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Research paper

Calculating the natural frequency of cantilever tapered beam using classical Rayleigh, modified Rayleigh and finite element methods

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Abstract

Beam is a structural element and can be used in different shapes according to its applications and the tapered beam is one of these structural elements. The frequency of tapered beam was investigated in this work using three calculation methods. These methods were Classical Rayleigh Method (CRM), Modified Rayleigh Method (MRM) and Finite Element Method (FEM) using ANSYS Workbench (17.2). The basic idea of Classical Rayleigh Method (CRM) and Modified Rayleigh Method (MRM) was changing the tapered beam into stepped beam with N-steps. The results showed that there was a good agreement between the natural frequency which was calculated by ANSYS and Modified Rayleigh Method (MRM) when the number of steps was (6) and the natural frequency increases when the larger width (or height) increases for different values of smaller width (or height).The frequency ratio is constant when the smaller width (or height) increases. Also, the frequency ratio increases when the width ratio (WL/WS) increases and when the number of divisions (N) increases, the slope of frequency ratio increases too.

Keywords: *Classical Rayleigh Method; Modified Rayleigh Method; Finite Element Method; ANSYS Workbench; Tapered Beam; Frequency.*

1. Introduction

Beam is a structural element and can be classified into different types depending on different attributes such as shape of cross-section (circular , rectangular and I-section), geometric profile (prismatic and Non-prismatic), boundary conditions(cantilever and simply support) etc. Non-prismatic beam has been used in many engineering applications like robotics, aeronautics, and other innovative engineering applications in order to provide suitable distribution of mass and strength. Therefore, many authors studied the vibration analysis of non-uniform beam and these studies considered generally one of the classical beam theories, such as Bernoulli-Euler [1-3] and Timoshenko [4-8] theory.

Several studies were carried out in order to derive the analytical solution for calculating natural frequency of tapered cantilever beam. Mabie and Rogers [9] studied the free vibration of a cantilever beam by developing new differential equation and they started their development from the Bernoulli-Euler equation. They [10], also, used the equations of Bernoulli - Euler in order study the free vibrations of non-uniform cantilever beams. They considered two configurations of tapered beam (a) constant thickness and linearly variable width and (b) constant width and linearly variable thickness in order to plot Charts for each configuration. Naguleswaran [11] utilized an infinite power series in order to find the solution for wedge and conic beams and he employed the method of Frobenius. Other analytical solutions were performed based on Bessel functions [12] orthogonal polynomials [13], and hyper-geometric functions [14].

The fundamental natural frequency of non-uniform beams with various end support conditions was calculated by simple formulas which presented by Abrate S. [15] while De Rosa M. A. et al. [16] studied the dynamic behavior of non-uniform beams using Bessel functions and Izabela Zamorska [17] studied the free vibration problem of non-uniform Bernoulli-Euler beams using the Green's function method. Stanisław Kukla and Izabela Zamojska [18], also, used the Green's function method for studying frequency analysis of non-uniform beam. Mahmoud A. A. et al. [19] used the differential transformation method in order to calculate Natural frequencies and corresponding normalized mode shapes for uniform and non-uniform Euler beam with different cases of cross section and boundary conditions.

Various approximation methods like Rayleigh-Ritz method, differential quadrature method, finite element method and mesh-free method have also been used in order to study the vibrations of beams with variable cross-section have been studied in several studies [20-26].

P.Nagalatha and P.sreenivas[27] used polynomial regression method for calculating natural frequencies by the reanalysis of structural modification of a beam element. They compared between the results of Regression method and Finite Element Method (FEM) and they found that there is a very good agreement between them. B. Rama Sanjeevasresta and Dr. Y. V. Mohan Reddy[28] and E Ozkaya [29] used reanalysis of simple beam structure using a polynomial regression method in their papers to calculate the natural frequencies of Euler-Bernoulli beam. Dhyai Hassan Jawad [30] considered non-uniform Euler-Bernoulli beam in his work and he used Finite Element Method (FEM) in order to study the buckling behavior and free vibration when the tapered parameter and degree of flexural bending change. Byoung Koo Lee et al. [31] solved, numerically, the ordinary differential governing equation of tapered beam and they used the combination of Runge Kutta method and the determinant search method in order

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to calculate the natural frequencies. Firouz-Abadi R. D. et al. [32] solved the governing equation of motion of the Euler-Bernoulli beam considering the effect of axial force distribution using the Wentzel, Kramers, Brillouin (WKB) approximation method in order to calculate the transverse free frequency of variable-cross section beams. Rossi R. E. and Laura P. A. A. [33] used finite element method (FEM) in order to study the dynamic behavior and the natural frequencies of tapered beams.
In this work, the fundamental natural frequency of cantilever tapered beam is calculated using three numerical methods (Classical Rayleigh, Modified Rayleigh and Finite Element Methods) for different dimensions of square cross section area.

