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Research paper

Fixed point results in tricomplex valued fuzzy metric spaces with application

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Abstract

In this article, accredit the innovative concept of complex valued the fuzzy set due to Ramot et al.[17], Singh et al.[20] and Choi et al.[7], we introduce the conceptualisation of tricomplex valued fuzzy metric spaces and this paper is inspired by Ismat Beg et al.[10]. Various related topological features have been established for tricomplex-valued fuzzy metric spaces, thereby reinforcing the foundational concept.

Keywords: *Fixed point, Fuzzy metric spaces, Bicomplex valued metric space, Tricomplex valued fuzzy metric space, Banach contraction principle.*

1. Introduction

Segre, in his seminal work [18], laid the groundwork for the exploration of special algebras, introducing a groundbreaking perspective in the form of commutative generalizations beyond complex numbers, such as bicomplex and tricomplex numbers. Building upon Segre's foundation, Price [16] further advanced the field by developing bicomplex algebra and function theory.

This line of inquiry, which was recently revitalized, has garnered renewed interest due to its far-reaching applications across various domains within mathematical sciences and diverse branches of science and technology. A substantial body of research has emerged from the efforts of numerous scholars in this field, contributing to an increasingly robust understanding of these specialized algebras and their practical implications.

Azam et al. [1] extended it to complex valued metric space and established a common fixed point theorem for a pair of self-contracting mappings. Choi et al. [7] proved some common fixed point theorems with two weakly compatible mappings in bicomplex valued metric spaces.

In the year 1922, S. Banach made a significant contribution to the field of fixed point theory with his seminal work [2], presenting a crucial result known as the 'Banach contraction principle.' This principle has since become a pervasive and powerful tool in addressing existence problems across various branches of mathematical analysis. Over the years, it has remained an active and fruitful area of research, continually providing valuable insights and solutions to mathematical problems.

On the other hand, important theoretical development in the fuzzy sets theory introduced by Zadeh [22]. Fuzzy sets theory is the way of defining the concept of fuzzy metric spaces by Kramosil and Michalek [11], which can be regarded as a generalization of the statistical metric spaces. Subsequently, M. Grabiec [9] defined G-complete fuzzy metric spaces and extended the complete fuzzy metric spaces. Following Grabiec's work, George and Veeramani [8] modified the notion of M-complete fuzzy metric spaces with the help of continuous t-norms. Many authors introduced and generalized the numerous types of fuzzy contractive mappings ([12], [13], [19], [21]) and investigated some fixed point theorems in fuzzy metric spaces.

Fuzzy complex numbers and fuzzy complex analysis were first introduced by Buckley [3] -[6]. Building on Buckley's contributions [3] -[6], several authors have pursued further research in the realm of fuzzy complex numbers. Notably, Ramot et al. [17] made significant strides by extending fuzzy sets to what they termed "complex fuzzy sets." In their work, a complex fuzzy set is defined by a membership function whose range extends beyond the conventional $[0,1]$ interval to encompass the entire unit circle within the complex plane.

As articulated by Ramot et al. [17], membership in a complex fuzzy set retains its inherent fuzziness, analogous to membership in a traditional fuzzy set. This extension into the complex domain introduces a nuanced and enriched perspective to the concept of fuzzy sets, opening avenues for a deeper understanding and application of fuzzy logic within the context of complex numbers.

Expanding upon the groundwork laid by Ramot et al. [17], Singh et al. [20] further developed the concept of complex fuzzy sets. They introduced and defined the novel notion of complex-valued fuzzy metric spaces by leveraging continuous t-norms. Additionally, Singh et al.

established a Hausdorff topology on these complex-valued fuzzy metric spaces.
In their work, they introduced the concept of Cauchy sequences within the context of complex-valued fuzzy metric spaces. Moreover, Singh et al. made significant contributions by formulating and proving the complex-valued fuzzy version of the vital Banach contraction principle, offering fixed point theorems through rational expressions, and demonstrating Jungck-type fixed point results. This comprehensive exploration of complex-valued fuzzy metric spaces and associated principles contributes to the ongoing advancements in the field of fuzzy mathematics.

2. Preliminaries

Let's initiate our exploration by establishing fundamental concepts and notations. Let the symbol $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ signify the set encompassing all non-negative real numbers, and $\mathbb{N}$ represent the set of natural numbers. With these foundational elements in place, we introduce the following definitions pertinent to the characterization of a fuzzy metric space.

