International Journal of Engineering and Technology, 1 (4) (2012) 298-304

www.sciencepubco.com/index.php/IJET

An Approach for Solving Linear Fractional Programming Problems

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Abstract

Linear fractional programming problems are useful tools in producti on planning, financial and corporate planning, health care and hospital planning and as such have attracted considerable research interest. The paper presents a new approach for solving a fractional linear programming problem in which the objective function is a linear fractional function, while the constraint functions are in the form of linear inequalities. The approach adopted is based mainly upon solving the problem algebraically using the concept of duality and partial fractions and an example is given to clarify the developed method.

Keywords: *Duality, Fractional Programming, Linear Inequalities, Objective Function, Partial Fractions, Production Planning.*

1 Introduction

Linear programming is a mathematical technique aimed at identifying optimal maximum or minimum values of a problem subject to certain constraints [1], while a linear fractional programming (LFP) problem is one whose objective function has a numerator and a denominator and are very useful in production planning, financial and corporate planning, health care and hospital planning. Several methods to solve this problem have been proposed [2]. Charnes and Kooper [3], have proposed a method which depends on transforming the LFP problem to an equivalent linear program. Another method which is called up dated objective function method was also derived to solve the linear fractional programming problems by re-computing the local gradient of the objective function [4]. Also some aspects concerning duality and sensitivity analysis in linear fraction program was discussed by Bitran and Magnant [5].

Nowadays, the most important issue considered by managers of industries is production planning. Manufacturers require a production policy to be globally competitive [6]. In production planning, managers, sometimes, may face up with goals to optimize inventory/sales, actual cost/standard cost, output/employee, etc., with respect to some constraints. Such types of problems are inherently multi objective fractional programming problems. Wide applications of fractional programming arise in different problems in operations research, for example, production, resource allocation [7]. This kind of mathematical programming problem has attracted considerable research and interest, since they are useful in production planning, financial planning corporate planning and health care planning. However for a single objective linear fractional programming the transformation by Charnes and Cooper [3], can be used to transform the problem into a linear programming problem.

The concept of Multi-Objective Programming (MOP), on the other hand, has become popular among researchers during the past few years due to the fact that many single objective optimization methods are not able to help practitioners reach desirable solutions [8-10]. The concept of MOP combined with fractional programming is an interesting area of research which incorporates many production planning applications [11, 12]. Few approaches have been reported for solving the multiple objective linear fractional programming (MOLFP) problem [13, 14]. Multi-objective linear programming is an extension of linear programming and it was introduced by Chaudhuri and De [15]. The problem was also considered and presented a simplex–based solution procedure to find all weakly efficient vertices of the augmented feasible region [16]. It was however showed that the procedure suggested by Kormbluth and Steuer for computing the numbers to find break points may not work all the time and a failsafe method for computing these numbers was proposed by Benson [17]. The objective space for multiple objectives linear fractional programming with equal denominators was given by Tantawy [2], using the concept of duality. The approach enables the transformation of a single objective linear fractional programming problem into a linear programming problem using partial fractions method with the concept of duality.

2 Definition of a Linear Fractional Programming Problem

A linear fractional programming problem occurs when a linear fractional function is to be minimized or maximized and the problem can be formulated mathematically as follows:
A linear - fractional programming problem is of the type;

$$\text{Maximize } P(x) = \frac{c^T x + \alpha}{f^T x + \beta}$$

$$\text{subject to } x \in X = (x, Ax \le b) \tag{1}$$
$$f^T x + \beta \ge 0$$
$$x \ge 0$$

where $x \in R^n$, A is an $(m+n) \times n$ matrix, c and d are n-vectors, $b \in R^{m+n}$, and α, β are scalars. It is assumed that the feasible solution set X is bounded and closed (compact set). Assume $\lim_{h \to 0} \frac{\alpha}{h} = +\infty \ \ if \ \ a > 0$ and $\lim_{h \to 0} \frac{\alpha}{h} = -\infty \ \ if \ \ a \le 0$.