2. Problem description

The tapered beam with square cross section area is shown in Fig. (1). Due to change in dimensions (i.e. area) and Second Moment of Inertia, the Euler-Bernoulli and Timoshenko equations , which described the equation of motion of beam, cannot be used in this case. Several researchers tried to derive new equation of motion but they cannot found the solution of this equation (Others have found a solution for tapered beam but it was for special cases).

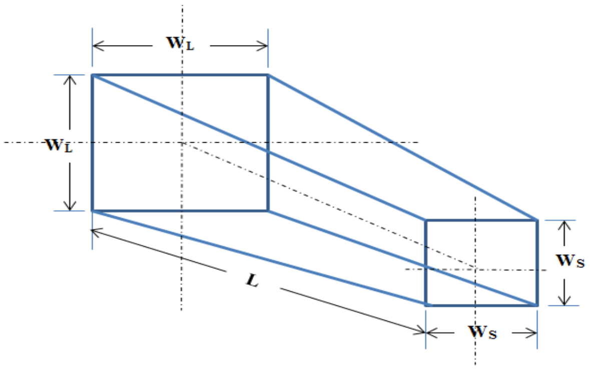


Fig. 1: Geometry of Tapered Beam Used in this Work.

In this work, Rayleigh method and FE method were adopted to obtain the natural frequency of the tapered beam. These methods are used in order to avoid the complexity in governing equation and its solution [34-36].

3. Rayleigh method (RM)

Rayleigh method bases on calculating the kinetic energy and potential energy of the system in order to find the fundamental natural frequency of the system. The general formula of Rayleigh method is [34-37]:

$$\omega^2 = \frac{\int_0^l EI\left(\frac{d^2y(x)}{dx^2}\right)^2 dx}{\int_0^l \rho A(y(x))^2 dx} = \frac{g\sum_{i=1}^{n+1} m_i y_i}{\sum_{i=1}^{n+1} m_i (y_i)^2} \quad (1)$$

Where: (ω) is frequency, (E) is Modulus of Elasticity, (I) is Second Moment of Inertia, (ρ)is Density, (A) is Cross Section Area, (m) mass, and (y) is Deflection.
From the equation (1), the second moment of inertia is the main problem in this method because of the changing in the cross section area along the beam. Therefore, the main idea of this work was converting the tapered beam into stepped beam with (N) steps. The tapered beam was divided into (N) parts with different width (or diameter) (see Fig.(2)) then the equivalent second moment of inertia can be calculated using the same procedure described in [34] and [35]. There are two methods in order to calculate the equivalent moment of inertia [34-36] and these methods are:

1) Classical method

In this method, the equivalent second moment of inertia for beam can be calculated using the following equation [34-36]:

$$I_{eq} = \frac{(L_{Total})^3}{\sum_{n=1}^{N}\left[\frac{(L_n)^3-(L_{n-1})^3}{I_n}\right]} \quad (2)$$

2) Modified method

The equivalent moment of inertia at any point in the beam can be calculated by applying the same idea described in [34] and [35]. For example, if the tapered beam is divided into two steps, the equivalent second moment of inertia can be written as:

$$I_{eq}(x) = \frac{(L_{Total})^3}{\left[\frac{(L_1(x))^3}{I_1}+\frac{(L_2)^3-(L_1(x))^3}{I_2}\right]} \quad (3)$$

And when the tapered beam is divided into three steps, the equivalent second moment of inertia can be written as:

$$I_{eq}(x) = \frac{(L_{Total})^3}{\left[\frac{(L_1(x))^3}{I_1}+\frac{(L_2)^3-(L_1(x))^3}{I_2}+\frac{(L_3)^3-(L_2)^3}{I_3}\right]} \quad (4)$$

