International Journal of Engineering & Technology, 7 (4.10) (2018) 570-572

International Journal of Engineering & Technology

Website: www.sciencepubco.com/index.php/IJET

Research paper

Some Results on Generating Graceful Trees

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Abstract

Let T and S be any two simple graphs. Then $T + S$ is the graph obtained by merging a vertex of each copy of S with every attachment vertices of T. Let T_0 be the one vertex union of copies of the given caterpillar T with the common vertex as one of the penultimate vertices. If S_0 is any caterpillar, then define $T_1 = T_0 + S_0$. Recursively for $i \geq 2$, construct T_i,that is, $T_i = T_{i-1} + S_{i-1}$. Here the tree S_{i-1} considered for attachment with T_{i-1} is a caterpillar, but not necessarily the same among the levels. In this paper we prove that the tree T_i is graceful for $i \geq 1$.

Keywords: *Trees; Graceful labeling; Recursive attachment.*

1. Introduction

In 1963, Ringel conjectured that T is a tree with m edges, then the complete graph K_{2m+1} can be decomposed into $2m + 1$ subgraphs isomorphic to T. Kotzig has given the strengthened form of this conjecture and in 1967, Rosa(see [2]) introduced various ways of numbering the vertices of T as a tool to attack the Ringel-Kotzig conjecture. In 1968, the word graceful is coined by Golomb (see [1]) and it is the term widely used. A graceful labeling of a graph $G(V, E)$ with m edges is an injection $f : V(G) \to \{0, 1, 2, \dots, m\}$ with the property that resulting edge labels are distinct, where an edge uv is assigned the label $|f(u) - f(v)|$. A graph which admits graceful labeling is called a graceful graph.

There are many graphs shown to be graceful. Some of the known family of trees are caterpillars, symmetrical trees(see [5]),[10]), banana trees(see [11], [12]), spiders(see [4]). Further, Stanton and Zarnke (see [3]) Koh, Rogers and Tan(see [9] provided methods for combining graceful trees to generate larger graceful trees from the known graceful trees. Burzio and Ferrarese(see [6]) have shown that subdivision of a graceful tree is graceful. Sethuraman and Jesintha (see [13]) have provided a method to generate graceful lobster from a graceful caterpillar. Sethuraman and Ragukumar (see [14]) provided an algorithm that generates a graceful tree from a given arbitrary tree by adding a sequence of new pendent edges to the given arbitrary tree thereby proving that every tree is a subtree of a graceful tree. A complete survey on graceful labeling is done by Gallian (see [8]). Graceful graphs have wide applications in coding theory, X-ray crystallography, circuit design, radio communication, networks and radio astronomy.

Definition 1.1: A vertex u is said to be an attachment vertex if $deg(u) \geq 2$.

Definition 1.2: A vertex of a caterpillar which has at most one adjacent vertex of degree greater than or equal to two is called a penultimate vertex and represent it as v_f and v_l.

Let T and S be any two simple graphs. Then $T + S$ is the graph obtained by merging the penultimate vertex (fixed) of each copy of S with every attachment vertices of T. Let T_0 be the one vertex union of copies of the given caterpillar T with the common vertex as one of the penultimate vertices. If S_0 is any caterpillar, then define $T_1 = T_0 + S_0$. For $i \geq 2$, define $T_i = T_{i-1} + S_{i-1}$ recursively by attaching each copy of S_{i-1} with every vertices of degree greater or equal to two of T_{i-1}. Here the tree considered for attachment with T_i is a caterpillar, but not necessarily the same among the levels. In this paper, we prove that the tree T_i is graceful for $i \geq 1$. In the next section let us first prove that the tree T_1 is graceful.

2. T_1 is graceful

Let T be any caterpillar with the penultimate vertices v_f and v_l. Consider r copies of T and let it be $T^1, T^2, T^3, \dots T^r$. Then T_0 is the one vertex union of T^i, for $1 \leq i \leq r$ with the common vertex as v_f , having t attachment vertices. Let S_0 be any caterpillar and $S_0^1, S_0^2, S_0^3, \dots, S_0^t$ be the copies of S_0. Now the tree $T_1 (= T_0 + S_0)$ is constructed by merging the penultimate vertex of each copy of S_0^j with every attachment vertex of T_0. With this construction, in the next theorem we prove that T_1 is graceful.

Theorem 2.1: The tree T_1 is graceful.