Definition 2.1. [8]. An ordered triple $(X,M,*)$ is called fuzzy metric space such that X is a nonempty set, $*$ defined a continuous t-norm and M is a fuzzy set on $X\times X\times(0,\infty)$, satisfying the following conditions, for all $x,y,z\in X, s,t>0$.
(FM-1) $M(x,y,t)>0$.
(FM-2) $M(x,y,t)=1$ iff $x=y$.
(FM-3) $M(x,y,t)=M(y,x,t)$.
(FM-4) $(M(x,y,t)*M(y,z,s))\leq M(x,z,t+s)$.
(FM-5) $M(x,y,\cdot):(0,\infty)\to(0,1]$ is left continuous.

Segre [18] defined the bicomplex number as $\xi=a_1+a_2i_1+a_3i_2+a_4i_1i_2=z_1+i_2z_2$, where $a_1,a_2,a_3,a_4\in\mathbb{C}_0$ (the set of reals) and $z_1=a_1+a_2i_1, z_2=a_3+a_4i_1\in\mathbb{C}_1$ (the set of complex numbers), the independent units i_1,i_2 are such that $i_1^2=i_2^2=-1$ and $i_1i_2=i_2i_1$. We denote the set of bicomplex numbers as $\mathbb{C}_2$.
Segre [18] defined the tricomplex number as $\xi=a_1+a_2i_1+a_3-a_4i_2=z_1-z_2'$, where $a_1,a_2,a_3,a_4\in\mathbb{C}_0$ (the set of reals) and $z_1=a_1+a_2i_1, z_2=a_3+a_4i_1\in\mathbb{C}_1$ (the set of complex numbers), the independent units i_1,i_2 are such that $i_1^2=i_2^2=-1$ and $i_1i_2=i_2i_1$. We denote the set of bicomplex numbers as $\mathbb{C}_2$.
Pal et al. [15] defined the partial order relation $i_2\preceq_{i_2}$ on $\mathbb{C}_2$ defined as:
For any $\xi=z_1+i_2z_2, \eta=w_1+i_2w_2\in\mathbb{C}_2$, $\xi\preceq_{i_2}\eta$ if and only if $z_1\preceq w_1$ and $z_2\preceq w_2$ and $\xi\preceq_{i_2}\eta$ if one of the following conditions is satisfied:
(i) $z_1=w_1, z_2=w_2$.
(ii) $z_1\prec w_1, z_2=w_2$.
(iii) $z_1=w_1, z_2\prec w_2$
(iv) $z_1\prec w_1, z_2\prec w_2$,
also defined two conditions
(1). write $\xi\not\preceq_{i_2}\eta$ if $\xi\preceq_{i_2}\eta$ and $\xi\neq\eta$ then one of $(ii),(iii)$ and (iv) is satisfied.
(2). write $\xi\prec_{i_2}\eta$if only (iv) is satisfied.

Choi et al. [7] defined the bicomplex valued metric space as

Definition 2.2. [7] Let X be a nonempty set. Suppose the mapping $d:X\times X\to\mathbb{C}_2$ satisfies the following conditions:
(1) $0\prec_{i_2}d(x,y)$ for all $x,y\in X$,
(2) $d(x,y)=0$ if and only if $x=y$,
(3) $d(x,y)=d(y,x)$for all $x,y\in X$,
(4) $d(x,y)\preceq_{i_2}d(x,z)+d(z,y)$ for all $x,y,z\in X$,
Then (X,d) is called a bicomplex valued metric spaces.

Singh et al. [20] defined the complex valued continuous t-norm and complex valued fuzzy metric space as

Definition 2.3. [20] A binary operation $*:r_se^{i\theta}\times r_se^{i\theta}\to r_se^{i\theta}$, where in $r_s\in[0,1]$ and a fix $\theta\in[0,\frac{\pi}{2}]$, is called complex valued continuous t-norm if it satisfies the following conditions:
(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $a*e^{i\theta}=a$, for all $a\in e^{i\theta}$, where $r_s\in[0,1]$,
(4) $a*b\preceq c*d$ whenever $a\preceq c$ and $b\preceq d$, for all $a,b,c,d\in r_se^{i\theta}$, where $r_s\in[0,1]$.