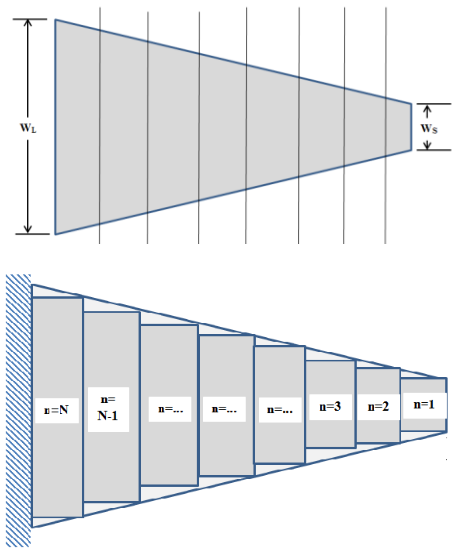


Fig. 2: Dividing the Tapered Beam into Stepped Beam with (N) Steps.

4. Programming Rayleigh methods

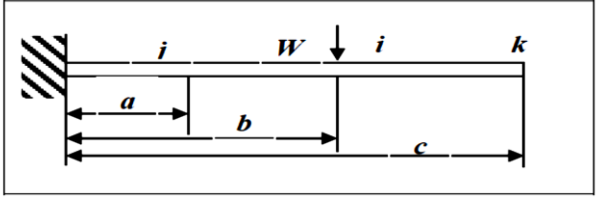
The Rayleigh Methods (i.e. Classical Rayleigh Method (CRM) and Modified Rayleigh Method (MRM)) were programing using MATLAB code. The general steps are:

1) Input the material properties (i.e. density and modulus of elasticity) and beam dimensions (see Fig. (1)).
2) Input number of divisions (N) and in this work N=2, 3, 4, 5, 6, 20 and 50.
3) Calculate the width (or diameter) of each steps.
4) Dividing each steps calculated in step (3) into (M) parts (i.e. (M+1) nodes) and M=25 in this work.
5) Calculate the equivalent second moment of inertia according to the method (i.e. CRM or MRM).
6) Calculate the mass matrix $[m]_{((M+1)*N)}$.
7) Calculate the delta matrix $[\delta]_{((M+1)*N)*((M+1)*N)}$ using Table (1)
8) Calculate the deflection at each node using the following equation and apply the boundary conditions:

$$[y]_{((M+1)*N)} = [\delta]_{((M+1)*N)*((M+1)*N)}\,[m]_{((M+1)*N)} \quad (5)$$

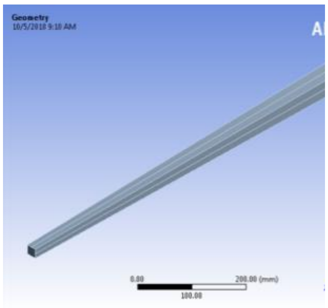
Table 1: Formula of the Deflections of the Cantilever Beam [34-36].

$$\delta_{ji} = \frac{Wa^2(3b-a)}{6EI},\ \delta_{ii} = \frac{Wb^3}{3EI},\ \delta_{ki} = \frac{Wb^2(3c-b)}{6EI}$$

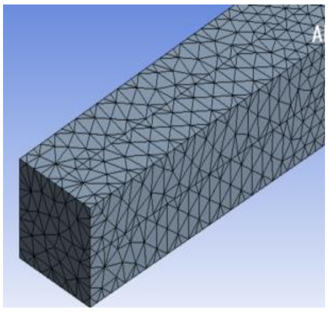


5. Finite element method (FEM)

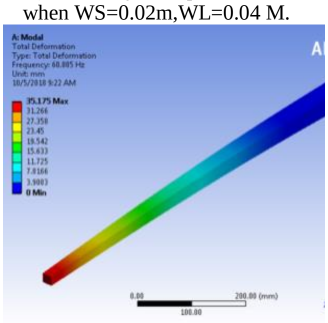
The finite elements method was applied in this work using the ANSYS – Workbench (17.2). The 3D model was built as shown in Fig. (3) and the Tetrahedrons element were used. Generally the number of elements were about (90,000) and the size of element was (1 mm).



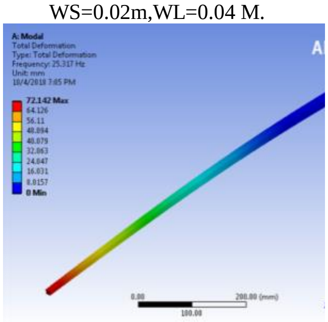
A) Geometry of Tapered Beam when WS=0.02m,WL=0.04 M.



