International Journal of Engineering & Technology, 7 (4.10) (2018) 883-887

International Journal of Engineering & Technology

Website: www.sciencepubco.com/index.php/IJET

Research paper

A Study on p-Cyclic Orbital Geraghty type Contractions

M.L.Suresh[1], T. Gunasekar[2], S. Karpagam[3*] and B. Zlatanov[4]

[1,2]*Vel Tech Rangarajan Dr.Raguntha la R & D Institute of Science and Technology, Avadi, Chennai - 62, Tamil Nadu, India.*
[3 3]*Department of Science and Humanities, "Saveetha School of engineering" Saveetha Institute of medical and Technical Sciences,*
[4]*Faculty of Mathematics and Information, University of Plovidu, Paisii , Hilendarski, Plovdiv, Bulgaria, bzlatanov@gmail.com*
**Corresponding author E-mail : karpagam.saravanan@gmail.com*

Abstract

Consider a metric space (M,ρ) and the non empty sub sets, $B_1, B_2, \ldots, B_p, (p \in \mathrm{N})$ of X. A map $S:\cup_{i=1}^{p} B_i \to \cup_{i=1}^{p} B_i$, called p-cyclic orbital Geraghty type of contraction is introduced. Convergence of a unique fixed point and a best proximity point for this map is obtained in a uniformly convex Banach space setting. Also, this best proximity point is the unique periodic point of such a map.

Keywords: *p-cyclic maps, Orbital contraction, Geraghty type contraction.*

1. Introduction

The theory of fixed points originated as a consequence of solving differential equations. Fixed point theorems assure the existence of solutions under certain conditions on the map and some condition on an topological space on which the map is defined. We now state the classical Banach contraction theorem which is used to solve problems of practical importance.

Theorem 1.1. Let (M,ρ) be a metric space which is complete and S be a self map on X such that

$$\rho(Sp,\mathrm{Sq}) \le k\rho(\mathrm{p},\mathrm{q}),\ 0<k<1,\ p,\mathrm{q}\in M$$

Then there exists a point $x_0 \in M$ such that for a point $x \in M$ the iterative sequence $\{S^n(x)\}$ converges to x_0, which is the unique fixed point of S .
In such a case, f is called a Banach contraction map where k is the contraction constant. The sequence $\{f^n(x)\}$ is called the iterative sequence of $x \in M$. This method of obtaining a fixed point as the limit of an iterative sequence is called the Picard-type integrative method. Note that, here *f* is uniformly continuous in M. A great number of attempts are made to generalize the BanachContraction Principle.
One of such an attempt is made by Geraghty in [1], by introducing a new class of test functions 'w' which is defined as below.

Definition 1.1. ([1]) w is the class of functions $\mathrm{u}:[0,\infty) \to [0,1)$ such that $u(t_n) \to 1 \Rightarrow t_n \to 0$.

Note that 'u' is not continuous in any sense. Using this, Geraghty proved a fixed point theorem which is stated as below:

Theorem: 1.2 ([1]) Let $\mathrm{S}:\mathrm{M} \to M$ be a map, on (M,ρ) which is a metric space which is complete and satisfy, for some $u \in S$

$$\rho(Sp,\mathrm{Sq}) \le u(\rho(\mathrm{p},\mathrm{q}))\rho(\mathrm{p},\mathrm{q}),\ p,\mathrm{q}\in M \qquad (1.1)$$

Then there exists a unique fixed point $x_0 \in M$ further for any $x \in M$ the sequence of iterates $\{S^n(x)\}$ converges to x_0.

As a generalization of Geraghty's theorem, Kirk et al. ([2]) gave a theorem which is stated as follows.

Theorem 1.3. Let $B_1, B_2, \ldots, B_p\ (p \ge 2)$ be non empty and which are closed subsets of a metric space (M,ρ) which is complete. Let $u \in S$ and suppose $\mathrm{S}:\cup_{i=1}^{p} B_i \to \cup_{i=1}^{p} B_i$, be a map satisfying,

1) $S(B_i) \subseteq B_{i+1}$, where $B_{i+1} = B_i$.

