Investigation of the dynamics of non linear operators generated from $\xi^{(as)}$-QSO

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A quadratic stochastic operator (QSO) exhibits the time development of various species in biology. Several QSOs have been examined by Lotka and Volterra. The main problem in a non linear operators is to explore their behavior. The behavior of a non linear operators have not been studied in comprehensively even QSOs which are the simplest a nonlinear operators. To address this problem, many classes of QSO were introduced. This paper aims to examine the behavior of six an operators selected from different classes of $\xi^{(as)}$-QSO.

1 Introduction

Many systems are presented by a nonlinear operators. One of the most simplest nonlinear case is quadratic one. A quadratic dynamical systems have been demonstrated to be exporter for various fields to study dynamical properties and modeling, such as biology [1, 24, 9], physics [14, 20, 28, 29], economics and mathematics [9, 10, 11, 21]

The theory of Markov processes is a quickly evolving field with various applications to numerous branches of physics and mathematics. Nevertheless, several physical and biological systems can't be studyed by Markov processes. One of these systems is given by quadratic stochastic operators (QSO). The system of a QSO which relates to genetics population has appeared in [1]. A QSO typically is utilized to exhibit the time evolution of various species in biology, which emerges as follows. Let a population consists of m species. Let $x^{(0)} = (x_1^{(0)}, \cdots, x_m^{(0)})$ taken as a probability distribution of species at an initial state and let $p_{ij,k}$ be probability which the species i^{th} and j^{th} will be interbreed to produce k^{th} species. Then, a probability distribution $x^{(1)} = (x_1^{(1)}, \cdots, x_m^{(1)})$ of the species in the first generation can be described as a total probability as below,

$$x_k^{(1)} = \sum_{i,j=1}^{m} P_{ij,k} x_i^{(0)} x_j^{(0)}, \quad k = \overline{1, m}.$$

Consequently, it shows that the association $x^{(0)} \to x^{(1)}$ represents a mapping V, which is known as evolution operator. Starting from initial state $x^{(0)}$, the population develops to the first generation $x^{(1)} = V(x^{(0)})$ and then to the second generation $x^{(2)} = V(x^{(1)}) = V(V(x^{(0)})) = V^{(2)}\left(x^{(0)}\right)$ and so on. Hence, the discrete the dynamical system debates the population system evolution states by follows:

$$x^{(0)}, \quad x^{(1)} = V\left(x^{(0)}\right), \quad x^{(2)} = V^{(2)}\left(x^{(0)}\right), \quad \cdots$$

In another sense, if the present generation distribution is given then, a QSO describes a distribution of the next generation. The a wonderful implementations of QSO to population genetics were provided in [11]. The new accomplishments and open problems in theory of the QSO have been described and explained in [26]. The difficulty of the problem relies on the given cubic matrix coefficient $(P_{ijk})_{i,j,k=1}^{m}$. Numerous researchers dedicated their study to present a special class of QSO and researched its behavior, such as, F-QSO [17], Volterra-QSO [6, 25], permutated Volterra-QSO [7, 8], ℓ-Volterra-QSO [15, 16], Quasi-Volterra-QSO [4], non-Volterra-QSO [5, 19], strictly non-Volterra-QSO [18] and non Volterra operators which produced by measurements [2, 3]. Nevertheless, all these classes jointly will not cover a system of QSOs. Therefore, there exist numerous classes of QSO need to be studyed. Recently, a new class of QSO was introduced depending on a partition of the coupled index set, $\mathbf{P}_m = \{(i,j) : i < j\} \subset I \times I$ and $\mathbf{\Delta}_m\{(i,i) : i \in I\} \subset I \times I$, where $I = \{1, \cdots, m\}$. In [22], the $\xi^{(s)}$-QSO related to $|\xi_1| = 2$ of $\mathbf{P_m}$ with a point partition of $\mathbf{\Delta_m}$ was investigated. In [12], the $\xi^{(a)}$-QSO related to $|\xi_1| = 2$ of $\mathbf{P_m}$ with a trivial partition of $\mathbf{\Delta_m}$ was studied. The $\xi^{(s)}$-QSO related to $|\xi_1| = 3$ of $\mathbf{P_m}$ with a point partition of $\mathbf{\Delta_m}$ was examined in [13]. Furthermore, the $\xi^{(s)}$-QSO and $\xi^{(a)}$-QS related to $|\xi_1| = 1$ of $\mathbf{P_m}$ with a point and a trivial partitions of $\mathbf{\Delta_m}$ respectively were also discussed in[23].

In [27], it has been classified the $\xi^{(as)}$-QSO into 18 non-conjugate classes which generated by the partitions of $\mathbf{P}_3$ with $|\xi_1| = 2$ related to the partitions of $\mathbf{\Delta_3}$ with $|\xi_2| = 2$. Therefore, we proceed with study the dynamics of some classes which are not studied. The paper is organized as follows. Section 2, provides several preliminary definitions. Sections 3 presents the study of the behavior of V_1 and V_2 obtained from class G_1 and G_2 respectively. Section 4 present the study of the dynamics of classes $G15$ and G_{17} by investigating the behavior of V_{27} and V_{29}. In the last section, the behavior of V_4, V_{10}, V_{16} and V_{18} selected from G_4, G_6,G_{10} and G_{12} respectively are studied.

2 Preliminaries

In this section, some essential ideas are reviewed.

Definition 1 *QSO is a mapping of the simplex*

$$S^{m-1} = \{(x_1, \cdots, x_m) \in \mathbb{R}^m : \quad (x_1 + x_2 + \cdots + x_m) = 1, \quad x_i \geq 0, \quad i = 1, \cdots, m\} \tag{1}$$

into itself of the form

$$x'_k = \sum_{i,j=1}^{m} P_{ij,k} x_i x_j, \quad k = \overline{1,m}, \tag{2}$$

where $V(x) = x' = (x'_1, \cdots, x'_m)$ *and* $P_{ij,k}$ *is a coefficient of heredity, which verifies the succeeding conditions*

$$\sum_{k=1}^{m} P_{ij,k} = 1 P_{ij,k} \geq 0, \quad P_{ij,k} = P_{ji,k}, \quad . \tag{3}$$

The above denition suggests that each QSO $V : S^{m-1} \to S^{m-1}$ is dened rather distinctively by a cubic matrix $\mathcal{P} = \left(P_{ijk}\right)_{i,j,k=1}^{m}$ with conditions (3).