B) Meshing of Tapered Beam when WS=0.02m,WL=0.04 M.



C) First Shape Mode When WS=0.02m,WL=0.04 M.



D) First Shape Mode When WS=0.01m,WL=0.0175 M.

Fig. 3: Samples of Geometry, Meshing and Results of Tapered Beam.

6. Results , discussion and conclusions

Generally, the length of beam was (0.84) m and five values of smaller width (width of small square)(W_S=0.01, 0.02, 0.03, 0.04 and 0.05 m). The width ratio was (1, 1.25, 1.5, 1.75, 2, 2.25, 2.5, 2.75 and 3) .The dimensions of tapered beams with square cross section area, used in this work, were summarized in Table (2).

Figure (4) shows the comparison among the fundamental natural frequency due to change in larger width (or height) for different values of smaller width (or height). Three methods were used for calculating natural frequency of tapered beam. From Figure (4), the natural frequency increases linearly when the larger width (or height) increases for different values of smaller width (or height). The increasing in natural frequency can be explain according to:

a) The increasing of the larger width (or height) leads to increase supported area which causes increasing of natural frequency.
b) The increasing of the larger width (or height) causes increasing the volume of beam (i.e. the mass of beam) and this leads to increase the natural frequency.

In CRM, the results show the coinciding in the results of natural frequency for the different values of smaller width (or height). In other words, the results appear as a single continues line. While a small shifting between the lines (i.e. values of smaller width (or height)) can be seen for the ANSYS and MRM calculation methods and this shifting increases when the number of divisions (N), used in MRM, increases.

In order to study the effect of number of divisions (N) on the natural frequency of tapered beam, Figures (5-9) show the comparison of the three calculation methods (ANSYS, CRM and MRM) for different numbers of divisions (N). In these Figures, the results of ANSYS and MRM converge to each other when the number of division increases till (N=6) and then the natural frequency calculated by MRM was greater than ANSYS results. While the results calculated by CRM are smaller than that of ANSYS and MRM specially at high value of larger width (or height).

The effect of number of divisions (N), smaller width (or height), calculation method and width ratio (W_L/W_S) on the frequency ratio

$((\omega f)_{Tapered} / ((\omega f)_{Uniform})$ can be seen in Figure (10). The frequency ratio was constant when the smaller width (or height) increases. Also, the frequency ratio increases when the width ratio (W_L/W_S) increases. Finally, the frequency ratio changes when calculation method changes and when the number of divisions (N) increases, the slope of frequency ratio increases too.

From the results, the following points can be concluded:

1) The ANSYS, MRM are suitable methods that can be used to calculate the natural frequency of Tapered beam.
2) In MRM, there is a critical number of divisions which can be used to get a good agreement with ANSYS results.
3) The natural frequency of tapered beam depends on smaller width, number of divisions and width ratio (Tapered ratio).

Finally, the experimental procedure can be studied in future work and a comparison between the results of these methods and the results of other theoretical and experimental methods can be done. Also, different length and different shape of tapered beam can be investigated theoretically and experimentally.

Table 2: Cases Studied in this Work.

No.	Length of Beam (m)	Smaller Width of Beam (m)	Larger Width of Beam (m)	Smaller Height of Beam (m)	Larger Height of Beam (m)	W_L/W_S
1			0.01		0.01	1
2			0.0125		0.0125	1.25
3			0.015		0.015	1.5
4			0.0175		0.0175	1.75
5	0.84	0.01	0.02	0.01	0.02	2
6			0.0225		0.0225	2.25
7			0.025		0.025	2.5
8			0.0275		0.0275	2.75
9			0.03		0.03	3
10			0.02		0.02	1
11			0.025		0.025	1.25
12			0.03		0.03	1.5
13			0.035		0.035	1.75
14	0.84	0.02	0.04	0.02	0.04	2
15			0.045		0.045	2.25
16			0.05		0.05	2.5
17			0.055		0.055	2.75
18			0.06		0.06	3
19			0.03		0.03	1
20			0.0375		0.0375	1.25
21			0.045		0.045	1.5
22			0.0525		0.0525	1.75
23	0.84	0.03	0.06	0.03	0.06	2
24			0.0675		0.0675	2.25
25			0.075		0.075	2.5
26			0.0825		0.0825	2.75
27			0.09		0.09	3
28			0.04		0.04	1
29			0.05		0.05	1.25
30			0.06		0.06	1.5
31	0.84	0.04	0.07	0.04	0.07	1.75
32			0.08		0.08	2
33			0.09		0.09	2.25
34			0.1		0.1	2.5