2) $$\rho(Sp,\mathrm{Sq}) \le u(\rho(\mathrm{p},\mathrm{q}))\rho(\mathrm{p},\mathrm{q}),\quad p \in B_i, \mathrm{q}\in B_{i+1},\ 1 \le i \le p$$.

Then S has a fixed point which is unique in $\cap_{i=1}^{p} B_i$.

Note that in $\cap_{i=1}^{p} B_i$, S is a self map and is map defined by Geraghty. In [3], Eldred and Veeramani introduced a concept of cyclic maps which is given as follows:

Definition 1.2. Let (M,ρ) be a metric space and let P and Q be non empty subsets of (M,ρ) A self map S on $P \cup Q$ is said to be a cyclic map if $S(P) \subseteq Q\ \&\ S(Q) \subseteq P$.

If $\rho(s,Ss)=$ dis(P,Q). where $dis(\mathrm{P},\mathrm{Q})=\inf\{(\mathrm{s},\mathrm{t}):s\in P,t\in Q\}$ then $s\in P\cup Q$ is called a best proximity point.

Note: Best proximity point will be abbreviated as b.p.pt, and fixed point as f.p. in this paper.

We note that the obtained b.p.pt. is a fixed point of S. if $dis(\mathrm{P},\mathrm{Q})=0$. F.p.theorems can be generalized as b.p.pt. theorems.

Theorem 1.4. ([3]) Consider $(\mathrm{M},\|.\|)$, a Banach space which is uniformly convex and P and Q be non empty convex subsets of M, which are closed. Let S, a self map on $P\cup Q$ be a cyclic map such that

$$\|Sp-Sq\|\le u(\|p-q\|)+\left[1-u(\|p-q\|)\right]dist(\mathrm{P},\mathrm{Q})$$

$p\in P,q\in Q,u\in w$. Then S has a unique b.p.pt. $\xi\in P$ further, if $x_0\in P$ and a sequence is defined as $x_{n+1}=Sx_n$ then $\{x_{2n}\}$ converges to this unique b.p.pt.

In [3], the following useful lemma is proved.

Lemma 1.3. Let M, P and Q be as said in Theorem 1.4. Let P be a convex subset. Let $\{p_n\}$and $\{q_n\}$ be sequences in the subset P and $\{Z_n\}$ be a sequence in the subset Q such that.

$\|p_n-z_n\| and \|q_n-z_n\|$ converge to dist (P,Q).

This implies that $\|p_n-z_n\|$ converge to zero.

In [6], the notion of p- cyclic map is introduced. Let (M,ρ) be a metric space. Let $B_i,\ i=1,2,...,p\,(p\ge 2)$ be non empty subsets of M. Then a map $\mathrm{S}:\cup_{i=1}^{p}B_i\to\cup_{i=1}^{p}B_i$, is called a p-cyclic map if $S(B_i)\subseteq B_{i+1}$, where $B_{i+1}=B_i$. Let us say that A is a **"p-cyclic representation"**, if the subsets are as said above and S is a p-cyclic map on the union. A point $x\in B_i$ is a b.p.pt. of S in B_i if $\rho(x,\mathrm{S}x)=dis(B_i,\mathrm{B}_{i+1})$. In [4], the following b.p.pt. theorem for a Geraghty type of contraction is obtained for a p- cyclic map.

Theorem 1.5. Let A be a p-cyclic representation on a Banach space which is uniformly convex. Let $B_1,\mathrm{B}_2,...,\mathrm{B}_p,(p\ge 2)$ be non empty, convex subsets of M, which are closed. Let $u\in w$ and suppose

$$\|Sp-Sq\|\le u(\|p-q\|)\|p-q\| +\left[1-u(\|p-q\|)\right]dist(B_i,\mathrm{B}_{i+1})$$
$$p\in B_i,\ q\in B_{i+1},1\le i\le p$$

Then for $x\in B_i$, $\{S^{pn}(x)\}$ converges to a point $z\in B_i$ such that z is a b.p.pt. of S in B_i. Also, $S^k z$ is a b.p.pt. and also unique periodic point of S in B_{i+k} .