Proof: Let m be the number of edges of T_1. Now for $1 \leq i \leq r$ name the unlabelled attachment vertices of each copy S_0^i in the direction from v_l to v_f or v_f to v_l as i is odd or even and let it be $u_1, u_2, u_3, \dots u_s$.
Let $P(u)$ denote the set of all pendant vertices which are adjacent to u. For $1 \leq i \leq s - 1$,

let us define $R(u_i) = \{u_i\} \cup P(u_{i+1})$

$$A = \cup_i R(u_i), \text{ for } 1 \leq i \leq s, i - odd,$$

and $B = (\cup_i R(u_i)) \cup P(u_1)$, for $2 \leq i \leq s, i - even$

From the above, clearly (A, B) defines a bipartition of T_1. Now label the vertices of A, that is, $a_1, a_2, \ldots, a_p$ by $m, m-1, m-2, \ldots, m-(p-1)$ and the vertices of B ,that is, $b_1, b_2, \ldots, b_q$ by $0, 1, 2, 3, \ldots, q-1$. Further observe that the edges values to be realized that are in transition between the copies are realized with the unique incident to the penultimate vertex. Clearly all the edge values of are distinct and varies from $1, 2, 3, \ldots, m$ and hence the above labeling gives the required graceful labeling of T_1. □

The illustration corresponding to the proof of the theorem 2.1 is provided through the figures 1,2,3 in appendix.
Now let us proceed to our main result.

For $i \geq 2$, consider the tree T_{i-1} having k attachment vertices and by induction assume that T_{i-1} is graceful.

Let S_{i-1} be any caterpillar not necessarily the same taken before. Let $S_{i-1}^1, S_{i-1}^2, S_{i-1}^3, \ldots, S_{i-1}^t$ be the copies of S_{i-1}. Consider the tree T_{i-1} without graceful numbering. Now construct the tree $T_i (= T_{i-1} + S_{i-1})$ by merging the penultimate vertex of each copy of S_{i-1}^j for $1 \leq j \leq t$ with every attachment vertex of T_{i-1}.

Theorem 2.2: The tree T_i is graceful for $i \geq 2$.

Proof: Consider the graph T_i with N edges. As same like theorem 2.1, for $1 \leq i \leq t$ name the unlabelled attachment vertices of each copy S_{i-1} and let it be $u_1, u_2, u_3, \ldots u_M$.

For $1 \leq i \leq M-1$,

let us define $R(u_i) = \{u_i\} \cup P(u_{i+1})$
$A = \cup_i R(u_i)$, for $1 \leq i \leq M, i - odd$,
and $B = (\cup_i R(u_i)) \cup P(u_1)$, for $2 \leq i \leq M, i - even$

From the above, clearly (A, B) defines a bipartition of T_i. Now label the vertices of A, that is, $a_1, a_2, \ldots, a_p$ by $N, N-1, N-2, \ldots, N-(p-1)$ and the vertices of B ,that is, $b_1, b_2, \ldots, b_q$ by $0, 1, 2, 3, \ldots, q-1$. Clearly all the edge values are distinct and varies from $1, 2, 3, \ldots, N$. Thus T_i is graceful. □

3. Conclusion

It is noted that for $i \geq 1$, the tree T_i is constructed recursively from T_{i-1} by attaching a caterpillar S_{i-1}. Thus if S_{i-1} is chosen to be any tree, not necessarily a caterpillar, whether still T_i admits graceful labeling.

Conjecture 1. The tree $T_i = T_{i-1} + S_{i-1}$ is graceful for $i \geq 1$ where S_{i-1} is any graceful tree.

As T_0 is graceful, the tree T_i generated by this attachment process also resulted the graceful labeling, thus by choosing T_0 as any graceful tree, whether T_i admits graceful labeling.

Conjecture 2. The tree $T_i = T_{i-1} + S_{i-1}$ is graceful for $i \geq 1$ where T_0 is any graceful tree.

Acknowledgement

The authors thankfully acknowledge the referee for their valuable comments in improving the presentation of the paper. Further the author thank the management of SASTRA University for providing support in publishing this paper.