35			0.11		0.11	2.75
36			0.12		0.12	3
37			0.05		0.05	1
38			0.0625		0.0625	1.25
39			0.075		0.075	1.5
40			0.0875		0.0875	1.75
41	0.84	0.05	0.1	0.05	0.1	2
42			0.1125		0.1125	2.25
43			0.125		0.125	2.5
44			0.1375		0.1375	2.75
45			0.15		0.15	3

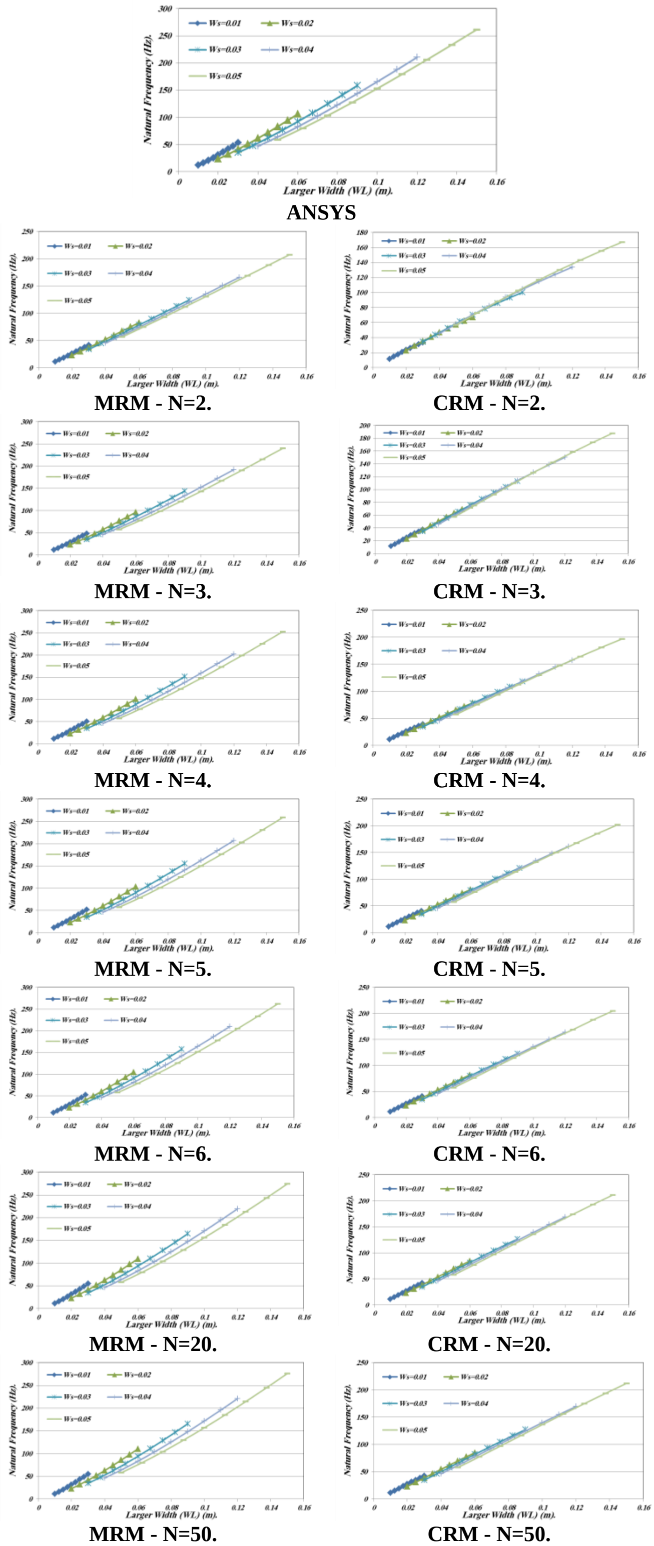


Fig. 4: The Variation of the Fundamental Natural Frequency Due to Change in Larger Width (WL) for Different Values of Smaller Width (WS).

