International Journal of Engineering & Technology, 7 (3.28) (2018) 339-342

International Journal of Engineering & Technology

Website: www.sciencepubco.com/index.php/IJET

Research paper

On Shamanskii-Like Iterative Method for Solving Fuzzy Nonlinear Equations

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Abstract

This paper proposes a Shamanskii-like method with fixed Jacobian matrix for solving fuzzy nonlinear equation. The method does not require evaluation of the Jacobian at every iteration. This is made possible by considering a fixed Jacobian at x_n. Numerical experimentation are carried out, which shows the superiority of the proposed method against other existing methods.

Keywords*: Fuzzy nonlinear equations; parametric form; Fixed Jacobian; Shamanskii method.*

1. Introduction

Many real-life problems require the numerical solution of a system of nonlinear equations of the form

$$F(x) = 0 \qquad (1)$$

where $F: R^n \to R^n$ and that it is required to find $x^* \in R^n$ such that $F(x^*) = 0$. When the coefficients of (1) are written in crisp number, it might be comforting to represent some or all of them with fuzzy numbers. Zadeh [3] was the inventor of the concept of fuzzy numbers where he introduced an application of fuzzy number arithmetic in nonlinear equations which the parametric form can be represented partially or wholly by fuzzy numbers [5, 8, 9]. The numerical solution to fuzzy nonlinear equation with fuzzy coefficient involving fuzzy variable is on when the Jacobian is nonsingular near exact root (x^*). Various methods have existed namely the Newton's method [4], the Broyden's method [7], the Chord method [10], Shamanskii-like acceleration method [1, 2], diagonally updating Shamanskii-like method [11] for solving fuzzy nonlinear equations is its variants. For recent findings, please refer to [12-14]. Each method poses both advantages and disadvantages. As an example, to overcome the problem of re-evaluating Jacobian matrix for each iteration in Newton's like method, the Jacobean matrix needs only be evaluated once throughout the whole process or once after every few predetermined iterations. This paper consider a fixed Shamanskii method for solving systems of nonlinear equations with the focus is on minimizing the computational cost due to the Jacobian matrix.
The structure of this paper is as the following. In section 2, we present some basic definitions in addition to brief overview of fuzzy nonlinear equations. In section 3, we present the description of the proposed method 3. In section 4, we include an alternative approach for solving fuzzy nonlinear equation. And finally, we report our numerical results in section 5 followed by a conclusion in section 6.

2. Preliminaries

For a start, the following discussion defines the fuzzy numbers for the purpose of completeness.

Definition 1
A fuzzy number is a set like $u: R \to I = [0,1]$, which satisfy the following conditions [3],
(1) u is upper semi-continuous,
(2) $u(x) = 0$ outside some interval $[c, d]$,
(3) There exist real numbers a, b such that $c \le a \le b \le d$ and
(3.1) $u(x)$ is monotonic increasing on $[c, a]$
(3.2) $u(x)$ is monotonic decreasing on $[b, d]$
(3.3) $u(x) = 1, a \le x \le b.$

Let E denotes the set of all fuzzy numbers. An equivalent parametric form can be found in [9].

Definition 2
A fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$ of function $\underline{u}(r), \overline{u}(r), 0 \le r \le 1$, which satisfies the following requirements:
(1) $\underline{u}(r)$ is bounded monotonic increasing left continuous function,
(2) $\overline{u}(r)$ is bounded monotonic decreasing left continuous function,
(3) $\underline{u}(r) < \overline{u}(r), 0 \le r \le 1.$

A crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha, 0 \le r \le 1$. A popular fuzzy number is the trapezoidal fuzzy number $u = (x_0, y_0, \alpha, \beta)$ with interval defuzzifier $[x_0, y_0]$ and left fuzziness α and right fuzziness β where the membership function is

$$u(x)=\begin{cases}\frac{1}{\alpha}(x-x_0+\alpha), & x_0-\alpha\le x\le x_0,\\ 1 & x\in[x_0,y_0]\\ \frac{1}{\beta}(y_0-x+\beta), & y_0\le x\le y_0+\beta,\\ 0 & \text{otherwise.}\end{cases}$$

and its parametric form is

$$\underline{u}(r)=x_0-\alpha+\alpha r,\quad \overline{u}(r)=y_0+\beta-\beta r.$$

Now, we introduce $TF(R)$ as set of all triangular fuzzy numbers. We deals with two operations namely the addition and scalar multiplication using the extension principle which is representeable as the following.