For $V : S^{m-1} \to S^{m-1}$, the set of fixed points, limiting point and k-periodic are denoted as $Fix(V)$, $\omega_V(x^{(0)})$ and $Per_k(V)$ respectively.

We reflect the fact that a Volterra-QSO is defined by (2), (3) and the extra assumption

$$P_{ij,k} = 0 \quad \text{if} \quad k \notin \{i, j\}. \tag{4}$$

The biological treatment of condition (4) is clear: *the offspring repeats the genotype (trait) of one of its parents.*. One can see that a Volterra-QSO adopts the following form:

$$x'_k = x_k \left(1 + \sum_{i=1}^{m} a_{ki} x_i\right), \quad k \in I, \tag{5}$$

where

$$a_{ki} = 2P_{ik,k} - 1 \;\; \text{for}\, i \neq k \text{ and } \; a_{ii} = 0, \; i \in I. \tag{6}$$

Moreover,

$$a_{ki} = -a_{ik} \;\; \text{and} \;\; |a_{ki}| \leq 1.$$

In [6, 25, 21], this sort of operators was studied in an intensive manner.
The ntation of ℓ-Volterra-QSO, was inserted in [13] which popularize a a concept of Volterra-QSO. The definition is as follows.

Let $\ell \in I$ be fixed, and we take it that the heredity coefficient $\{P_{ij,k}\}$ verify the following conditions

$$P_{ij,k} = 0 \;\; \text{if} \;\; k \notin \{i, j\} \;\; \text{for any} \;\; k \in \{1, \ldots, \ell\}, \;\; i, j \in I, \tag{7}$$

$$P_{i_0 j_0,k} > 0 \;\; \text{for some } (i_0, j_0), \; i_0 \neq k, \; j_0 \neq k, \;\; k \in \{\ell + 1, \ldots, m\}. \tag{8}$$

Remark 1 *The following properties are established:*

(i)) Easy to note that ℓ-Volterra-QSO is a Volterra-QSO if and only if $\ell = m$.

(ii) A periodic trajectory does not exist for Volterra-QSO [7]. However,]. However, periodic trajectories can be found for ℓ-Volterra-QSO [13].

Complying with [22], each element $x \in S^{m-1}$ is considered as a probability distribution of the set $I = \{1, ..., m\}$. Let $x = (x_1, \cdots, x_m)$ and $y = (y_1, \cdots, y_m)$ be two vectors selected from S^{m-1}. We call that x *and* y *are equivalent* if $x_k = 0 \Leftrightarrow y_k = 0$ and this relation is

symbolized by $x \sim y$. Let $supp(x) = \{i : x_i \neq 0\}$ be a support of $x \in S^{m-1}$. We call that x *and* y *are singular* if $supp(x) \cap supp(y) = \emptyset$, and this relation is symbolized by $x \perp y$.
We denote sets of coupled indexes by

$$\mathbf{P}_m = \{(i,j) : \ i < j\} \subset I \times I, \quad \Delta_m = \{(i,i) : i \in I\} \subset I \times I.$$

For a given pair $(i,j) \in \mathbf{P}_m \cup \Delta_m$, we set a vector $\mathbb{P}_{ij} = (P_{ij,1}, \cdots, P_{ij,m})$. Clearly, because of condition (3), $\mathbb{P}_{ij} \in S^{m-1}$.

Let $\xi_1 = \{A_i\}_{i=1}^N$ and $\xi_2 = \{B_i\}_{i=1}^M$ be some fixed partitions of $\mathbf{P}_m$ and Δ_m, respectively, i.e. $A_i \bigcap A_j = \emptyset$, $B_i \bigcap B_j = \emptyset$, and $\bigcup_{i=1}^{N} A_i = \mathbf{P}_m$, $\bigcup_{i=1}^{M} B_i = \Delta_m$, where $N, M \leq m$.

Definition 2 *[22] QSO $V : S^{m-1} \to S^{m-1}$ given by (2), (3), is said a $\xi^{(as)}$-QSO w.r.t. the partitions ξ_1, ξ_2, if the following conditions are satisfied:*

(i) For each $k \in \{1, \ldots, N\}$ and any (i,j), $(u,v) \in A_k$, one has $\mathbb{P}_{ij} \sim \mathbb{P}_{uv}$;

(ii) For any $k \neq \ell$, $k, \ell \in \{1, \ldots, N\}$ and any $(i,j) \in A_k$ and $(u,v) \in A_\ell$ one has $\mathbb{P}_{ij} \perp \mathbb{P}_{uv}$;

(iii) For each $d \in \{1, \ldots, M\}$ and any (i,i), $(j,j) \in B_d$, one has $\mathbb{P}_{ii} \sim \mathbb{P}_{jj}$;

(iv) For any $s \neq h$, $s, h \in \{1, \ldots, M\}$ and any $(u,u) \in B_s$ and $(v,v) \in B_h$, one has that $\mathbb{P}_{uu} \perp \mathbb{P}_{vv}$.

3 Dynamics of classes G_1 and G_2

This section examines the dynamics of the classes G_1 and G_2. This section needs some assisting facts about properties of the function $f_\alpha : [0,1] \to [0,1]$ given by

$$f_\alpha(x) = \alpha x^2 + 2x(1-x). \tag{9}$$

where $\alpha \in [0,1]$.

Proposition 1 *Let $f_\alpha : [0,1] \to [0,1]$ be a function given by (9). Then, the following statements are true:*

(i) $Fix(f_\alpha) = \left\{0, \frac{1}{2-\alpha}\right\}$,

(ii) $\omega_{f_\alpha}(x_0) = \{\frac{1}{2-\alpha}\}$, where $x_0 \notin Fix(f_\alpha)$.