In [5], an idea of cyclic orbital contraction is given which is such that the contraction condition need not be satisfied for all the points. For such a map, fixed points and b.p.pt.are obtained. In [10], a contraction map called p- cyclic orbital non expansive map is defined.

Definition 1.4, Consider a p-cyclic representation A on a metric space (M,ρ) . Let for some $s\in B_i,\ (1\le i\le p)$, for each $k=0,1,2,...,p-1$

$$\rho\left(S^{pn+k}\mathrm{s},\mathrm{S}^{k+1}t\right)\le\rho\left(S^{pn+k-1}\mathrm{s},\mathrm{S}^{k}t\right)..........(2)$$

for all $t\in B_i$ and for all $n\in N$. Then the map S is called p-cyclic non expansive of orbital type.

In [10], the following proposition is given.

Proposition : 1.6. *Let M* be a normed space which is a strictly convex. Let A be a p-cyclic representation and S be a p-cyclic orbital non expansive map which satisfies equation (2) for som $s\in B_i,\ (1\le i\le p)$. Suppose for each $k=0,1,2,...,p-1$ and $t\in B_i$, $\lim_{n\to\infty}\rho\left(S^{pn+k-1}\mathrm{s},\mathrm{S}^{k}t\right)=dis(B_{i+k-1},\mathrm{B}_{i+k})$ and $\{S^{pn+k}(x)\}$ converges to $z_k\in B_{i+k}$. Then the following hold:

(a) $$dis(B_1,\mathrm{B}_2)=dis(B_2,\mathrm{B}_3)=...=dis(B_1,\mathrm{B}_2)=dis(B_{p-1},\mathrm{B}_p)=dis(B_p,\mathrm{B}_1),$$

(b) z_k is a b.p.pt. of S in B_{i+k} and $z_k=S^k z_0$ for k=1,2,…,p.

(c) z_k is a unique periodic point of S with period p in B_{i+k} .

2. Main Results

We give a notion of a contraction map, called as p- cyclic orbital Geraghty type, which is given below:

Definition 2.1. Let A be a p-cyclic representation on a metric space M. Let S satisfy the following: For some $x\in B_i,\ (1\le i\le p)$, for all $y\in B_i$ and for all $n\in\mathrm{N}$, the following inequality holds:

$$\rho\left(S^{pn+k}x,\mathrm{S}^{k+1}y\right)$$
$$\le u\left(\rho\left(S^{pn+k-1}x,\mathrm{S}^{k}y\right)\right)\rho\left(S^{pn+k-1}x,\mathrm{S}^{k}y\right) \qquad (2.1)$$
$$+\left[1-u\left(\rho\left(S^{pn+k-1}x,\mathrm{S}^{k}y\right)\right)\right]dist(B_{i+k-1},\mathrm{B}_{i+k})$$

where $u\in w$. Then S is called "**p-cyclic orbital Geraghty type map**".

Proposition 2.1. Let A be a p-cyclic representation and S be a p - cyclic orbital Geraghty type map with a $x\in B_i,\ (1\le i\le p)$, for which (2.1) hold. Then we have the following results:

(a) S is a p - cyclic orbital non expansive map.

(b) $\rho\left(S^{pn+k}\mathrm{s},\mathrm{S}^{pn+k+1}t\right)\to dis(B_{i+k},\mathrm{B}_{i+k+1})$ $t\in B_i$, for each k={1,2,...,p}.

(c) $\rho\left(S^{pn-1}\mathrm{s},\mathrm{S}^{pn}t\right)\to dis\left(B_{i-1},\mathrm{B}_i\right),\ t\in B_i$.

(d) $\rho\left(S^{pn+p}\mathrm{s},\mathrm{S}^{pn+1}t\right)\to dis\left(B_i,\mathrm{B}_{i+1}\right),\ t\in B_i$.

(e) $\rho\left(S^{pn-p}\mathrm{s},\mathrm{S}^{pn+1}t\right)\to dis\left(B_i,\mathrm{B}_{i+1}\right),\ t\in B_i$.

