International Journal of Engineering & Technology, 7 (3.28) (2018) 321-324

International Journal of Engineering & Technology

Website: www.sciencepubco.com/index.php/IJET

Research paper

On New Properties of Differential Transform via Difference Equations

S. Al-Ahmad[1], M. Mamat[1]*, R. AlAhmad [2,3], I. M. Sulaiman[1], Puspa Liza Ghazali[4], Mohamad Afendee Mohamed[1]

[1]*Faculty of Informatics and Computing, Universiti Sultan Zainal Abidin, Terengganu, Malaysia*
[2]*Department of Mathematics, Yarmouk University, Irbid, Jordan*
[3]*Faculty of Engineering, Higher Colleges of Technology, Ras Alkhaimah, UAE, P.O. box 4793*
[4]*Faculty of Economy and Management Science, Universiti Sultan Zainal Abidin, Terengganu, Malaysia*
**Corresponding author Email: must@unisza.edu.my*

Abstract

The area of differential transform has been enjoying vivid growth recently, with a lot of emphasis on linear and nonlinear ordinary differential/difference equations. In this paper, we construct and prove new properties of differential transform; particularly, in the case of certain quotients of functions. We further presented interesting relations between the differential transform, the difference operator, and incomplete gamma functions. Numerical examples with encouraging results have been presented to illustrate the efficiency of the method.

Keywords: *differential transform; differential equations; polynomial approximation.*

1. Introduction

In 1986, Zhou [23] introduced the differential transform method (DTM) as a new idea in solving differential equations. The differential transform method was further applied to obtain the solution of initial value problems, difference equations, and boundary value problems. The idea of DTM is based on the concept of Taylor [20-22], and it usually gets the solution in a series form. This method constructs an analytical solution in the form of a polynomial. It uses the form of polynomials as the approximation to exact solutions which are sufficiently differentiable. The DTM is an iterative procedure for obtaining Taylor series solutions of differential equations.

This paper is structured as follow. Section 2 discusses brief overview and some fundamental results of differential/difference transform. In section 3, we present the new properties of differential transform followed by its applications. Numerical example of well-known benchmark problem is presented in section 4. Finally, we present the conclusion and discussion in section 5.

2. Preliminaries

This section presents some useful definitions of differential transform, incomplete gamma function, and difference operators.

Definition 2.1 [18]: If a function $f(x)$ is analytical with respect to x in the domain of interest, then

$$F(k) = \frac{f^{(k)}(x_0)}{k!}. \tag{2.1}$$

The inverse differential transform of $F(k)$ is defined as:

$$f(x) = \sum_{k=0}^{\infty} F(k)(x - x_0)^k. \tag{2.2}$$

From (2.1) and (2.2), we have

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k. \tag{2.3}$$

Let $U(k), G(k)$, and $H(k)$ be the differential transforms of $u(x), g(x)$, and $h(x)$ respectively at $x_0 = 0$. Then, the main operations of the DTM is presented in Table 1.

Table 1: Differential Transform

Original Function	Transformed Function
$u(x) = g(x) + h(x)$	$U(k) = G(k) + H(k)$
$u(x) = cg(x)$	$U(k) = cG(k)$
$u(x) = \frac{d^n g(x)}{dx^n}$	$U(k) = \frac{(k+n)!}{k!} G(k+n)$
$u(x) = g(x)h(x)$	$U(k) = \sum_{i=1}^{k} G(i)H(k-i)$
$u(x) = x^n$	$U(k) = \delta(k-n)$
$u(x) = \exp(cx)$	$U(k) = \frac{c^k}{k!}$
$u(x) = \cos(\omega x)$	$U(k) = \frac{\omega^k}{k!} \cos\left(\frac{k\pi}{2}\right)$
$u(x) = \sin(\omega x)$	$U(k) = \frac{\omega^k}{k!} \sin\left(\frac{k\pi}{2}\right)$

For further reference on differential transform and DTM see [8, 12-14, 17-18, 27].

The incomplete gamma function plays a vital role in differential transform. The following is the definition of the incomplete gamma function which is needed in this paper.