Proof. (i) The set of fixed points of f_α are obtained by solving the following equation:

$$\alpha x^2 + 2x(1-x) = x. \tag{10}$$

By finding the solution for equation (10) with respect to variable x, we obtain that $x = 0$ and $x = \frac{1}{2-\alpha}$. Therefore, $Fix(f_\alpha) = \{0, \frac{1}{2-\alpha}\}$.

(ii) One can see f_α is an increasing on $[0, \frac{1}{2-\alpha}]$ and decreasing on $[\frac{1}{2-\alpha}, 1]$. Let us divide interval $[0,1]$ into tow intervals $I_1 = [0, \frac{1}{2-\alpha}]$ and $I_2 = [\frac{1}{2-\alpha}, 1]$. One can easily check that $f_\alpha(I_2) \subseteq I_1$, which means I_1 is invariant interval under f_α. Therefore, exploring the behavior of f_α over I_1 is sufficient. Let $x_0 \in I_1$, then $f_\alpha^{(n+1)}(x_0) \geq f_\alpha^{(n)}(x_0)$ for any $n \in \mathbb{N}$, which indicates $\{f_\alpha^{(n)}(x_0)\}_{n=1}^\infty$ is a bounded increasing sequence that converges to x^* and x^* should be a fixed point, that is $\frac{1}{2-\alpha}$. Hence, $\omega_{f_\alpha}(x_0) = \{\frac{1}{2-\alpha}\}$, this process completes the proof. $\square$

Now, we are going to explore the dynamics of $\xi^{(as)}$-QSO $V_1 : S^2 \to S^2$ selected from G_1. To begin, V_1 is rewritten as follows:

$$V_1 := \begin{cases} x' = \alpha(x^{(0)})^2 + 2x^{(0)}(1 - x^{(0)}), \\ y' = (1-\alpha)(x^{(0)})^2, \\ z' = (z^{(0)})^2 + (y^{(0)})^2 + 2y^{(0)}z^{(0)}. \end{cases} \tag{11}$$

Theorem 1 *Let $V_1 : S^2 \to S^2$ be a $\xi^{(as)}$-QSO given by (11)and $x_1^{(0)} = (x^{(0)}, y^{(0)}, z^{(0)}) \notin Fix(V_1)$ be any initial point in simplex S^2. Then, the following statements are true:*

(i) One has

$$Fix(V_1) = \{e_3, (x^*, y^*, z^*)\},$$

where $x^* = \frac{1}{2-\alpha}$, $y^* = \frac{1-\alpha}{(2-\alpha)^2}$ and $z^* = \frac{1-2\alpha+\alpha 2}{(2-\alpha)^2}$.

(ii) One has

$$\omega_{V_1}(x_1^{(0)}) = \begin{cases} e_3 & , if \quad x_1^{(0)} \in L_1, \\ (x^*, y^*, z^*), if & x_1^{(0)} \notin L_1. \end{cases}$$

Proof. Let $V_1 : S^2 \to S^2$ be a $\xi^{(as)}$-QSO given by (11), $x_1^{(0)} \notin Fix(V_1)$ be any initial point in simplex S^2 and $\{x_1^{(n)}\}_{n=1}^\infty$ be a trajectory of V_1 starting from point $x_1^{(0)}$.

(i) The set of fixed points of V_1 are obtained by finding the solution for the following system of equations:

$$\begin{cases} x = \alpha x^2 + 2x(1-x), \\ y = (1-\alpha)x^2, \\ z = y^2 + z^2 + 2yz. \end{cases} \tag{12}$$

By depending on the first equation in system (12), we find $x = 0$ or $x = \frac{1}{2-\alpha}$. It follows by using the second and third equations in system (12) respectively that if $x = 0$, then $y = 0$ and $z = 1$; if $x = \frac{1}{2-\alpha} = x^*$, then $y = \frac{1-\alpha}{(2-\alpha)^2} = y^*$ and $z = \frac{1-2\alpha+\alpha 2}{(2-\alpha)^2} = z^*$. Therefore, $Fix(V_1) = \{e_3, (x^*, y^*, z^*)\}$.

(ii) Let $x_1^{(0)} \notin Fix(V_1)$. We are going to explore the dynamics of V_1 when $x_1^{(0)} \in L_1$ and $x_1^{(0)} \notin L_1$, where $L_1 = \{x_1^{(0)} \in S^2 : x^{(0)} = 0\}$. Thus, two cases should be discussed separately:

(a) Let $x_1^{(0)} \in L_1$. It is clear that $x^{(n)} = f_\alpha^{(n)}(x^{(0)})$ and L_1 is invariant line under V_1. Therefore, sequence $\{x^{(n)}\}_{n=1}^\infty$ converges to zero. It follows by using the second and third coordinate of V_1, we conclude easily that sequence $\{y^{(n)}\}_{n=1}^\infty$ converges to zero and sequence $\{z^{(n)}\}_{n=1}^\infty$ converges to one. Hence, the limiting point in this case is $\omega_{V_1}(x_1^{(0)}) = \{e_3\}$.

(b) Let $x_1^{(0)} \notin L_1$. Due to Proposition (1), we have sequence $\{x^{(n)}\}_{n=1}^\infty$ converging to x^*. By depending on the second and third coordinates of V_1, we conclude that sequence $\{y^{(n)}\}_{n=1}^\infty$

converges to y^* and sequence $\{z^{(n)}\}_{n=1}^{\infty}$ converges to z^*. Hence, the limiting point in this case is $\omega_{V_1}(x_1^{(0)}) = \{(x^*, y^*, z^*)\}$, this process completes the proof.