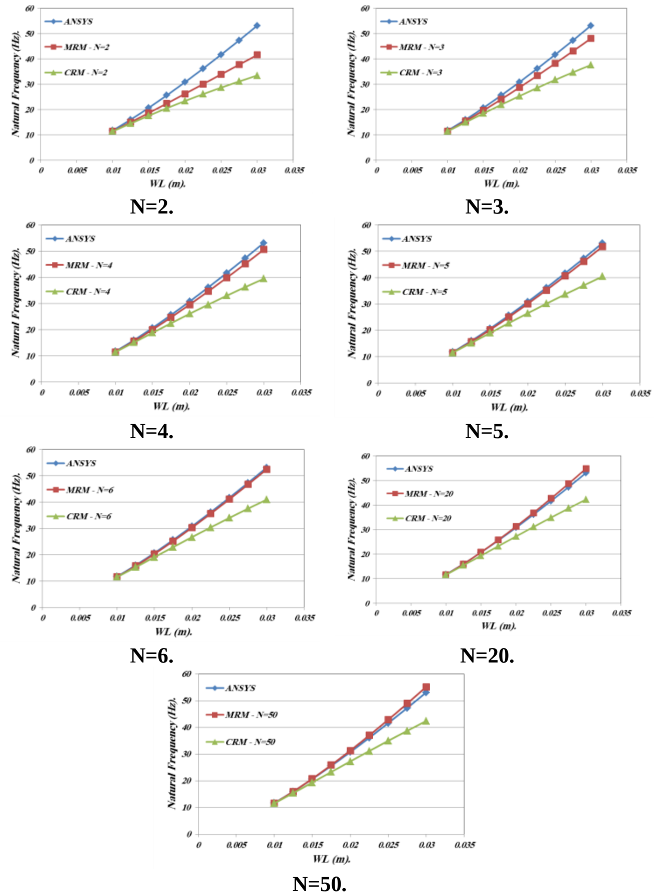


Fig. 5: The Comparison Among the Fundamental Natural Frequencies Calculated by ANSYS, Classical Rayleigh Method and Modified Rayleigh Method for Different Values of Larger Width (WL) when the Smaller Width (WS) is (0.01) M.

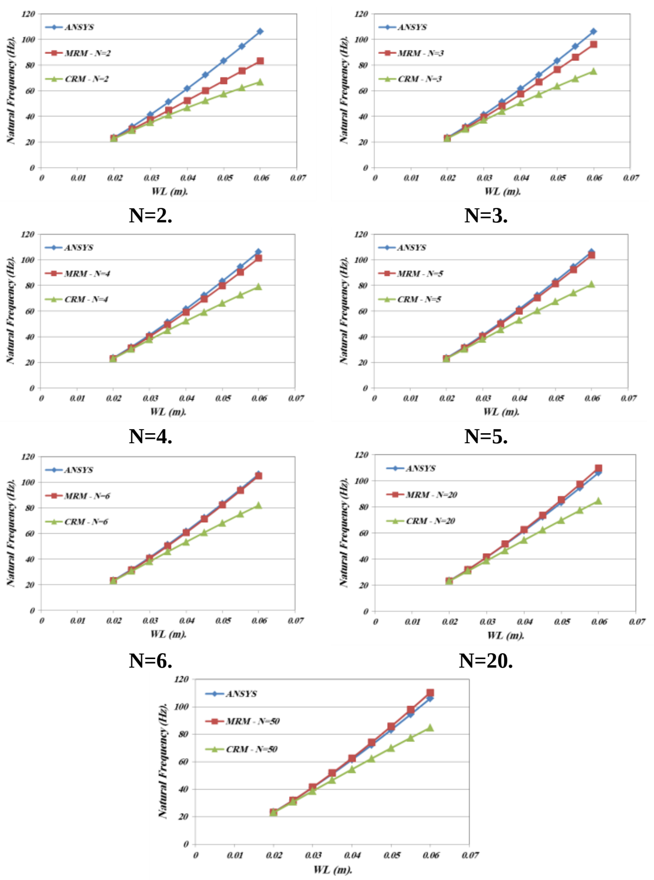
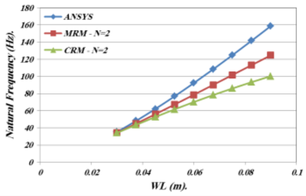
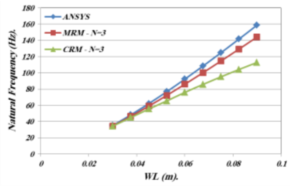


Fig. 6: The Comparison among the Fundamental Natural Frequencies Calculating by ANSYS, Classical Rayleigh Method and Modified

Rayleigh Method for Different Values of Larger Width (WL) when the Smaller Width (WS) is (0.02) M.