By generalizing the linear fractional programming problem we have;

$$\text{maximize } F(x) = \frac{c_i^T x - d_i}{f_i^T + g_i} \tag{2}$$
$$\text{subject to } Ax \le b$$
$$f_i^T x + g_i \ge 0, \quad i = 1, \cdots\cdots, k,$$

$$\text{maximize } F(x) = (c_i - \frac{d_i}{g_i} f^T) \frac{x}{f^T x + g} + \frac{d_i}{g_i}$$
$$\text{subject to } (A + \frac{d_i}{g_i} f^T) \bullet \frac{x}{f^T x + g} \le \frac{b_i}{g_i} \tag{3}$$

Defining $\frac{x}{f_i^T + g_i} \ge 0$, then Equation (3) can be written in the form

$$\text{Maximize } F(y) = (c_i - \frac{d_i}{g_i} f^T) + \frac{d_i}{g_i}$$
$$\text{subject to } (A + \frac{d_i}{g_i} f^T) \le \frac{b_i}{g_i} \tag{4}$$

Equation 4 can simply be written in the form

$$\text{Maximize } F(y) = P^T y + \frac{b_i}{g_i}$$
$$\text{Subject to } Gy \le t \tag{5}$$

where $P^T = (c_i - \frac{d_i}{g_i} f^T)$, $G = (A + \frac{d_i}{g_i} f^T)$ and $t = \frac{b_i}{g_i}$.

From (4) which defines y, we have $x = g_i \frac{y}{1 - f^T y}$ (6)

Now, consider the dual of the linear programming in (5) in the form

$$\text{Minimize } w = u^T t$$
$$\text{Subject to } u^T G = P^T \tag{7}$$
$$u \ge 0$$

On multiplying the set of constraints of this dual problem by T = (T_1 | T_2), $T_1 = p(p^T p),$
the column of the matrix T_2 constitutes the base of

$$N(p^T) = \{v;\ p^T v = 0\} \tag{8}$$

We have $u^T GT_1 = 1,\ u^T GT_2 = 0$ *and* $u \geq 0$. In this case when $GT_2 \neq 0$, *an* $s \times (m+n)$ *matrix* Q of non- negative entries is defined such that QGT = 0. This matrix will play an important role to find the optimal value of the above problem as the maximum value of w on the interval on the real line defined by

$$W = \{w \in R \mid \mathrm{QGT_1} w \leq Qg\}$$

The above representation can simply be written as W= {w $\in$ R | Zw $\leq$ z},
where Z = QGT_1 and z = Qg . (9)

Also a sub matrix $\bar{Q}$ of the given matrix Q satisfying $\bar{Q}\,GT_1$ = 1 will be important for specifying the dual values needed for solving the linear fraction programming problem (1). The dual values satisfy the well known Kuhn-Takucer condition [2] and for a point y^k to be an optimal solution of the above program, (5) must exist.

$u \geq 0$ such that $G^T u = p,$ or simply

$$u = (G, G^T)^{-1} G, p \tag{10}$$

3 New Method for Solving Linear Fractional Programming (LFP) Problems

The new method for solving LFP problems is summarized as follows:

1. Compute $T_1 = (p^T p)^{-1} p$ and the matrix T_2 as in Equation (8)
2. Find the matrix Q of non-negative entries such that QGT_2 = 0,
3. Find a sub matrix $\bar{Q}$ of the given matrix Q satisfying $\bar{Q}\,GT_1$ = 1
4. In the rows of $\bar{Q}$ for every positive entry, determine the corresponding active constraint in the given matrix GT_1
5. Solve an $n \times n$ system of linear equations for these set of active constraints to get the optimal solution y*. Then use (6) to get the optimal solution of the Linear Fractional Programming (LFP) problem defined by Equation (1).