(f) $\rho\left(S^{pn}\mathrm{s},\mathrm{S}^{pn+p+1}t\right)\to dis\left(B_i,\mathrm{B}_{i+1}\right),\ t\in B_i$.

Proof. Let $x\in B_i,\ (1\le i\le p)$, satisfy (2.1) and $y\in B_i$.

(a) Let $\rho\left(S^{pn+k-1}x,\mathrm{S}^k y\right)>dist\left(B_{i+k-1},\mathrm{B}_{i+k}\right)$. Let

$$\mu_k=\rho\left(S^{pn+k}x,\mathrm{S}^{k+1}y\right),$$

$$\mu_{k-1}=\rho\left(S^{pn+k-1}x,\mathrm{S}^k y\right)\ and$$

$$d=dist\left(B_{i+k-1},\mathrm{B}_{i+k}\right),$$

$$\begin{aligned}\mu_k&\le u\left(\mu_{k-1}\right)\mu_{k-1}+\left[1-u\left(\mu_{k-1}\right)\right]d\\&=u\left(\mu_{k-1}\right)\left[\mu_{k-1}-d\right]+d--->(*)\\&<\mu_{k-1}-d+d\\&=\mu_{k-1}\end{aligned}$$

Henc $\rho\left(S^{pn+k}x,\mathrm{S}^{k+1}y\right)<\rho\left(S^{pn+k-1}x,\mathrm{S}^k y\right)$. (2.2)

Let $\rho\left(S^{pn+k-1}x,\mathrm{S}^k y\right)=dist\left(B_{i+k-1},\mathrm{B}_{i+k}\right)$.

Then from (*),

$$\begin{aligned}\mu_k&\le\alpha\left(\mu_{k-1}\right)\left[\mu_{k-1}-d\right]+d--->(*)\\&<\alpha\left(\mu_{k-1}\right)(0)+d\\&=d\\&=\rho\left(S^{pn+k-1}x,\mathrm{S}^k y\right).\end{aligned}$$

Hence S is a p-cyclic orbital non expansive map.

(b) Let and $k\in\{1,2,3,...,p\}$. Since S is a p-cyclic orbital non expansive map, the sequence $\left\{\rho(\mathrm{S}^{pn+k}x,\mathrm{S}^{pn+k+1}y)\right\}$ is bounded below by $dist\left(B_{i+k},\mathrm{B}_{i+k+1}\right)$ and non increasing. Therefore, this sequence converges to $r\ge dis\left(B_{i+k},\mathrm{B}_{i+k+1}\right)$, where $r=\inf\rho\left(S^{pn+k}x,\mathrm{S}^{pn+k+1}y\right)$.

If $r=dis\left(B_{i+k},\mathrm{B}_{i+k+1}\right)$, then the result is obvious. Hence, let $r>dist\left(B_{i+k},\mathrm{B}_{i+k+1}\right)$. Now

$$\le\rho\left(S^{pn+k+1}x,\mathrm{S}^{pn+k+2}y\right)$$

$$\le u\left(\rho\left(S^{pn+k+1}x,\mathrm{S}^{pn+k+2}y\right)\right)\rho\left(S^{pn+k+1}x,\mathrm{S}^{pn+k+2}y\right)$$

$$+\left(u\left(\rho\left(S^{pn+k+1}x,\mathrm{S}^{pn+k+2}y\right)\right)\right)dist\left(B_{i+k},\mathrm{B}_{i+k+1}\right)$$

$$\rho\left(S^{p(n+1)+k}x,\mathrm{S}^{p(n+1)+k+1}y\right)-dist\left(B_{i+k},\mathrm{B}_{i+k+1}\right)$$

$$\le u\left(\rho\left(S^{pn+k}x,\mathrm{S}^{pn+k+1}y\right)\right)$$

$$\times\left[\rho\left(S^{pn+k}x,\mathrm{S}^{pn+k+1}y\right)-dist\left(B_{i+k},\mathrm{B}_{i+k+1}\right)\right]$$

$$\frac{\rho\left(S^{p(n+1)+k}x,\mathrm{S}^{p(n+1)+k+1}y\right)-dist\left(B_{i+k},\mathrm{B}_{i+k+1}\right)}{\rho\left(S^{pn+k}x,\mathrm{S}^{pn+k+1}y\right)-dist\left(B_{i+k},\mathrm{B}_{i+k+1}\right)}$$

$$\le u\left(\rho\left(S^{pn+k}x,\mathrm{S}^{pn+k+1}y\right)\right)<1,\ u\in w.-->(2.3)$$