Let $u=(\underline{u},\overline{u}), v=(\underline{v},\overline{v}) and\ k>0$, we define $u+v$ and multiplication by real number $k>0$ as

$$(\underline{u+v})(r)=\underline{u}(r)+\underline{v}(r),\quad (\overline{u+v})(r)=\overline{u}(r)+\overline{v}(r),$$
$$(\underline{ku})(r)=k\underline{u}(r),\quad (\overline{ku})(r)=k\overline{u}(r).$$

3. The Shamanskii method for solving nonlinear equation

Now we will use the fixed point iteration method for finding the solution to (1) as a fixed point of some function $G:R^n\to R^n$.

$$x_{k+1}=G(x_k),\qquad k=0,1,\dots \tag{2}$$

where x_0 is the initial guess. Until now, the Classical Newton is the most popular fixed point method, expressable by

$$x_{n+1}=x_n-J'(x_n)^{-1}F(x_n),\qquad n=0,1,2,\dots$$

where $J'(x_n)$ represent the Jacobian matrix evaluated at x_n.
However, Newton's method comes with a disadvantage in that, it requires a solution and invertion of the Jacobian matrix per iteration basis.
One method in tackling this is the Shamanskii's method which is indeed a modified Newton's method of the type in which the Jacobian is replaced by a new Jacobian after m_k steps of the iteration.
Shamanskii method generates a sequence of points via, given an integer m and an initial iterate x_0, we move from x_n to x_{n+1} through an intermediate sequence $\{y_{n,p}\}_{p=1}^{m}$ which is one Newton iterate followed by several chord iterates,

$$y_{n,1}=x_n-F'(x_n)^{-1}F(x)$$

$$y_{n,p+1}=y_{n,p}-F'(x_n)^{-1}F(y_{n,p}),\ 1\le p\le m-1,$$
(3)
$$x_{n+1}=y_{n,m}$$

In this paper, we consider the fixed Shamanskii's method which we present as follows: using fixed Shamanskii's method, a sequence $\{y_{n,p}\}_{p=1}^{m}$ will be generated iteratively, as shown in Algorithm 1.

Algorithm 1 (Fixed Shamanskii's method)

S1: Given x_0
S2: Solve $V_{k,1}=x_k-J(x_0)^{-1}F(x_k)$
S3: Compute $x_{k,p+1}=x_{k,p}-J(x_0)^{-1}F(V_{k,p})$, (4)
$x_{n+1}=V_{k,m}\ where\ k=1,2,\dots and\ p\in(1,m-1)$

4. Iterative Approach for solving fuzzy nonlinear equations

This section is intended to implement a new iterative approach for finding a solution for fuzzy nonlinear equation

$$F(x)=0$$

In parametric form, it can be expressed as follows:

$$\underline{F}(\underline{x},\overline{x};r)=0$$
$$\overline{F}(\underline{x},\overline{x};r)=0\quad \forall r\in[0,1]. \tag{5}$$

Assume that $\alpha=(\underline{\alpha},\overline{\alpha})$ is the solution to the nonlinear system (5), that is

$$\underline{F}(\underline{\alpha},\overline{\alpha};r)=0,$$
$$\overline{F}(\underline{\alpha},\overline{\alpha};r)=0,\qquad \forall r\in[0,1]$$

