, ρ_{13} *and* ρ_{23} representing pairwise correlations.

The expression for the dynamics of futures prices are given by

$$\frac{dF(t,T)}{F(t,T)} = \sqrt{\upsilon(t)}\left(\sigma_S dW_1(t) + \int_t^T \sigma_y(t,u)du dW_2(t)\right) \qquad (94)$$

The Volatility of futures prices be contingent on $\upsilon(t)$, but since futures prices are only exposed to $W_1(t)$ and $W_2(t)$, while $\upsilon(t)$ is only exposed to $W_3(t)$, it is immediately clear that volatility risk

can´t be completely hedged by trading in futures (or spot) contracts. To the extent that $W_1(t)$ and $W_2(t)$ are correlated with $W_3(t)$ (i.e. ρ_{13} and/or ρ_{23} are nonzero), volatility contains a covered component, and volatility risk is partially hedgeable. The volatility risk is entirely unhedgeable, if these correlations are both zero.

From the above, one can derive the condition on $\mu_y(t,T)$ that must hold to ensure absence of arbitrage opportunities. [11] give this condition.

$$\mu_y(t,T) = -\upsilon(t)\sigma_y(t,T)\left(\rho_{12}\sigma_S + \int_t^T \sigma_y(t,u)du\right) \quad (95)$$

The dynamics of the future curve in terms of a low dimensional affine state vector, again assume that $\sigma_y(t,T)$ is given by (81).

In this case, $y(t,T)$ is given by

$$y(t,T) = y(0,T) + \frac{\alpha}{e^{\gamma(T-t)}} + \frac{\alpha}{e^{2\gamma(T-t)}}\phi(t) \quad (96)$$

Where

$$x(t) = -\int_0^t \frac{\left(\rho_{12}\sigma_S + \frac{\alpha}{\gamma}\right)\frac{1}{e^{\gamma(t-u)}}\upsilon(u)du}{+\int_0^t \frac{1}{e^{\gamma(t-u)}}(\upsilon(u))^2 dW_2(u)} \quad (97)$$

$$\phi(t) = \int_0^t \frac{\alpha}{\gamma}e^{-2\gamma(t-u)}\upsilon(u)du, \quad (98)$$

With the following dynamics

$$dx(t) = \left(-\gamma x(t) - \left(\frac{\alpha}{\gamma} + \rho_{12}\sigma_S\right)\upsilon(t)\right)dt + \sqrt{\upsilon(t)}dW_2(t), \quad For \quad x(0) = 0 \quad (99)$$

$$d\phi(t) = \left(-2\gamma\phi(t) + \frac{\alpha}{\gamma}\upsilon(t)\right)dt, \quad \phi(0) = 0 \quad (100)$$

Consequently, futures prices are given by

$$F(t,T) = S(t)\frac{F(0,T)}{F(0,t)}e^{\{B(T-t)x(t)+C(T-t)\phi(t)\}}, \quad (101)$$

Where $B(T-t) = \frac{\alpha}{\gamma}\left(1 - \frac{1}{e^{\gamma(T-t)}}\right) \quad (102)$

$$C(T-t) = \frac{\alpha}{2\gamma}\left(1 - \frac{1}{e^{2\gamma(T-t)}}\right) \quad (103)$$

Obtaining the expression for $\delta(t)$ from (96), the dynamics of the log spot price, $s(t) \equiv \log(S(t))$, are given by

$$ds(t) = (y(0,t) + \alpha(x(t) + \phi(t) -)) - \frac{1}{2}\sigma_S^2\upsilon(t)dt + \sigma_S\sqrt{\upsilon(t)}dW_1(t) \quad (104)$$

It follows that futures prices are exponentially affine in $s(t), x(t)$ and $\phi(t)$ which, along with $\upsilon(t)$, jointly establish an affine state vector. Note that $\phi(t)$ is an "auxiliary," nearby deterministic, state variable that capture the path information of $\upsilon(t)$. The pricing of European options on futures contract is highly tractable. Here, we continue with the case in which $\sigma_y(t,T)$ is given in equation (81). For the most exchange – traded products, option expires slightly before the expiry of the underlying futures contracts.