Letting $n\to\infty$ in (2.3), we get $1\le u\left(\rho\left(S^{pn+k}x,\mathrm{S}^{pn+k+1}y\right)\right)<1$.

Since $u\in w,u\left(\rho\left(S^{pn+k}x,\mathrm{S}^{pn+k+1}y\right)\right)\to1$

$$\Rightarrow\rho\left(S^{pn+k}x,\mathrm{S}^{pn+k+1}y\right)\to0=r,$$

this is a contradiction. Similarly c), d) e) and f) can be proved.
The proposition given below is useful to prove the main result, the proof of which follows from lemma 1.3 and proposition 2.1.

Proposition 2.2. Let A be a p-cyclic representation on a Banach space M, which is uniformly convex, where the subsets are non empty, and convex which are closed, and S be a p-cyclic orbital Geraghty type map with an $s\in B_i,\ (1\le i\le p)$, for which (2.1) hold. Then the following hold :

(a) $\left\|S^{pn}s-S^{pn+p}s\right\|\to0$

(b) $\left\|S^{pn}s-S^{pn-p}s\right\|\to0$

(c) $\left\|S^{pn+1}s-S^{pn+p+1}s\right\|\to0$.

Theorem 2.1. Let A be a p-cyclic representation on a complete metric space M, such that for some $s\in B_i,\ (1\le i\le p)$ the following is satisfied for all $y\in A_i$ and for all $x\in\mathrm{N}$:

$$\rho\left(S^{pn+k}\mathrm{s},\mathrm{S}^{k+1}t\right)$$

$$\le u\left(\rho\left(S^{pn+k-1}\mathrm{s},\mathrm{S}^k t\right)\right)\rho\left(S^{pn+k-1}\mathrm{s},\mathrm{S}^k t\right)........\ (2.4)$$

$u\in w$. Then $\cap_{i=1}^p B_i$ is non empty and $\left\{S^{pn}s\right\}$ converges to $z_0\in\cap_{i=1}^p B_i$ which the unique is f.p. of S.

Proof. When $dis\left(B_{i+k},\mathrm{B}_{i+k+1}\right)=0$ in (2.1), we get (2.4). Let $s\in B_i,\ (1\le i\le p)$ satisfy (2.4). Let us prove by induction on m, that given $\varepsilon>0$, there exists an $n\in\mathrm{N}$ such that $\rho\left(S^{pn}\mathrm{s},\mathrm{S}^{pm}s\right)<\varepsilon$, for all $n,m\ge n_0$, (2.5)

Let $\varepsilon>0$ be given. Now

$$\rho\left(S^{pn}\mathrm{s},\mathrm{S}^{pm}s\right)\le\rho\left(S^{pn}\mathrm{s},\mathrm{S}^{pm+1}s\right)+\rho\left(S^{pm+1}\mathrm{s},\mathrm{S}^{pm}s\right).$$

By similar argument as in Proposition 2.1 b), it can be shown that by putting

$$dis\left(B_{i+k},\mathrm{B}_{i+k+1}\right)=0,\quad\lim_{n\to\infty}\rho\left(S^{pm}\mathrm{s},\mathrm{S}^{pm+1}s\right)=0\ .$$

Hence there exists an $n_0\in\mathrm{N}$ such that

$$\rho\left(S^{pm+1}\mathrm{s},\mathrm{S}^{pm}s\right)<\left(\delta/p\right),0<\delta<(\varepsilon/2),m\ge n_0\ (2.6)$$

Hence it is enough that if we prove that

$$\rho\left(S^{pn}\mathrm{s},\mathrm{S}^{pm+1}s\right)<(\varepsilon/2),m,n\ge n_0.\qquad(2.7)$$

Fix $n\ge n_0$ such that (2.6) holds. Now (2.7) is true for m=n.