$\square$

Now, we are going to explore the dynamics of $\xi^{(as)}$-QSO $V_2 : S^2 \to S^2$ selected from G_2. To begin, V_2 is written as follows:

$$V_2 := \begin{cases} x' = \alpha(x^{(0)})^2 + 2x^{(0)}(1 - x^{(0)}), \\ y' = (1 - \alpha)(x^{(0)})^2 + 2y^{(0)}z^{(0)}, \\ z' = (z^{(0)})^2 + (y^{(0)})^2. \end{cases} \tag{13}$$

Theorem 2 *Let $V_2 : S^2 \to S^2$ be a $\xi^{(as)}$-QSO given by* (13)*and $x_1^{(0)} = (x^{(0)}, y^{(0)}, z^{(0)}) \notin Fix(V_2)$ be any initial point in simplex S^2. Then, the following statements are true:*

(i) One has

$$Fix(V_2) = \left\{e_3, (0, \frac{1}{2}, \frac{1}{2})(\hat{x}, \hat{y}, \hat{z})\right\},$$

where $\hat{x} = \frac{1}{2-\alpha}$, $\hat{y} = \frac{\alpha-\sqrt{\alpha^2-8\alpha+8}}{4(\alpha-2)}$ and $\hat{z} = \frac{3\alpha-4+\sqrt{\alpha^2-8\alpha+8}}{4(\alpha-2)}$.

(ii) One has

$$\omega_{V_2}(x_1^{(0)}) = \begin{cases} (0, \frac{1}{2}, \frac{1}{2}) \quad , if \quad x_1^{(0)} \in L_1, \\ \\ (\hat{x}, \hat{y}, \hat{z}) \quad , if \quad x_1^{(0)} \notin L_1. \end{cases}$$

Proof. Let $V_2 : S^2 \to S^2$ be a $\xi^{(as)}$-QSO given by (11), $x_1^{(0)} \notin Fix(V_2)$ be any initial point in simplex S^2 and $\{x_1^{(n)}\}_{n=1}^{\infty}$ be a trajectory of V_2 starting from point $x_1^{(0)}$.

(i) The set of fixed points of V_2 are obtained by finding the solution for the following system of equations:

$$\begin{cases} x{=}\alpha x^2 + 2x(1 - x), \\ y{=}(1 - \alpha)x^2 + 2yz, \\ z{=}y^2 + z^2. \end{cases} \tag{14}$$

By depending on the first equation in system(14), we find $x = 0$ or $x = \frac{1}{2-\alpha}$. By using the second and third equation in system(14) respectively, that if $x = 0$, then $y = 0$ and $z = 1$ or $y = z = \frac{1}{2}$; if $x = \frac{1}{2-\alpha}$, then $y = \hat{y}$ and $z = \hat{z}$. Therefore, $Fix(V_2) = \{e_3, (0, \frac{1}{2}, \frac{1}{2})(\hat{x}, \hat{y}, \hat{z})\}$.

(ii) Let $x_1^{(0)} \notin Fix(V_2)$. We are going to explore the dynamic of V_2 when $x_1^{(0)} \in L_1$ and $x_1^{(0)} \notin L_1$. Thus, two cases should be discussed separately:

(a) Let $x_1^{(0)} \in L_1$. Since the first coordinate of V_2 is equal to the first coordinate of V_1, we obtain L_1 is invariant line under V_2. Therefore, $\{x^{(n)}\}_{n=1}^{\infty}$ converges to zero. Now, we intend to explore the dynamics of the third coordinate. To achieve this objective, consider the third coordinate of V_2, namely, $z' = k(z^{(0)}) = 2(z^{(0)})^2 - 2z^{(0)} + 1$ and divide interval $[0, 1]$ into two intervals as follows: $I_1 = [0, \frac{1}{2}]$ and $I_2 = [\frac{1}{2}, 1]$. One can easily see that $k(z^{(0)})$ is an increasing when $z^{(0)} \in I_1$ and decreasing when $z^{(0)} \in I_2$ and $k(I_1) \subseteq I_2$. Therefore, I_2 is invariant interval under k, which indicates exploring the dynamics of k over I_2 is sufficient.

Let $z^{(0)} \in I_2$. Then, $k^{(n+1)}(z^{(0)}) \leq k^{(n)}(z^{(0)})$, which means $k^{(n)}(z^{(0)})$ is a decreasing bounded sequence that converges to a fixed point of k that is $\frac{1}{2}$. Therefore, sequence $z^{(n)}$ converges to $\frac{1}{2}$. By using $x^{(n)} + y^{(n)} + z^{(n)} = 1$, we conclude sequence $\{y^{(n)}\}_{n=1}^{\infty}$ converges to $\frac{1}{2}$. Hence, the limiting point in this case is $\omega_{V_2}(x_1^{(0)}) = \{(0, \frac{1}{2}, \frac{1}{2})\}$.

(b) Let $x_1^{(0)} \notin L_1$. Due to proposition (1), we have $\{x^{(n)}\}_{n=1}^{\infty}$ converging to $\hat{x}$. It can be easily to check $\hat{x} \geq \frac{1}{2}$. Thus, it is sufficient to explore the dynamics of the second and third coordinates of V_2 over interval $[0, \frac{1}{2})$. To achieve this objective, we consider the following function:

$$N_\alpha(y^{(0)}) = \frac{1-\alpha}{(2-\alpha)^2} + \frac{2(1-\alpha)}{2-\alpha} y^{(0)} - 2(y^{(0)})^2, \tag{15}$$

where $\alpha \in [0, 1]$ and $y^{(0)} \in [0, \frac{1}{2})$.