Definition 2.2 [11, 16]: For $\Re(s) > 0$, the lower incomplete gamma function is defined as:

$$\gamma(s,x) = \int_0^x t^{s-1} e^{-t} dt,$$

and the upper incomplete gamma function is defined as:

$$\Gamma(s,x) = \int_0^\infty t^{s-1} e^{-t} dt,$$

Clearly,

$$\Gamma(s,z) = \Gamma(s) - \gamma(s,x) \tag{2.4}$$

Moreover, $\gamma(s,x) \to \Gamma(s)$ as $x \to \infty$ and $\Gamma(s,0) = \Gamma(s)$.

Proposition 2.1 [11, 16]: For $N = 0,1,2,\ldots$

$$\sum_{n=0}^{N} \frac{a^n}{n!} e^a \frac{\Gamma(N+1,a)}{N!}.$$

Proof: By integration by parts

$$\Gamma(n+1,a) = \int_a^\infty t^n e^{-t} dt$$
$$a^n e^{-a} + \Gamma(n,a).$$

Therefore,

$$\Gamma(n+1,a) - n\Gamma(n,a) = a^n e^{-a}.$$

Dividing both sides by $n!$, we have

$$\frac{\Gamma(n+1,a)}{n!} - \frac{\Gamma(n,a)}{(n-1)!} = \frac{a^n e^{-u}}{n!}.$$

Thus,

$$\Delta \frac{\Gamma(n,a)}{(n-1)!} = \frac{a^n e^{-u}}{n!}.$$

Taking the sum of both sides from $n = 0$ to N gives

$$\frac{\Gamma(N,a)}{(N-1)!} = \sum_{n=0}^{N-1} \frac{a^n e^{-a}}{n!}.$$

Consequently,

$$e^a \frac{\Gamma(N+1,a)}{N!} = \sum_{n=0}^{N} \frac{a^n}{n!}. \qquad \blacksquare$$

For more properties of the incomplete gamma functions, function integral, and their applications, please refer to [1-7, 9-11, 15-16].

Definition 2.3: Let $S(N)$ be the set of all complex-valued sequences over N define

$$\Delta: S(N) \to S(N)$$

by

$$(\Delta u)(n) = u(n+1) - u(n).$$

Table 2 is the forward difference operator for some functions.

Table 2: Differential Transform

Original Function	Transformed Function
$f(n)$	$(\Delta f)(n)$
C	0
n	1
$n(n-1)$	n
$n(n-1)(n-2)$	$3n(n-1)$
$(n)_m$	$m(n)_{m-1}$
a^n	$a^n(a-1)$

Here,

$$(n)_m = \frac{n!}{(n-m)!}.$$

Also, the forward difference operator Δ satisfies the following proposition.

Proposition 2.2

$$\sum_{n=m}^{N} (\Delta f)(n) = f(N+1) - f(m).$$

Proof

$$\sum_{n=m}^{N} (\Delta f)(n) = (\Delta f)(m) + (\Delta f)(m+1) + (\Delta f)(m+2) + \cdots + (\Delta f)(N-1) - (\Delta f)(N)$$
$$= \big(f(m+1) - f(m)\big) + \big(f(m+2) - f(m+1)\big) + \cdots + \big(f(N+1) - f(N)\big)$$
$$= f(N+1) - f(m). \qquad \blacksquare$$

For further properties of the difference operators, see [19].

3. New Properties of Differential Transform and Its Application

The proposition (2.2) will allow us to solve differential equations by finding the differential transform via difference equations. For instance, Newton's Law of Cooling states that the temperature of a body changes at a rate proportional to the difference in temperature between its own temperature and the temperature of its surroundings.
We can therefore write

$$\frac{dT}{dt} = k(T - T_1), T(0) = T_0$$

where,
T = temperature of the body at any time, t
T_1 = temperature of the surroundings (also called ambient temperature)
T_0 = initial temperature of the body
k = constant of proportionality
Applying the differential transform to both sides will give

$$(n-1)T(n+1) = k(T(n) - T_1\delta(n)).$$

This implies,

$$(n-1)T(n+1) - KT(n) = -kT_1\delta(n)$$

After simplification, we have

$$T(n+1) - \frac{k}{n+1}T(n) = \frac{k}{n+1}T_1\delta(n). \tag{3.1}$$

Multiplying equation (3.1) by $\frac{(n+1)!}{k^{n+1}}$ gives

$$\frac{(n+1)!}{k^{n+1}}T(n+1)-\frac{n!}{k^n}T(n)=-\frac{n!}{k^n}T_1\delta(n)$$
$$\Delta\left(\frac{n!\,T(n)}{k^n}\right)=-\frac{n!}{k^n}T_1\delta(n)$$