Now, let $x_0=(\underline{x}_0,\overline{x}_0)$ be an approximate solution for this nonlinear equation, then $\forall r\in[0,1]$, we have $h(r),k(r)$ such that

$$\underline{\alpha}(r)=\underline{x}_0(r)+h(r),$$
$$\overline{\alpha}(r)=\overline{x}_0(r)+k(r).$$

Applying the Taylor series of $\underline{F},\overline{F}$ about $(\underline{x}_0,\overline{x}_0)$, then $\forall r\in[0,1]$,

$$\underline{F}(\underline{\alpha},\overline{\alpha};r)=\underline{F}(\underline{x}_0,\overline{x}_0,r)+h\,\underline{F}_{\underline{x}}(\underline{x}_0,\overline{x}_0,r)+g\,\underline{F}_{\overline{x}}(\underline{x}_0,\overline{x}_0,r)+0(h^2+hk+h^2)=0$$

$$\overline{F}(\underline{\alpha},\overline{\alpha};r)=\overline{F}(\underline{x}_0,\overline{x}_0,r)+h\,\overline{F}_{\underline{x}}(\underline{x}_0,\overline{x}_0,r)+g\overline{F}_{\overline{x}}(\underline{x}_0,\overline{x}_0,r)+0(h^2+hk+h^2)=0$$

And if $\underline{x}_0\ and\ \overline{x}_0$ are near $\underline{\alpha}\ and\ \overline{\alpha}$ respectively, then $h(r)\ and\ k(r)$ can be sufficiently small. Consider the case where all the needed partial derivatives exist are bounded. Hence, for $h(r)\ and\ k(r)$, where $\forall r\in[0,1]$ we have,

$$\underline{F}(\underline{x}_0,\overline{x}_0,r)+h\,\underline{F}_{\underline{x}}(\underline{x}_0,\overline{x}_0,r)+g\,\underline{F}_{\overline{x}}(\underline{x}_0,\overline{x}_0;r)=0$$
$$\overline{F}(\underline{x}_0,\overline{x}_0,r)+h\,\overline{F}_{\underline{x}}(\underline{x}_0,\overline{x}_0,r)+g\overline{F}_{\overline{x}}(\underline{x}_0,\overline{x}_0;r)=0$$

Thus, the values of $h(r)\ and\ k(r)$ can be found by solving (6), $\forall r\in[0,1]$.

$$J(\underline{x}_0,\overline{x}_0,r)\begin{pmatrix}h(r)\\ g(r)\end{pmatrix}=\begin{pmatrix}-\underline{F}(\underline{x}_0,\overline{x}_0,r)\\ -\overline{F}(\underline{x}_0,\overline{x}_0,r)\end{pmatrix} \tag{6}$$

where

$$J(\underline{x}_0,\overline{x}_0,r)=\begin{bmatrix}\underline{F}_{\underline{x}}(\underline{x}_0,\overline{x}_0,r) & \underline{F}_{\overline{x}}(\underline{x}_0,\overline{x}_0,r)\\ \overline{F}_{\underline{x}}(\underline{x}_0,\overline{x}_0,r) & \overline{F}_{\overline{x}}(\underline{x}_0,\overline{x}_0,r)\end{bmatrix}$$

is the Jacobian matrix of the function $F=(\underline{F},\overline{F})$ evaluated at $x_0=(\underline{x}_0,\overline{x}_0)$. Hence, the next approximations for $\underline{x}(r) and\ \overline{x}(r)$ are as follows

$$\underline{x}_1(r)=\underline{x}_0(r)+h(r),$$
$$\overline{x}_1(r)=\overline{x}_0(r)+k(r),$$

for all $r\in[0,1]$.
Employing the recursive method, we can further attain an approximated solution, $r\in[0,1]$,

$$\underline{x}_{n+1}(r)=\underline{x}_n(r)+h_n(r),$$

$$\overline{x}_{n+1}(r) = \overline{x}_n(r) + k_n(r), \qquad (7)$$

when $n = 1,2,\ldots$ Analogous to (5)

$$J(\underline{x}_n, \overline{x}_n, r)\begin{pmatrix} h(r) \\ g(r) \end{pmatrix} = \begin{pmatrix} -\underline{F}(\underline{x}_0, \overline{x}_0, r) \\ -\overline{F}(\underline{x}_0, \overline{x}_0, r) \end{pmatrix}$$