The interval $[0, \frac{1}{2})$ is divided into four intervals as follows:

$$I_1 = [0, \tfrac{1-\alpha}{2(2-\alpha)}],\ I_2 = [\tfrac{1-\alpha}{2(2-\alpha)}, \hat{y}],\ I_3 = [\hat{y}, \tfrac{\alpha^2-4\alpha+3}{2(2-\alpha)^2}] \text{ and } I_4 = [\tfrac{\alpha^2-4\alpha+3}{2(2-\alpha)^2}, \tfrac{\alpha-1}{\alpha-2}).$$

One can easily see that $N_\alpha(I_1) \subseteq \bigcup_{m=2}^{4} I_m$. Thus, exploring the dynamics of $N_\alpha(y^{(0)})$ over I_2, I_3 and I_4 is sufficient. To start, let $y^{(0)} \in I_2 \cup I_3$. Evidently, if $y^{(0)} \in I_2$, then $N_\alpha(y^{(0)}) \in I_3$ and $N_\alpha^{(2)}(y^{(0)}) \in I_2$; if $y^{(0)} \in I_3$, then $N_\alpha(y^{(0)}) \in I_2$ and $N_\alpha^{(2)}(y^{(0)}) \in I_3$. One can check $N_\alpha^{(2)}(y^{(0)}) \geq y^{(0)}$ for all $y^{(0)} \in I_2$ and $N_\alpha^{(2)}(y^{(0)}) \leq y^{(0)}$ for all $y^{(0)} \in I_3$. Therefore, two cases should be discussed separately:

(1) Let $y^{(0)} \in I_2$. We derive $N_\alpha^{(2n+2)}(y^{(0)}) \geq N_\alpha^{(2n)}(y^{(0)})$ for any $n \in \mathbb{N}$, which means $\{N_\alpha^{(2n)}\}_{n=1}^{\infty}$ is a bounded increasing sequence. Accordingly, $\{N_\alpha^{(2n)}\}_{n=1}^{\infty}$ converges to a fixed point of $N_\alpha^{(2)}$. One finds $\hat{y}$ is fixed point for $N_\alpha^{(2)}$ and it is the only possible fixed point of the convergence trajectory. Therefore, $\{N_\alpha^{(2n)}\}_{n=1}^{\infty}$ converges to $\hat{y}$.

(2) Similarly, let $y^{(0)} \in I_3$. We derive $N_\alpha^{(2n+2)}(y^{(0)}) \leq N_\alpha^{(2n)}(y^{(0)})$ for any $n \in \mathbb{N}$, which indicates $\{N_\alpha^{(2n)}\}_{n=1}^{\infty}$ is a bounded decreasing sequence. Moreover, $\{N_\alpha^{(2n)}\}_{n=1}^{\infty}$ converges to a fixed point of $N_\alpha^{(2)}$. One finds that $\hat{y}$ is fixed point for $N_\alpha^{(2)}$ and it is the only possible fixed point of the convergence trajectory. Therefore, $\{N_\alpha^{(2n)}\}_{n=1}^{\infty}$ converges to $\hat{y}$.

To explore the dynamics of V_2 when $y^{(0)} \in I_4$, the following claim is required.

Claim 1 *Let $y^{(0)} \in I_4$, then there is $n_k \in \mathbb{N}$, such that $N_\alpha^{(n_k)}(y^{(0)}) \in I_2 \cup I_3$.*

Proof. By contradiction, suppose that I_4 is invariant interval i.e., $y^{(n)} \in I_4$ for any $n \in \mathbb{N}$. Clearly, $\{y^{(n)}\}_{n=1}^{\infty}$ is a bounded decreasing sequence that converges to a fixed point of $N_\alpha(y)$. However, $Fix(N_\alpha) \cap I_4 = \emptyset$, which is a contradiction. Hence, there exist $n_k \in \mathbb{N}$, such that $N_\alpha^{(n_k)}(y^{(0)}) \in I_2 \cup I_3$. □

accordance with what have been proven in the above, we conclude $\{y^{(n)}\}_{n=1}^{\infty}$ converges to $\hat{y}$. Since $x^{(n)} + y^{(n)} + z^{(n)} = 1$, we obtain $\{z^{(n)}\}_{n=1}^{\infty}$ converges to $\hat{z}$. Hence, the limiting point in this case is $\omega_{V_2}(x_1^{(0)}) = \{(\hat{x}, \hat{y}, \hat{z})\}$, this process completes the proof. □

4 Dynamics of classes G_{15} and G_{17}

This section examines the dynamics of classes G_{15} and G_{17} by studying the dynamics of V_{27} and V_{29} selected from G_{15} and G_{17} respectively.

Now, we are going to explore the dynamics of $\xi^{(as)}$-QSO $V_{27}: S^2 \to S^2$. To begin, V_{27} is written as follows:

$$V_{27} := \begin{cases} x' = (y^{(0)})^2 + (z^{(0)})^2 + 2y^{(0)}z^{(0)}, \\ y' = \alpha(x^{(0)})^2 + 2x^{(0)}(1 - x^{(0)}), \\ z' = (1-\alpha)(x^{(0)})^2. \end{cases} \tag{16}$$

Theorem 3 *Let $V_{27}: S^2 \to S^2$ be a $\xi^{(as)}$-QSO given by (16)and $x_1^{(0)} = (x^{(0)}, y^{(0)}, z^{(0)}) \notin Fix(V_1) \cup Per_2(V_{27})$ be any initial point in simplex S^2. Then, the following statements are true:*

(i) One has $Fix(V_{27}) = \emptyset$. Moreover, $Per_2(V_{27}) = \{e_1, (\frac{3}{2} - \frac{1}{2}\sqrt{5}, \dfrac{7\alpha}{2} - \dfrac{3\alpha}{2}\sqrt{5} + 3 - \sqrt{5}, \dfrac{-1}{16}(1-\alpha)(-1+\sqrt{5})^4), (0, \alpha, 1-\alpha)\}$.

(ii) One has $\omega_{V_{27}}(x_1^{(0)}) = \{e_1, (0, \alpha, 1-\alpha)\}$.

Proof. Let $V_{27}: S^2 \to S^2$ be a $\xi^{(as)}$-QSO given by (16), $x_1^{(0)} \notin Fix(V_{27}) \cup Per_2(V_{27})$ be any initial point in simplex S^2 and $\{x_1^{(n)}\}_{n=1}^{\infty}$ be a trajectory of V_{27} starting from point $x_1^{(0)}$.