Taking the sum from $n=0$ to $n=N-1$, $N\geq 1$ and using proposition (2.2), we get

$$\frac{N!\,T(N)}{k^N}-T(0)=-T_1\sum_{n=0}^{N-1}\frac{n!}{k^n}\delta(n)$$
$$\frac{N!\,T(N)}{k^N}-T(0)=-T_1.$$

Thus,

$$T(N)=\frac{(T_0-T_1)k^N}{N!}\text{for } N\geq 1$$

Next, taking the inverse transformation, it becomes

$$T(t)=T(0)+\sum_{N=1}^{\infty}T(N)t^N$$
$$=T_0+\sum_{N=1}^{\infty}\frac{(T_0-T_1)}{N!}(Kt)^N$$
$$=T_0-(T_0-T_1)+\sum_{N=0}^{\infty}\frac{(T_0-T_1)}{N!}(Kt)^N$$
$$=T_0+(T_0-T_1)e^{kt}.$$

Another application of the difference equations is to find the differential transform of certain quotients.
For example, to find the differential transform of

$$f(t)=\frac{e^t}{2t+1}$$

This gives

$$2tf(t)+f(t)=e^t$$

Considering the fact that if $h(x)=x^m g(x)$, then

$$H(n)=\sum_{j=0}^{n}\delta(j-m)G(n-j)=G(n-m)$$
$$2F(n-1)+F(n)=\frac{1}{n!}$$
$$F(n+1)+2F(n)=\frac{1}{(n+1)!}$$

Dividing both side by $(-2)^{n+1}$

$$\frac{F(n+1)}{(-2)^{n+1}}-\frac{F(n)}{(-2)^n}=\frac{(-\frac{1}{2})^{n+1}}{(n+1)!}$$

This becomes

$$\Delta\left(\frac{F(n)}{(-2)^n}\right)=\frac{(-\frac{1}{2})^{n+1}}{(n+1)!}$$

Taking the sum from $n=0$ to $n=N-1$, $N\geq 1$ and using proposition (2.2), we get

$$\frac{F(N)}{(-2)^n}-F(0)=\sum_{n=0}^{N-1}\frac{(-\frac{1}{2})^{n+1}}{(n+1)!}$$

but for $F(0)=f(0)=1$, we have

$$\frac{F(N)}{(-2)^n}-1=\sum_{n=1}^{N}\frac{(-\frac{1}{2})^n}{n!}.$$

Therefore, for $N\geq 1$, we get

$$F(N)=(-2)^N\left(1-1+\sum_{n=1}^{N}\frac{(-\frac{1}{2})^n}{n!}\right)$$

Thus, using proposition (2.1)

$$F(N)=\frac{(-2)^N e^{-\frac{1}{2}}\Gamma(N+1,-\frac{1}{2})}{N!}$$
$$\begin{cases}\underline{F}(\underline{x},\overline{x},r)=\underline{c}(r),\\ \overline{F}(\underline{x},\overline{x},r)=\overline{c}(r).\end{cases}\quad\forall\, r\in[0,1]$$

4. Results and Discussion

In this section, we solve an example of linear system of first order differential equations.

Example 4.1

$$\frac{dx}{dt}=2y(t)-x(t)\tag{4.1}$$

$$\frac{dy}{dt}=2x(t)-y(t)\tag{4.2}$$

The above problem is a model for many applications in physics. We wish to find the differential transform for $x(t)$ and $y(t)$, using the properties of difference operator. Now, taking the differential transform of both sides, equations (4.1) and (4.2) are transformed into

$$(n+1)X(n+1)=2Y(n)-X(n).\tag{4.3}$$
$$(n+1)Y(n+1)=8X(n)-Y(n).\tag{4.4}$$

After some simplification, we get

$$X(n+1)+\frac{X(n)}{n+1}=\frac{2}{n+1}Y(n)$$
$$Y(n+1)+\frac{Y(n)}{n+1}=\frac{8}{n+1}X(n).$$