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APPENDIX

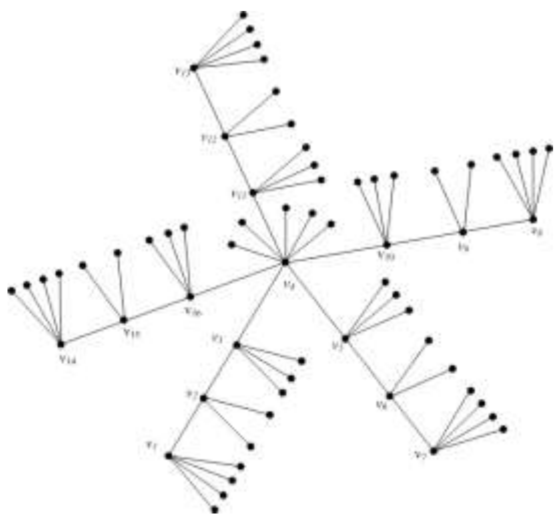

Fig.1:. The tree T_0 with the attachment vertices

Fig.2:. The tree S_0

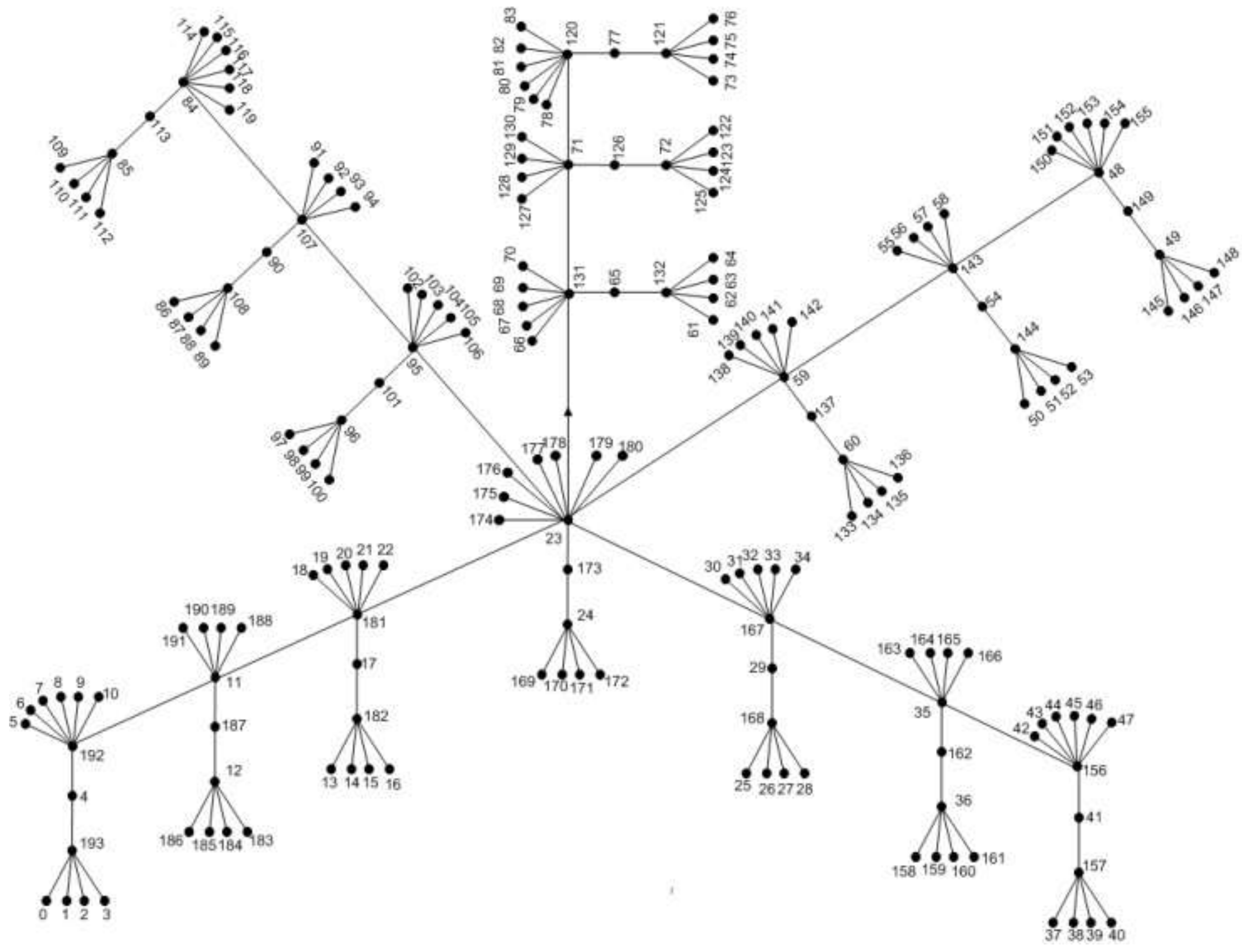

Fig.3:. The tree $T_1 = T_0 + S_0$