(i) The set of fixed points of V_{27} are obtained by finding the solution for the following system of equations:

$$\begin{cases} x = y^2 + z^2 + 2yz, \\ y = \alpha x^2 + 2x(1-x), \\ z = (1-\alpha)x^2. \end{cases} \tag{17}$$

By depending on the first equation in system (23), we find $x = \frac{3-\sqrt{5}}{2}$. Then, $y = \dfrac{-3}{2} - \dfrac{\sqrt{5}}{2} - \alpha(\dfrac{-3}{2} + \dfrac{\sqrt{5}}{2})^2 + 2\alpha(\dfrac{3}{2} - \dfrac{\sqrt{5}}{2})$ and $z = (1-\alpha)(\dfrac{3}{2} - \frac{\sqrt{5}}{2})^2$. One can check $(\frac{3-\sqrt{5}}{2}, \dfrac{-1}{2} - \dfrac{\sqrt{5}}{2} - \alpha(\dfrac{-1}{2} + \dfrac{\sqrt{5}}{2})^2, \alpha(-\dfrac{1}{2} + \frac{\sqrt{5}}{2})^2) \notin [0,1]$. Therefore, $Fix(V_{27}) = \emptyset$. To find 2−periodic points, the following system of equations should be solved:

$$\begin{cases} x = (1-(1-x)^2)^2, \\ y = \alpha(1-x)^4 + 2(1-x)^2(1-(1-x)^2), \\ z = (1-\alpha)(1-x)^4. \end{cases} \tag{18}$$

By depending on the first equation (18), we find $x = 0$, $x = \frac{3}{2} - \frac{1}{2}\sqrt{5}$ or $x = 1$. It follows by using the second and third equations in system (18) that if $x = 0$, then $y = \alpha$ and

$z = 1-\alpha$; if $x = 1$, then $y = z = 0$; if $x = \frac{3}{2} - \frac{1}{2}\sqrt{5}$, then $y = \dfrac{7\alpha}{2} - \dfrac{3\alpha}{2}\sqrt{5} + 3 - \sqrt{5}$ and $z = \dfrac{-1}{16}(1-\alpha)(-1+\sqrt{5})^4$. Hence, $Per_2(V_{27}) = \{e_1, (\frac{3}{2} - \frac{1}{2}\sqrt{5}, \dfrac{7\alpha}{2} - \dfrac{3\alpha}{2}\sqrt{5} + 3 - \sqrt{5}, \dfrac{-1}{16}(1-\alpha)(-1+\sqrt{5})^4), (0, \alpha, 1-\alpha)\}$.

(ii) Let $x_1^{(0)} \notin Fix(V_{27}) \cup Per_2(V_{27})$. The first coordinate of V_{27} can be redrafted as $\phi^{(1)}(x^{(0)}) = (1-x^{(0)})^2$. It is obvious that $\phi^{(1)}$ and $\phi^{(2)}$ are a decreasing and increasing on $[0,1]$ respectively, where $\phi^{(2)} = (1-(1-x^{(0)})^2)^2$. One can observe that $Fix(\phi^{(1)}) \cap [0,1] = \{\frac{3-\sqrt{5}}{2}\}$ and $Fix(\phi^{(2)}) \cap [0,1] = \{0, \frac{3-\sqrt{5}}{2}, 1\}$, which indicates intervals $[0, \frac{3-\sqrt{5}}{2}]$ and $[\frac{3-\sqrt{5}}{2}, 1]$ are invariant under the function $\phi^{(2)}$. Evidently, $\phi^{(2)}(x^{(0)}) \le x^{(0)}$ for all $x^{(0)} \in [0, \frac{3-\sqrt{5}}{2}]$ and $\phi^{(2)}(x^{(0)}) \ge x^{(0)}$ for all $x^{(0)} \in [\frac{3-\sqrt{5}}{2}, 1]$. Accordingly, if $x^{(0)} \in [0, \frac{3-\sqrt{5}}{2}]$, then the limiting point is $\omega_{\phi^{(2)}}(x^{(0)}) = \{0\}$; if $x^{(0)} \in [\frac{3-\sqrt{5}}{2}, 1]$, then the limiting point is $\omega_{\phi^{(2)}}(x^{(0)}) = \{1\}$. In another way,

$$V_{27}^{(n)}(x_1^{(0)}) = \begin{cases} \left(\phi^{(2n)}(x^{(0)}), H(x^{(0)}), H^*(x^{(0)})\right) & , if \;\; n \;\; is \;\; even \\ \\ \left(\phi^{(2n)}(\phi(x^{(0)})), H(\phi(x^{(0)})\right), H^*(\phi(x^{(0)})) & , if \;\; n \;\; is \;\; odd \end{cases} \tag{19}$$

where $H(x^{(0)}) = \left(1 - \sqrt{\phi^{(2n)}(x^{(0)})}\right)\left((\alpha-1)(1-\sqrt{\phi^{(2n)}(x^{(0)})}) + 1\right)$ and $H^*(x^{(0)}) = (1-\alpha)\left(1 - \sqrt{\phi^{(2n)}(x^{(0)})}\right)^2$.

In the previous formula, we have proven, if $x^{(0)} \in [0, \frac{3-\sqrt{5}}{2}]$, then $\phi(x^{(0)}) \in [\frac{3-\sqrt{5}}{2}, 1]$ and $\phi^{(2)}(x^{(0)}) \in [0, \frac{3-\sqrt{5}}{2}]$, which indicates that $\phi^{(2n)}(x^{(0)})$ converges to zero and $\phi^{(2n+1)}(x^{(0)})$ converges to one. Similarly, if $x^{(0)} \in [\frac{3-\sqrt{5}}{2}, 1]$, then $\phi^{(2n)}(x^{(0)})$ converges to one and $\phi^{(2n+1)}(x^{(0)})$ converges to zero. Hence, the set of limiting points in two cases is $\omega_{V_{27}}(x_1^{(0)}) = \{e_1, (0, \alpha, 1-\alpha)\}$. □

Now, we are going to explore the dynamics of $\xi^{(as)}$-QSO $V_{29} : S^2 \to S^2$. Let us rewrite V_{29} as follows:

$$V_{29} := \begin{cases} x' = (y^{(0)})^2 + (z^{(0)})^2 + 2y^{(0)}z^{(0)}, \\ y' = \alpha(x^{(0)})^2, \\ z' = (1-\alpha)(x^{(0)})^2 + 2x^{(0)}(1-x^{(0)}). \end{cases} \tag{20}$$