N=2.



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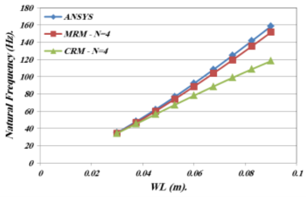

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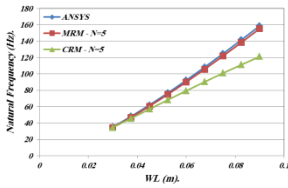

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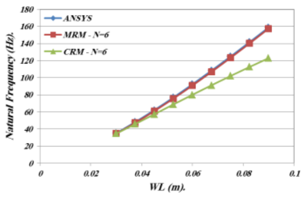

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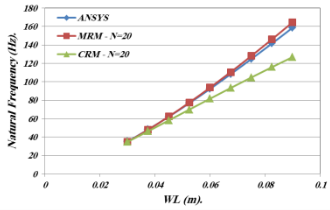
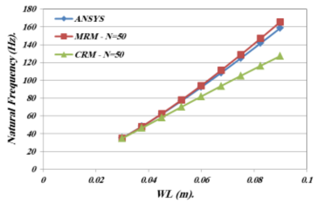
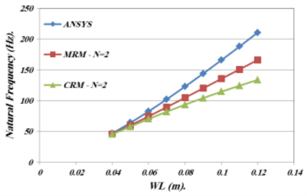


N=20.

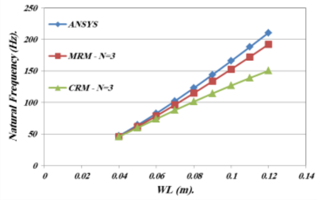


N=50.

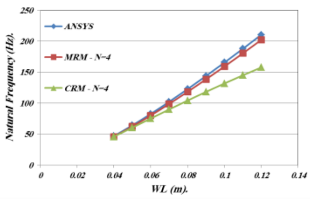
Fig. 7: The Comparison among the Fundamental Natural Frequencies Calculated by ANSYS, Classical Rayleigh Method and Modified Rayleigh Method for Different Values of Larger Width (WL) when the Smaller Width (WS) is (0.03) M.



N=2.



N=3.



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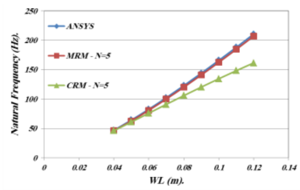

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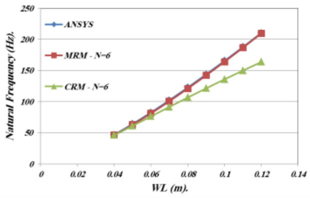

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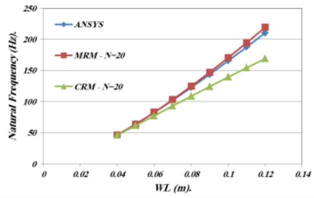
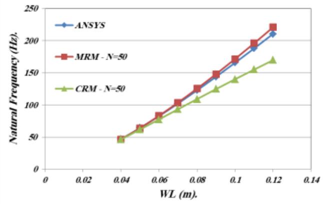
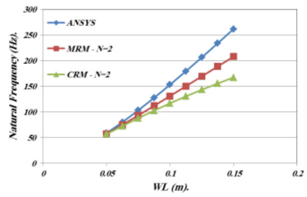


N=20.



N=50.

Fig. 8: The Comparison among the Fundamental Natural Frequencies Calculated by ANSYS, Classical Rayleigh Method and Modified Rayleigh Method for Different Values of Larger Width (WL) when the Smaller Width (WS) Is (0.04) M.