3.1 Remarks

1. The matrix Q of non-negative entries such that QGT = 0, is considered as the a polar matrix of the given matrix GT_2
2. With d = 0 in (LFP), the above problem reduces to linear programming (LP) problem, and hence the method can be used to solve the (LP) as a special case of this (LFP) using the same argument.

3.2 Numerical Examples

Some numerical examples is now considered in other to illustrate the problem:
Example 1.

Maximize $Z = \dfrac{x_1 + 3x_3 + 2x_3}{2x_1 + x_2 + 4x_3 + 1}$

Subject to $x_1 + 3x_2 + 6x_3 \le 8$

$2x_1 + x_2 + 4x_3 \le 5$

$x_1, x_2, x_3 \ge 0$

Table 1: The final table for $x_1 = x_3 = 0,\ x_2 = \dfrac{8}{3}$ to be an optimal solution.

a_1	a_2	a_3	a_4	a_5	X_B
1/3	1	2	1/3	0	8/3
5/3	0	2	1/3	1	7/3
40/3		92/3	1		

For the solution $x_2 = 8/3$, $x_1 = x_3 = 0$, we observe that the first constraint

$\dfrac{1}{3}x_1 + x_2 + 2x_3 \le \dfrac{8}{3}$ holds as an equality,

while the second constraint $\dfrac{5}{3}x_1 + 0x_2 + 2x_3 \le \dfrac{7}{3}$ holds as an inequality.

This second constraint is an invalid constraint. We therefore combine this invalid constraint with the objective function to generate the following parametric fractional programming problem.

$$Z = \frac{x_1 + 3x_2 + 2x_3 + \lambda(\frac{5}{3}x_1 + 2x_3)}{2x_1 + x_2 + 4x_3 + 1 + \mu(\frac{5}{3}x_1 + 2x_3)}$$

Subject to $\dfrac{1}{3}x_1 + x_2 + 2x_3 = \dfrac{8}{3}$

$x_1, x_2, x_3 \ge 0$

Example 2: Consider the following combination of linear fractional programming (LFP) problems,

Maximize $z_1 = \dfrac{x_1 + x_3 + 2}{x_1 + 2}$ and Maximize $z_2 = \dfrac{-x_1 + 2x_2 + 4}{x_1 + 2}$

Subject to $x_1 + x_2 + x_3 \le 1$

$x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$

Using partial fractions with duality concept we have,

c_1^t = (1 0 1), c_2^t = (1 2 0), d^t = (1 0 0), also α_1 = 2, α_2 = 4 and β = 2

Example 3: Consider the following linear fractional programming (LFP) problem

$$\text{Maximize } Z = \frac{x_1 + x_2 + 3}{x_2 + 1}$$

$$\text{Subject to} \quad x_1 + x_2 \leq 6$$
$$-x_1 \leq 0,$$
$$-x_2 \leq 0$$

For this LFP we have $c^t = (1 \quad 1)$, $d^t = (0 \quad 1)$, $b_i = 3$, $g_i = 1$, (c and d are matrices) and then we have

$$T_1 = \begin{pmatrix} 1/5 \\ -2/5 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \text{ and } GT = \begin{pmatrix} 10 \\ -2 \\ -1 \end{pmatrix}$$

Which gives Q = $\begin{pmatrix} 1 & 5 & 0 \\ 1 & 0 & 10 \end{pmatrix}$

The second row in Q satisfies $\bar{Q}GT_1 = 1$. This indicates that the first and the third constraints in G are the only active set of constraints, on solving $y_1 + 8y_2 = 6$, $y_2 = 0$, we get $y^{t*} = (6 \quad 0)$ as the optimal solution for the equivalent problem which finally on using (6) gives $x^{t*} = (6 \quad 0)$ as the optimal solution of our linear fraction program with optimal value z* = 9.

4 Conclusion

A method for solving linear fractional functions with constraint functions in the form of linear inequalities is given. The proposed method differs from the earlier methods as it is based upon solving the problem algebraically using the concept of duality with partial fractions approach. The method appears simple to solve any linear fractional programming problem of any size.

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