Assume that (2.7) is true for some m, m > n_0. We shall prove that (2.7) is true for (m+1) is place of m. Now

$$\rho\left(S^{pn}s, S^{p(m+1)+1}s\right)$$
$$\le \rho\left(S^{pn}s, S^{pm+1}s\right)+\rho\left(S^{pm+1}s, S^{pm+2}s\right)+...$$
$$+\rho\left(S^{pm+p}s, S^{pm+p+1}s\right)$$
$$<(\varepsilon/2)+\left(\delta/p\right)<\varepsilon$$

Hence $\left\{S^{pn}s\right\}$ is a Cauchy sequence. Let it converge to a limit say, $\xi \in A_i$, By putting $dist\left(B_{i-1}, B_i\right)=0$, in proposition 2.1.(c) it can be proved that $\rho\left(S^{pn-1}s, S^{pn}s\right)\to 0$. Now $\rho\left(\xi, S\xi\right)=0$. Since S is p-cyclic, $\xi \in \cap_{i=1}^{p} B_i$ and is f.p. To prove that ξ is unique, consider η be such that $\eta = S\eta$. It can be shown by similar argument as in Proposition 2.1.(b) $\rho\left(S^{pn}\eta, S^{pn+1}\eta\right)\to 0\, as\, n\to\infty$. Hence

$$\rho\left(\xi,\eta\right)=\rho\left(S^{pn}\xi, S^{pn+1}\eta\right)\to 0\, as\, n\to\infty.$$

Hence $\xi=\eta$.

Remark 2.2. We see that theorem 1.3 is a corollary to the theorem 2.3.

Theorem 2.2. Let A be a p-cyclic representation on a Banach space M, which is uniformly convex and let the subsets be non empty, and convex subsets of M which are closed. such that S is a p- cyclic orbital Geraghty type map. Then for every $s \in B_i$, $\left(1\le i\le p\right)$ satisfying equation (2.1). $\left\{S^{pn}s\right\}$ converges to a unique $z_i \in B_i$ which is a b.p.pt of S in B_i and it is also a unique periodic point of S in B_i. Also, $S^i z_i = z_{i+j}$ is a b.p.pt. and unique periodic point of S in B_{i+j} for j =1,2,...,(p-1).

Proof. If $dis\left(B_i, B_{i+1}\right)=0$, in (2.1), then by Theorem (2.1) we have a unique f.p. of S in $\cap_{i=1}^{p} B_i$. Hence assume $dis\left(B_i, B_{i+1}\right)>0$ .Claim : For every $\varepsilon>0$ there exists $n_0 \in \mathrm{N}$ such that for all $m>n>n_0$,

$$\left\|S^{pm}s - S^{pn+1}s\right\| \le dis\left(B_i, B_{i+1}\right)+\varepsilon \qquad (2.8)$$

Suppose not, then there exists an $\varepsilon_0>0$ such that for all $k\in \mathrm{N}$, there exists $m_k>n_k>k$, for which

$$\left\|S^{pmk}s - S^{pnk+1}s\right\| > dis\left(B_i, B_{i+1}\right)+\varepsilon_0 \qquad (2.9)$$

Now for each k,

$$dis\left(B_i, B_{i+1}\right)+\varepsilon_0$$
$$\le \left\|S^{pmk}s - S^{pnk+1}s\right\|$$
$$\le \left\|S^{pmk}s - S^{pmk-p}s\right\| + \left\|S^{pmk-p}s - S^{pnk+1}s\right\|$$
$$< \left\|S^{pmk}s - S^{pmk-p}s\right\| + dis\left(B_i, B_{i+1}\right)+\varepsilon_0 \;by\; (2.7).$$