Multiplying both side by $(-1)^{n+1}(n+1)!$ gives

$$(-1)^{n+1}(n+1)!X(n+1)-(-1)^n n!X(n)=2(-1)^{n+1}n!Y(n)$$
$$(-1)^{n+1}(n+1)!Y(n+1)-(-1)^n n!Y(n)=8(-1)^{n+1}n!X(n).$$

This implies

$$\Delta((-1)^n n!X(n))=2(-1)^{n+1}n!Y(n)$$
$$\Delta((-1)^n n!Y(n))=8(-1)^{n+1}n!X(n).$$

Let $z(n)=(-1)^n n!X(n)$ and $w(n)=(-1)^n n!Y(n)$.

The resulting system is

$$(\Delta z)(n)=-2w(n).$$
$$(\Delta w)(n)=-8z(n).$$

These equations can be re-written in matrix notations as $z(n)$ and $w(n)$.

$$\begin{pmatrix}z(n+1)\\ w(n+1)\end{pmatrix}=\begin{pmatrix}1 & -2\\ -8 & 1\end{pmatrix}\begin{pmatrix}z(n)\\ w(n)\end{pmatrix}.\tag{4.5}$$

Let $\vec{Q}(n) = \begin{pmatrix} z(n) \\ w(n) \end{pmatrix}$. Then, $\vec{Q}(n)$ satisfied the system

$$(\Delta\vec{Q})(n) = \begin{pmatrix} 0 & -2 \\ -8 & 0 \end{pmatrix}\vec{Q}(n)$$

As a matrix of $\begin{pmatrix} 0 & -2 \\ -8 & 0 \end{pmatrix}$ has eigenvalues 4 and -4 with corresponding eigenvectors $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ respectively.

Let $a(n)$ and $b(n)$ be

$$\begin{pmatrix} z(n) \\ w(n) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 0 & -2 \\ -8 & 0 \end{pmatrix}\begin{pmatrix} a(n) \\ b(n) \end{pmatrix}$$

$$\begin{pmatrix} z(n) \\ w(n) \end{pmatrix} = \begin{pmatrix} -8 & -2 \\ -16 & 4 \end{pmatrix}\begin{pmatrix} a(n) \\ b(n) \end{pmatrix}. \tag{4.6}$$

$$\begin{pmatrix} -8 & -2 \\ -16 & 4 \end{pmatrix}\begin{pmatrix} (\Delta a)(n+1) \\ (\Delta b)(n+1) \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ -8 & 0 \end{pmatrix}\begin{pmatrix} -8 & -2 \\ -16 & 4 \end{pmatrix}\begin{pmatrix} a(n) \\ b(n) \end{pmatrix}$$

Therefore,

$$\begin{pmatrix} (\Delta a)(n+1) \\ (\Delta b)(n+1) \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} a(n) \\ b(n) \end{pmatrix}$$

Thus,

$$a(n+1) = -3a(n) \text{ and } b(n+1) = 5a(n).$$

This gives

$$\frac{a(n+1)}{(-3)^{n+1}} = \frac{a(n)}{(-3)^n} \text{ and } \frac{b(n+1)}{5^{n+1}} = \frac{b(n)}{5^n}.$$

Hence,

$$\Delta\left(\frac{a(n)}{(-3)^n}\right) = 0 \text{ and } \Delta\left(\frac{b(n)}{5^n}\right) = 0.$$

Therefore,

$$a(n) = C_1(-3)^n \text{ and } b(n) = C_2 5^n.$$

Using (4.6), we have

$$z(n) = -8C_1(-3)^n - 2C_2 5^n \text{ and } z(n) = -16C_1(-3)^n + 4C_2 5^n$$

Hence, using (4.3) and (4.4)

$$X(n) = \frac{-8C_1 3^n}{n!} - \frac{2C_2(-5)^n}{n!} \text{ and } Y(n) = \frac{-16C_1 3^n}{n!} + \frac{4C_2(-5)^n}{n!}.$$

Finally, we obtain

$$x(t) = -8C_1 e^{3t} - 2C_2 e^{-5t}$$
$$y(t) = -16C_1 e^{3t} + 4C_2 e^{-5t}.$$

5. Conclusion

Differential transform is an important tool for solving differential equations. One of the tools to be used, especially for quotients, to find the differential is the difference operators. Thus, in this paper, we investigated the performance of new properties of the differential transform, studying their efficiency on some problems. Numerical result was presented to illustrate the practical performance of the proposed scheme.