Corollary 1 *Let $V_{29} : S^2 \to S^2$ given by (20) be a $\xi^{(as)}$-QSO and $x_1^{(0)} = x^{(0)}, y^{(0)}, z^{(0)}) \notin Fix(V_{29}) \cup Per_2(V_{29})$ be any initial point in simplex S^2. Then, the following statements are true:*

(i) One has $Fix(V_{29}) = \phi$. Moreover, $Per_2(V_{27}) = \{e_1, (\frac{3}{2} - \frac{1}{2}\sqrt{5}, \dfrac{-1}{16}(1-\alpha)(-1+\sqrt{5})^4, \dfrac{7\alpha}{2} - \dfrac{3\alpha}{2}\sqrt{5} + 3 - \sqrt{5}), (0, 1-\alpha, \alpha)\}$.

(ii) One has $\omega_{V_{29}}(x_1^{(0)}) = \{e_1, (0, 1-\alpha, \alpha)\}$.

5 Dynamics of classes G_4, G_6, G_{10} and G_{12}

This section examines the dynamics for the classes G_4, G_6, G_{10} and G_{12} by studying the dynamics of V_4, V_{10}, V_{16} and V_{18} selected from G_4, G_6, G_{10} and G_{12} respectively. Let us start and rewrite V_4 as follows:

$$V_4 := \begin{cases} x' = \alpha(x^{(0)})^2, \\ y' = (1-\alpha)(x^{(0)})^2 + 2x^{(0)}(1-x^{(0)}), \\ z' = (y^{(0)})^2 + (z^{(0)})^2 + 2y^{(0)}z^{(0)}. \end{cases} \tag{21}$$

Theorem 4 *Let $V_4 : S^2 \to S^2$ be a $\xi^{(as)}$-QSO given by (21)and $x_1^{(0)} = (x^{(0)}, y^{(0)}, z^{(0)}) \notin Fix(V_4)$ be any initial point in simplex S^2. Then, the following statements are true:*

(i) One has

$$Fix(V_4) = \begin{cases} e_1, e_3 & , if\ \alpha = 1, \\ \\ e_3 & , if\ \alpha \neq 1. \end{cases} \tag{22}$$

(ii) One has

$$\omega_{V_4}(x_1^{(0)}) = \{e_3\}.$$

Proof. *Let $V_4 : S^2 \to S^2$ be a $\xi^{(as)}$-QSO given by (21), $x_1^{(0)} \notin Fix(V_4)$ and $\{x_1^{(n)}\}_{n=1}^{\infty}$ be a trajectory of V_4 starting from point $x_1^{(0)}$.*

(i) The set of fixed points of V_4 are obtained by finding the solution for the following system of equations:

$$\begin{cases} x = \alpha x^2, \\ y = (1-\alpha)x^2 + 2x(1-x), \\ z = y^2 + z^2 + 2yz. \end{cases} \tag{23}$$

Two cases will be taken separately $\alpha = 1$ and $\alpha \neq 1$.
Let $\alpha = 1$. By depending on the first equation in system (23), we find $x = 0$ or $x = 1$. If $x = 1$, then $y = z = 0$; if $x = 0$, then $y = 0$ and $z = 1$. Therefore, the fixed points are e_1 and e_3.
Let $\alpha \neq 1$. By depending on the first equation in system (23), we find $x = 0$ or $x = \frac{1}{\alpha}$. It can be easily to see that $\frac{1}{\alpha} > 1$, hence the possible value for x is zero. Thus, if $x = 0$, then $y = 0$ and $z = 1$. Therefore, the fixed point is e_3.

(ii) On the basis of the first coordinate in (21), we have $x' = \alpha(x^{(0)})^2$. Define $x' = \theta_\alpha(x^{(0)}) = \alpha(x^{(0)})^2$, where $\alpha, x^{(0)} \in [0,1]$. One can find that $Fix(\theta_\alpha) = \{0,1\}$ and check θ_α is a

decreasing on $[0,1]$. *Therefore,* $\theta_\alpha^{(n+1)}(x^{(0)}) \le \theta_\alpha^{(n)}(x^{(0)})$ *for any* $n \in \mathbb{N}$*, which indicates* $\{\theta_\alpha^{(n)}(x^{(0)})\}_{n=1}^{\infty}$ *is a bounded decreasing sequence. Accordingly,* θ_α *converges to a fixed point* x^**, that is* $x^* = 0$. *Therefore, sequence* $\{x^{(n)}\}_{n=1}^{\infty}$ *converges to zero. Owing to the first coordinate in this operator,* $\{y^{(n)}\}_{n=1}^{\infty}$ *converges to zero and* $\{z^{(n)}\}_{n=1}^{\infty}$ *converges to one. Hence, the limiting point is* $\omega_{V_4}(x_1^{(0)}) = \{e_3\}$ *, this process completes the proof.* □

Now, we are going to explore the dynamics of $\xi^{(as)}$*-QSO* $V_{10} : S^2 \to S^2$. *Let us rewrite* V_{10} *as follows:*

$$V_{10} := \begin{cases} x' = \alpha(x^{(0)})^2, \\ y' = (y^{(0)})^2 + (z^{(0)})^2 + 2x^{(0)}(1 - x^{(0)}), \\ z' = (1-\alpha)(x^{(0)})^2 + 2y^{(0)}z^{(0)}. \end{cases} \tag{24}$$

Theorem 5 *Let* $V_{10} : S^2 \to S^2$ *be a* $\xi^{(as)}$*-QSO given by* (24)*and* $x_1^{(0)} = (x^{(0)}, y^{(0)}, z^{(0)}) \notin Fix(V_{10})$ *be any initial point in simplex* S^2. *Then, the following statements are true:*

(i) One has

$$Fix(V_{10}) = \begin{cases} e_1, e_2, (0, \frac{1}{2}, \frac{1}{2}) & , if\ \alpha = 1, \\ \\ e_2, (0, \frac{1}{2}, \frac{1}{2}) & , if\ \alpha \neq 1. \end{cases} \tag{25}$$

(ii) One has

$$\omega_{V_{10}} = \{(0, \frac{1}{2}, \frac{1}{2})\}. \tag{26}$$

Proof. *Let* $V_{10} : S^2 \to S^2$ *be a* $\xi^{(as)}$*-QSO given by* (24), $x_1^{(0)} \notin Fix(V_{10})$ *be any initial point in simplex* S^2 *and* $\{x_1^{(n)}\}_{n=1}^{\infty}$ *be a trajectory of* V_{10} *starting from point* $x_1^{(0)}$.