If $J(\underline{x}_n, \overline{x}_n, r)$ be nonsingular, then from (6), we obtain the recursive scheme of Newton's method as follows

$$\begin{bmatrix} \underline{x}_{n+1}(r) \\ \overline{x}_{n+1}(r) \end{bmatrix} = \begin{bmatrix} \underline{x}_n(r) \\ \overline{x}_n(r) \end{bmatrix} - J(\underline{x}_n, \overline{x}_n, r)^{-1} \begin{bmatrix} \underline{F}(\underline{x}_n, \overline{x}_n, r) \\ \overline{F}(\underline{x}_n, \overline{x}_n, r) \end{bmatrix}$$

The proposed approach (Fixed Shamanskii's method) in algorithmic form can be found in Algorithm 2.

Algorithm 2 (Fixed Shamanskii's method for Fuzzy Nonlinear problem)
S1: Transform the fuzzy nonlinear equations into parametric form
S2: Determine the initial guess x_0 by solving the parametric equations for $r = 0$ and $r = 1$. And for $k = 0,1,2\ldots$
S3: Solve $J(x_0)^{-1}$ and $F(x_k)$.
S4: Calculate $V_{k,1} = x_k - J(x_0)^{-1}F(x_k)$.
S5: Calculate $x_{k+1} = x_{k,p} + [J(x_0)]^{-1}F(V_{k,p})$. $x_{k+1} = V_{k,m}$ $where\ k = 1,2,\ldots and\ p \in (1, m-1)$
S6: Repeat S3 to S5 and continue with the next k until $\epsilon \leq 10^{-4}$ are satisfied.

5. Results and discussion

By considering two problems, we will show and compare the performance of the iterative method. The experimentation used MATLAB 7.0 (R2013a) having the capability of double precision. Whereas, the standard test problems can be found in [4].

Example 1: Consider a fuzzy nonlinear equation
$(2,2,1,1)x^3 + (3,3,1,1)x^2 + (4,1,1) = (8,8,3,5)$

For once, let the values of x be positive, as such the parametric form of this equation can be simplified as:

$$(1+r)\underline{x}^3(r) + (2+r)\underline{x}^2(r) + (3+r) = (5+3r)$$
$$(3-r)\overline{x}^3(r) + (4-r)\overline{x}^2(r) + (5-r) = (13-5r)$$

which can be re-written as

$$(1+r)\underline{x}^3(r) + (2+r)\underline{x}^2(r) = (2+2r)$$
$$(3-r)\overline{x}^3(r) + (4-r)\overline{x}^2(r) = (8-4r)$$

let $r = 0$ and $r = 1$. We then obtain the initial guess as follows

$$\underline{x}^3(0) + 2\underline{x}^2(0) = 2$$
$$3\overline{x}^3(0) + 4\overline{x}^2(0) = 8$$
And
$$2\underline{x}^3(1) + 3\underline{x}(1) = 4$$
$$2\overline{x}^3(1) + 3\overline{x}(1) = 4$$

We consider the initial guess $x_0 = (0.8,0.8,0.9,0.9)$. By implementing Algorithm 2, we obtain the solution after four iterations, and the maximum error is recorded to be less than 10^{-5}. Fig. 1 shows the performance profiles of the positive solution for $r \in [0,1]$.