Acknowledgement

The authors would like to thank the government of Malaysia for funding this research under the Fundamental Research Grant Scheme (FRGS/1/2017/STG06/Unisza/01/1) and also Universiti Sultan Zainal Abidin, Terengganu, Malaysia.

References

[1] R. AlAhmad (2015). Products of incomplete gamma functions. Analysis, 36(3), 199-203.
[2] R. AlAhmad (2016). Products of Incomplete gamma functions Integral representations. Math. Sci. Appl. E-Notes, 4(2), 47-51.
[3] P. Agarwal, J.J. Nieto (2015). Some fractional integral formulas for the Mittag-Leffer type function with four parameters. Open Mathematics, 13(1), 537-546.
[4] P. Agarwal, M. Chand, S. Jain (2015). Certain integrals involving generalized Mittag-Leffer functions. Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci., 85(3), 359-371.
[5] P. Agarwal, S. Jain (2015). A new class of integral relations involving a general class of polynomials and I-functions. Walailak J Sci and Tech., 12(1)1, 1009-1018.
[6] Apelblat (1983). Table of definite and infinite integrals. P North-Holland.
[7] Cetinkaya (2013). The incomplete second Appell hypergeometric functions. Applied Mathematics and Computation, 219(15), 8332-8337.
[8] J. Choi, P. Agarwal (2015). Certain integral transforms for the incomplete functions. Appl. Math. Inf. Sci., 9(4), 2161-2167.
[9] I.S. Gradshteyn, I.M. Ryzhik (1966). Tables of integrals, series, and products. Academic Press.
[10] R. Metzler, J. Klafter, J. Jortner (1999). Hierarchies and logarithmic oscillations in the temporal relaxation patterns of proteins and other complex systems. Proc. Nat. Acad. Sci., 96(20), 11085-11089.
[11] M.R. Allen, M.S. Ira (1998). On certain Generalized incomplete Gamma functions. J. Comp. Appl. Math., 91(2),179-190.
[12] Shavitt (1963). The Gaussian function in calculations of statistical mechanics and quantum mechanics. Methods in Computational Physics, 2, 1-45.
[13] Shavitt, M. Karplus (1965). Gaussian-transform method for molecular integrals. I. Formulation for energy integrals. J. Chem. Phys., 43, 398-414.
[14] D. Sornette (1998). Multiplicative processes and power laws. Phys. Rev. E, 57(4), 4811-4813.
[15] H.M. Srivastava, P. Agarwal (2013). Certain fractional integral operators and the generalized incomplete hypergeometric functions. Applications and Applied Mathematics, 8(2), 333-345.
[16] T. Nico. (1975). Uniform asymptotic expansions of the incomplete gamma functions and the incomplete beta function. Math. Comp., 29(132), 1109-1114.
[17] Arikoglu, I. Ozkol (2006). Solution of difference equation by using differential transform method. Applied Mathematics and Computation, 174(2), 1216-1228.
[18] E. Elmabrouk, F. Abdelwahid (2016). Useful formulas for one-dimensional differential transform. British Journal of Applied Science and Technology, 18(3), 1-8.
[19] A.J. Jerri (2013). Linear difference equations with discrete transform methods. Springer Science and Business Media.
[20] C.L. Chen, S.H. Lin, C.K. Chen (1996). Application of Taylor transformation to nonlinear predictive control problem. Appl. Math. Model., 20(9), 699-710.
[21] F. Ayaz (2004). Solution of the system of differential equations by differential transform method. Appl. Math. Comput., 147(2), 547-567.
[22] Z. Odibat (2008). Differential transform method for solving volterra integral equation with separable kernels. Math. Comput. Model., 48(7-8), 1144-1149.
[23] J.K. Zhou (1986). Differential transform and its applications for electrical circuits. Huarjung University Press.