(i) The set of fixed points of V_1 *are obtained by finding the solution for the following system of equations:*

$$V_{10} := \begin{cases} x = \alpha x^2, \\ y = y^2 + z^2 + 2x(1 - x), \\ z = (1-\alpha)x^2 + 2yz. \end{cases} \tag{27}$$

Two cases will be taken separately $\alpha = 1$ *and* $\alpha \neq 1$.
Let $\alpha = 1$. *Depending on the first equation in system* (27)*, we find* $x = 0$ *or* $x = 1$. *If* $x = 1$*, then* $y = z = 0$*; if* $x = 0$*, then* $y = 1$ *and* $z = 0$ *or* $z = y = \frac{1}{2}$ *. Therefore, the fixed points are* e_1*,* e_3 *and* $(0, \frac{1}{2}, \frac{1}{2})$.
Let $\alpha \neq 1$. *Depending on the first equation in system* (27)*, we find* $x = 0$ *or* $x = \dfrac{1}{\alpha}$. *It can be easily to see that* $\dfrac{1}{\alpha} > 1$*, hence the possible value for* x *is zero. Thus, If* $x = 0$*, then* $y = 1$ *and* $z = 0$ *or* $z = y = \frac{1}{2}$. *Therefore, the fixed points are* e_2 *and* $(0, \frac{1}{2}, \frac{1}{2})$.

(ii) Since the first coordinate of V_{10} is equal to the first coordinate of V_4, we derive $\{x^{(n)}\}_{n=1}^{\infty}$ converges to zero. Now, we want to explore the dynamics of the third coordinate. To achieve this objective, consider the third coordinate of V_{10}, namely, $z' = h(z^{(0)}) = 2z^{(0)}(1 - z^{(0)})$ and divide the interval $[0, 1]$ into two intervals as follows: $I_1 = [0, \frac{1}{2}]$ and $I_2 = [\frac{1}{2}, 1]$. One can easily check $h(z^{(0)})$ is an increasing when $z^{(0)} \in I_1$ and decreasing when $z^{(0)} \in I_2$ and $h(I_2) \subseteq I_1$. Therefore, I_1 is invariant interval under h. Thus, it is adequate to explore the dynamics of $h(z^{(0)})$ over $[0, 1]$. Let $z^{(0)} \in I_1$. Then, $h^{(n+1)}(z^{(0)}) \geq h^{(n)}(z^{(0)})$, which indicates $h^{(n)}(z^{(0)})$ is an increasing bounded sequences. Moreover, $\{h^{(n)}(z^{(0)})\}_{n=1}^{\infty}$ converges to a fixed point of h. One finds fixed point of $h = \{0, \frac{1}{2}\}$. Therefore, sequence $\{z^{(n)}\}_{n=1}^{\infty}$ converges to $\frac{1}{2}$. By using $x^{(n)} + y^{(n)} + z^{(n)} = 1$, we conclude $\{y^{(n)}\}_{n=1}^{\infty}$ converges to $\frac{1}{2}$. Hence, the limiting point is $\omega_{V_{10}}(x_1^{(0)}) = \{(0, \frac{1}{2}, \frac{1}{2})\}$, this process completes the proof.

□

Now, we are going to explore the dynamics of $\xi^{(as)}$-QSO $V_{16,18} : S^2 \to S^2$. let us rewrite V_{16} and V_{18} as follows:

$$V_{16} := \begin{cases} x' = (1-\alpha)(x^{(0)})^2, \\ y' = \alpha(x^{(0)})^2 + 2x^{(0)}(1-x^{(0)}), \\ z' = (y^{(0)})^2 + (z^{(0)})^2 + 2y^{(0)}z^{(0)}. \end{cases} \tag{28}$$

$$V_{18} := \begin{cases} x' = (1-\alpha)(x^{(0)})^2, \\ y' = \alpha(x^{(0)})^2 + 2y^{(0)}z^{(0)}, \\ z' = (y^{(0)})^2 + (z^{(0)})^2 + 2x^{(0)}(1-x^{(0)}). \end{cases} \tag{29}$$

Corollary 2 *Let $V_{16,18} : S^2 \to S^2$ given by (28) and (29) are a $\xi^{(as)}$-QSO and $x_1^{(0)} = (x^{(0)}, y^{(0)}, z^{(0)}) \notin Fix(V_{16})$ and $x_1^{(0)} = (x^{(0)}, y^{(0)}, z^{(0)}) \notin Fix(V_{18})$ be any initial point in simplex S^2. Then, the following statements are true:*

i One has

$$Fix(V_{16}) = \begin{cases} e_1, e_3 & , if \ \alpha = 0, \\ e_3 & , if \ \alpha \neq 0. \end{cases} \tag{30}$$

ii One has

$$\omega_{V_{16}}(x_1^{(0)}) = \{e_3\}.$$

iii One has

$$Fix(V_{18}) = \begin{cases} e_1, e_3, (0, \dfrac{1}{2}, \dfrac{1}{2}) & , if \ \alpha = 0, \\ \\ e_3, (0, \dfrac{1}{2}, \dfrac{1}{2}) & , if \ \alpha \neq 0. \end{cases} \tag{31}$$

iv One has

$$\omega_{V_{18}}(x_1^{(0)}) = \left\{(0, \frac{1}{2}, \frac{1}{2})\right\}.$$

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