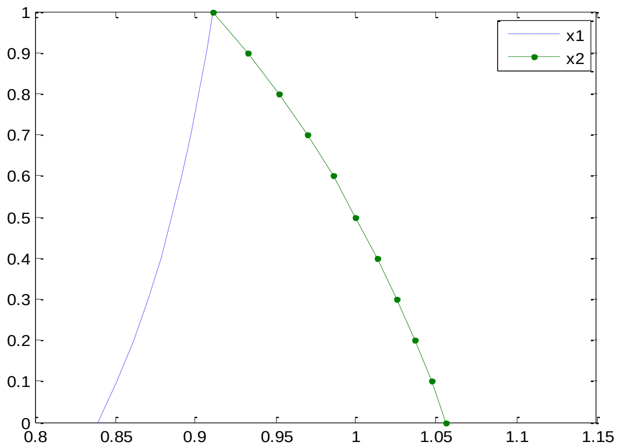

Fig. 1. Iterative solution of example 1

Example 2: Consider the following fuzzy nonlinear equation

$$(3,3,4,5)x^2 + (1,2,3)x = (1,1,2,3)$$

For once, let the values of x be positive, as such the parametric form of this equation can be simplified as:

$$(3+r)\underline{x}^2(r) + (1+r)\underline{x}(r) = (1+r)$$
$$(5-r)\overline{x}^2(r) + (3-r)\overline{x}(r) = (3-r)$$

To find the value of the initial guess, we let $r = 0$ and $r = 1$ such that
$r = 1$

$$4\underline{x}^2(1) + 2\underline{x}(1) = 2$$
$$4\overline{x}^2(1) + 2\overline{x}(1) = 2$$

$r = 0$

$$3\underline{x}^2(0) + \underline{x}(0) = 1$$
$$5\overline{x}^2(0) + 3\overline{x}(0) = 3$$

When $r = 0$, we have $\underline{x}(0) = 0.4343, \overline{x}(0) = 0.5307$ and when $r = 1$, we have $\underline{x}(1) = \overline{x}(1) = 0.5000$. We consider $x_0 = (0.4,0.4,0.5,0.6)$, as our initial guess. Via Algorithm 2 with $x_0 = (0.4,0.4,0.5,0.6)$ and fixed Jacobian $(J(\underline{x}_0, \overline{x}_0; r))$, we obtain the solution after 2 iterations with maximum error less than 10^{-5}. Fig. 2 concludes the finding of our method for $\forall r \in [0,1]$.

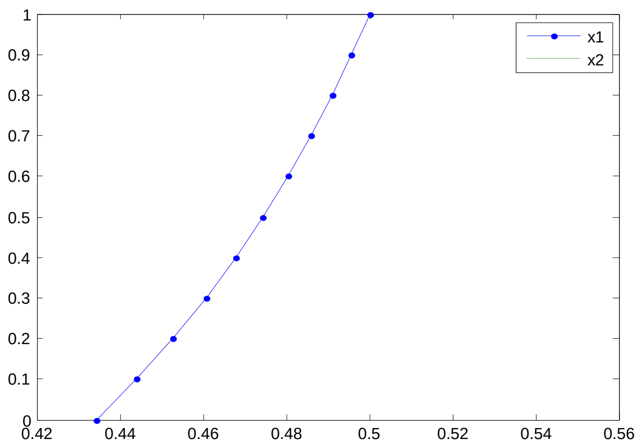

Fig. 2. Iterative solution of example 2

6. Conclusion

The purpose of this paper was to seek for an alternative iterative method that can be used to solve fuzzy nonlinear equation. Our main goal is to reduce computational complexity in the implementation of Jacobian matrix, of which we achieved by limiting to only one computation (at x_0) throughout the whole iteration process. This was achieved by transforming the fuzzy nonlinear equation into parametric form and then solved via fixed Shamanskii method. The numerical result presented has partially proved that our method is significantly much better than other existing ones.

Acknowledgement

This project is funded by the Malaysian Fundamental Research Grant Scheme (FRGS/1/2017/STG06/Unisza/01/1) and Research Management, Innovation & Commercialization Center, Universiti Sultan Zainal Abidin.

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