By letting $k\to\infty$ any by the proposition (2.1), we have

$$dis\left(B_i, B_{i+1}\right)+\varepsilon_0 \le \lim_{k\to\infty}\left\|S^{pmk}s - S^{pnk+1}s\right\|$$
$$+ dis\left(B_i, B_{i+1}\right)+\varepsilon_0.$$

That is,

$$\lim_{k\to\infty}\left\|S^{pmk}s - S^{pnk+1}s\right\| = dis\left(B_i, B_{i+1}\right)+\varepsilon_0. (2.11)$$

Now,

$$\left\|S^{pmk}s - S^{pnk+1}s\right\| \le \left\|S^{pmk}s - S^{pmk+p}s\right\|$$
$$+\left\|S^{pmk+p}s - S^{pnk+p+1}s\right\|$$
$$+\left\|S^{pnk+p+1}s - S^{pnk+1}s\right\|. \qquad (2.12)$$

Now by using p-cyclic orbital non expansiveness of S, (p-1) times to $\left\|S^{pmk+p}s - S^{pnk+p+1}s\right\|$, we get

$$\left\|S^{pmk+p}s - S^{pnk+p+1}s\right\|$$
$$\le \left\|S^{pmk+1}s - S^{pnk+2}s\right\|$$
$$\le u\left(\left\|S^{pmk}s - S^{pnk+1}s\right\|\right)\left\|S^{pmk}s - S^{pnk+1}s\right\|$$
$$+\left[1-u\left(\left\|S^{pmk}s - S^{pnk+1}s\right\|\right)\right]dis\left(B_i, B_{i+1}\right).$$

Let us denote $dis\left(B_i, B_{i+1}\right)$ by d and $\left\|S^{pmk}s - S^{pnk+1}s\right\|$ by μ_k. Therefore from (2.6)

$$\mu_k \le \left\|S^{pmk}s - S^{pmk+p}s\right\| \le u\left(\mu_k\right)\mu_k + \left(1-u\left(\mu_k\right)\right)d$$
$$+\left\|S^{pnk+p+1}s - S^{pnk+1}s\right\|.$$

That is,

$$\mu_k - u\left(\mu_k\right) - \left(1-u\left(\mu_k\right)\right)d$$
$$\le \left\|S^{pmk}s - S^{pmk+p}s\right\| + \left\|S^{pnk+p+1}s - S^{pnk+1}s\right\|.$$

$$\left(1-u\left(\mu_k\right)\right)\left(\mu_k - d\right)$$
$$\le \left\|S^{pmk}s - S^{pmk+p}s\right\| + \left\|S^{pnk+p+1}s - S^{pnk+1}s\right\|. \quad (2.13)$$

Since for each k, $\left(1-u\left(\mu_k\right)\right)>0$ and $\mu_k > d+\varepsilon_0$ by (2.6), we have $\left(1-u\left(\mu_k\right)\right)\left(\mu_k - d\right)>0$. Now letting $k\to\infty$ and using proposition 2.1. (a) and (c) in (2.10) we have $\lim_{k\to\infty}\left(1-u\left(\mu_k\right)\right)\left(\mu_k - d\right)=0$.

Since $\lim_{k\to\infty} \mu_k = d + \varepsilon_0$ by(2.8), we have, $\lim_{k\to\infty}(1-u(\mu_k))\varepsilon_0 = 0$,

which implies $\lim_{k\to\infty}(1-u(\mu_k)) = 0$, which in turn implies $u(\mu_k) \to 1$. Since $u \in w, u(\mu_k) \to 1 \Rightarrow \mu_k \to 0$, which is contradiction to the fact that $\mu_k \to d + \varepsilon_0$.

Hence the claim. Now by proposition 2.1. (b), $\|S^{pn}s - S^{pn+1}s\| \to dis(B_i, \mathrm{B}_{i+1})$. Combining this with the claim, by lemma 1.7, we have the following:

For every $\varepsilon > 0$ there exist an $n_1 \in \mathrm{N}$, such that

$\|S^{pm}s - S^{pn}s\| \le \varepsilon, m > n > n_1$.