Let $C(t,T_0,T_1,K)$ for the European call option and an option can be price quasi – analytically within the framework of this is paper.

First, the dynamics of the future price,
$$f(t,T_1) \equiv \log(F(t,T_1)) \quad (105)$$

Are given by

$$df(t,T_1) = -\frac{1}{2}\left(\begin{array}{c}\sigma_S^2 + B(T_1-t)^2 \\ +2\rho_{12}\sigma_S B(T_1-t)\end{array}\right)\upsilon(t)dt + \sqrt{\upsilon(t)}(\sigma_S dW_1(t) + B(T_1-t)dW_2(t)) \quad (106)$$

Next, using standard argument, one can show that the characteristic functions of $f(T_0,T_1)$ define as

$$\varphi(u,t,T_0,T_1) \equiv E_t\left[e^{iuf(T_0,T_1)}\right], \quad i = \sqrt{-1} \quad (107)$$

Has the exponentially affine solution

$$\varphi(u,t,T_0,T_1) = e^{M(T_0,t)+N(T_0-t)\upsilon(t)+iuf(t,T_1)} \quad (108)$$

Where $M(\tau)$ and $N(\tau)$ resolve the following system of PDEs

$$\frac{dM(\tau)}{d\tau} = N(\tau)k\theta \quad (109)$$

$$\frac{dN(\tau)}{d\tau} = N(\tau)\begin{pmatrix} -k + iu\sigma_\upsilon \\ (\rho_{13}\sigma_2 + \rho_{23}B(T_1 - T_0 + \tau)) \end{pmatrix}$$
$$+ \frac{1}{2}N(\tau^2)\sigma_\upsilon^2 - \frac{1}{2}(u^2 - iu)\begin{pmatrix} \sigma_S^2 + B(T_1 - T_0 + \tau)^2 \\ + 2\rho_{12}\sigma_S B(T_1 - T_0 + \tau) \end{pmatrix}$$

(110)

The boundary condition $N(0) = 0$ and $M(0) = 0$.

Finally, following [23], one can show that the Fourier translation of the call price (modified)

$$\hat{C}(t, T_0, T_1, K) = e^{\varphi \log(K)} C(t, T_0, T_1, K)$$

(111)

The expression of the above equation (111) in terms of the characteristic function of $f(T_0, T_1)$. In particular, $C(t, T_0, T_1, K)$ is given by

$$C(t, T_0, T_1, K) = P(t, T_0)\frac{e^{-\varphi \log(K)}}{\pi}$$
$$\int_0^\infty \mathrm{Re}\left[\frac{e^{-iu\log(K)}\varphi(u - (\varphi + 1)i, t, T_0, T_1)}{\varphi^2 + \varphi - u^2 + i(2\varphi + 1)u}\right]du,$$

(112)

Where $P(t, T_0)$ - price of a zero – coupon bond maturing at time T_0.

The pricing approach here differs from the one in [11] and has two advantages: First, it permits the computationally efficient Fast Fourier Transform algorithm. Second, it only requires the evaluation of one integral (as opposed to two integrals).

8. Conclusion

In this paper, I introduced and constructed the mathematical model for the option-pricing of energy derivative market. Here I presented and introduced the energy derivative market, describing some simple structure and outline of energy derivatives and how we can derive this energy derivative from other underlying market. I.e. financial derivatives. From the above discussion, I travelled the traditional energy option contract, which was used in the energy derivative market. Then, I extended energy option into option pricing model, which was construct and derive from Black Scholes Equation. After that, I constructed the stochastic model for energy spot price by using of Ordinary Least Square Regression Model. Finally, I constructed the mathematical model for efficient Pricing of Energy Derivative by using of stochastic volatility.

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