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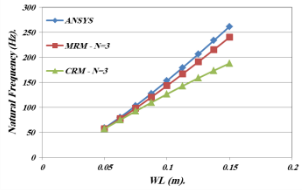

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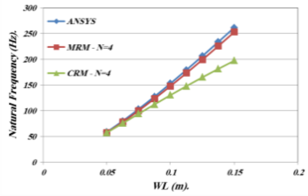

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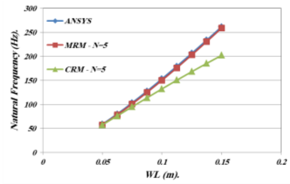

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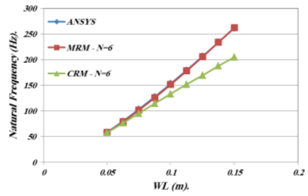

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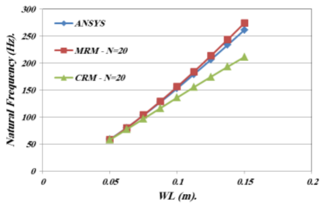

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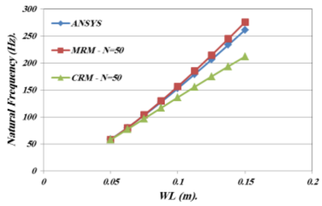
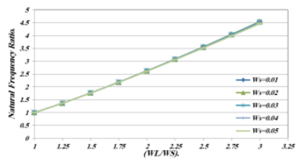
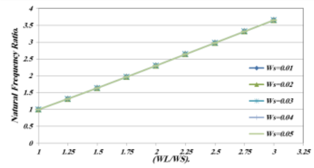


N=50.

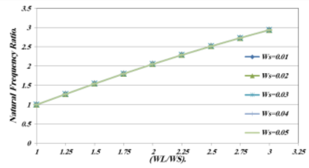
Fig. 9: The Comparison among the Fundamental Natural Frequencies Calculated by ANSYS, Classical Rayleigh Method and Modified Rayleigh Method for Different Values of Larger Width (WL) when the Smaller Width (WS) Is (0.05) M.



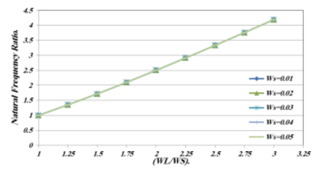
ANSYS



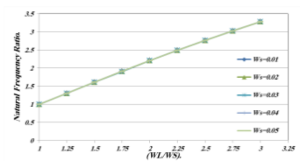
MRM - N=2.



CRM - N=2.



MRM - N=3.



CRM - N=3.

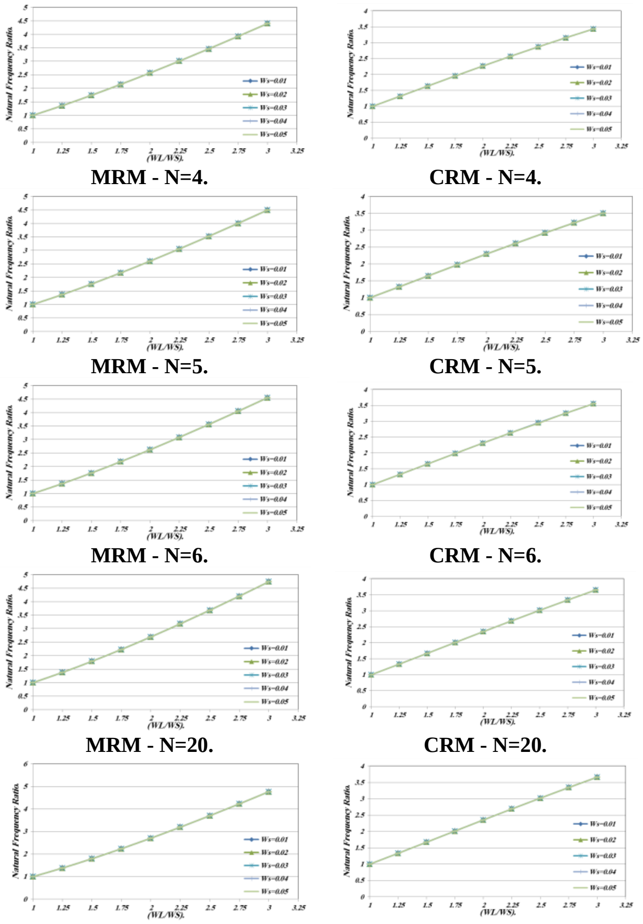


Fig. 10: The Variation of the Frequency Ratio Due to Change in Width Ratio (WL/WS) for Different Values of Smaller Width (WS).

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