Hence $\{S^{pn}s\}$ is a Cauchy sequence in B_i and converges to a $\omega_i \in B_i$. By proposition 1.6, b) and (c), is a b. p.pt. of S in B_i and also unique periodic point of S in B_i.

Further, $S^i\omega_i = \omega_{i+j}$ is a b.p.pt. in B_{i+j}.

Now we prove that, for any $t \in B_i, t \ne s$ satisfying the inequality (2.1), the sequence $\{S^{pn}t\}$ converges to the same $\omega_i \in B_i$.

Let $S^{pn}t$ converge to $\lambda \in B_i$. By what we have proved $\|\lambda - S\lambda\| = dis(B_i, \mathrm{B}_{i+1})$.

Suppose $\|\lambda - S\lambda\| > dis(B_i, \mathrm{B}_{i+1})$. Then

$$
\begin{aligned}
\|S\omega_i - S^2\lambda\| &\le u(\|\omega_i - S\lambda\|)\|\omega_i - S\lambda\| \\
&\quad + (1-u(\|\omega_i - S\lambda\|))dist(B_i, \mathrm{B}_{i+1}) \\
&< u(\|z_i - S\eta\|)\|z_i - S\eta\| \\
&\quad + (1-u(\|z_i - S\eta\|))\|z_i - S\eta\| \\
&= \|\omega_i - S\lambda\| \\
&= \|S^p\omega_i - S^{p+1}\lambda\| \\
&\le \|S\omega_i - S^2\lambda\|
\end{aligned}
$$

Thus we arrive at a contradiction.

Therefore $\|\omega_i - S\lambda\| = dis(B_i, \mathrm{B}_{i+1}) = \|\lambda - T\lambda\|$. Since the underlying space is uniformly convex Banach space and the sets are convex, we have $\lambda = \omega_i$.

3. Conclusion

In this paper, a new contraction map is introduced on a metric space, where the contraction condition need not be satisfied for all the points in the space. It is enough if the contraction condition hold for just a point in the space, to get a unique fixed point and b.p.pt.. It is easy to see that Theorems1.4 and 1.5 are corollaries to theorem 2.4.

References

[1] M.A. Geraghty, On contractive maps, Proc. Amer. Math. Soc 40 (1973) 604-608.

[2] W.A. Kirk, P.S.Srinivasan and P. Veeramani, Fixed points for mappings satisfying cyclical contractive conditions, Fixed point theory, 4:1 (2003), 79-89.

[3] A.Eldred, Existence and Convergence of Best proximity points, Ph.D Thesis, 2006, Indian Institute of Technology Madras, Chennai, India.

[4] S. Karpagam, On fixed point theorems and best proximity point theorems, Ph.D Thesis 2009, The Ramanujan Institute for Advanced study in Mathematics, University of Madras, Chennai , Indai.

[5] S. Karpagam and S. Agrawal, Best proximity point theorems for cyclic orbital Meir- Keeler contraction maps, Nonlinear Anal. 74 (2011) 1040-1046.

[6] S. Karpagam and S. Agrawal, Best Proximity point for p-cyclic Meir- Keeler contractions, Fixed Point Theory Appl., Volume 29, Article ID 197308, 9 pages.

[7] S. Karpgam and S. Agrawal, Existence of best proximity points for p- cyclic contractions, Fixed point Theory, 13 :99 - 105, 2012.

[8] M. Petric, BZlatanov, Best proximity points and Fixed points for p- summing maps, Fixed point theory appl., 2012:2012:86

[9] B. Zlatanov, Best Proximity points for p- Summing Cyclic Orbital Meir - Keeler Contractions, Nonlinear Anal. Model. Control 20 (4) 2015 528-544.

[10] S. Karpagam and B. Zlztanov, Best proximity points of p-cyclic orbital Meir - Keeler contraction maps, Nonlinear Anal. Model. Control, 21 (6) :790 -806,2016.

[11] S. Karpagam and B. Zlatanov, A note on p- summing orbital Meir - Keeler contraction maps, Int. J. Pure and Applied Math. Volume 107 no. 1, 2016, 225-243.