Example 2.4. *[20] $a*b$ = $min(a,b)$.*

Example 2.5. *[20] $a*b=max(a+b-e^{i\theta},0)$, for a fix $\theta\in[0,\frac{\pi}{2}]$.*

Example 2.6. *[20]*

$$a*b=\begin{cases}min\{a,b\}, & if\ max\{a,b\}=e^{i\theta},\\ 0 & otherwise\end{cases}$$

for a fix $\theta\in[0,\frac{\pi}{2}]$. Where min and max are defined in definition [17].

Definition 2.7. [20] The triplet $(X,M,*)$ is said to be complex valued fuzzy metric space if X is an arbitrary non empty set, $*$ is a complex valued continuous t norm and $M:X\times X\times(0,\infty)\to r_se^{i\theta}$ is a complex valued fuzzy set, where $r_s\in[0,1]$ and $\theta\in[0,\frac{\pi}{2}]$ satisfying the following conditions:
(BCF-1) $M(x,y,t)\succ 0$.

(BCF-2) $M(x,y,t) = e^{i\theta}$ for all $t > 0$ iff $x = y$.
(BCF-3) $M(x,y,t) = M(y,x,t)$.
(BCF-4) $M(x,y,t) * M(y,z,s) \succeq M(x,z,t+s)$.
(BCF-5) $M(x,y,\cdot) : (0,\infty) \to r_s e^{i\theta}$ is continuous.
for all $x,y,z \in X, s,t > 0, r_s \in [0,1]$ and $\theta \in [0, \frac{\pi}{2}]$. $(M,*)$ is called a complex valued fuzzy metric spaces.

This study aims to introduce and define the concept of tricomplex valued fuzzy metric spaces, utilizing continuous t-norms and establishing a Hausdorff topology specific to tricomplex valued fuzzy metric spaces. Our objective further involves proving a Banach-type fixed-point theorem and deriving additional fixed-point results within this framework.
In the course of our investigation, we extend and enhance several existing fixed-point theorems documented in recent literature, including contributions by Choi [7], George [8], Segre [18], and Singh [20]. To validate the efficacy of our results, we provide a concrete example, demonstrating their applicability in practice.
Additionally, we present an application that underscores the practical utility of our findings. Through these endeavors, our work contributes to the ongoing development of the theory of tricomplex valued fuzzy metric spaces and offers valuable insights for further exploration in this mathematical domain.

3. Tricomplex valued fuzzy metric spaces

Within this section, we embark on the introduction of novel definitions crucial to our study. We commence by establishing the concept of a tricomplex valued continuous t-norm, laying the foundation for subsequent developments in our exploration. Building upon this, we delve into the definition of tricomplex valued fuzzy metric spaces, providing a framework for understanding distance and relationships within this context.
To elucidate and validate our definitions, we present illustrative examples that serve as concrete instances, attesting to the coherence and applicability of our introduced concepts. Through these definitions and accompanying examples, we aim to provide a clear and robust framework for the subsequent analyses and theorems in our study.

Definition 3.1. A binary operation $* : r_s(1+i_2)(1+i_3)e^{i_1\theta} \times r_s(1+i_2)(1+i_3)e^{i_1\theta} \to r_s(1+i_2)(1+i_3)e^{i_1\theta}$, where in $r_s \in [0,1]$ and a fix $\theta \in [\theta, \frac{\pi}{2}]$, is called tricomplex valued continuous t-norm if it satisfies the following conditions:
(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $a * (1+i_2)(1+i_3)e^{i_1\theta} = a$, for all $a \in (1+i_2)(1+i_3)e^{i_1\theta}$, where $r_s \in [0,1]$,
(4) $a * b \preceq_{i_2} c * d$ whenever $a \preceq c$ and $b \preceq d$, for all $a,b,c,d \in r_s(1+i_2)(1+i_3)e^{i_1\theta}$, where $r_s \in [0,1]$.

Definition 3.2. The triplet $(X,M,*)$ is said to be tricomplex valued fuzzy metric space if X is an arbitrary non empty set, $*$ is a tricomplex valued continuous t norm and $M : X \times X \times (0,\infty) \to r_s(1+i_2)(1+i_3)e^{i_1\theta}$ is a tricomplex valued fuzzy set, where $r_s \in [0,1]$ and $\theta \in [0, \frac{\pi}{2}]$ satisfying the following conditions:
(BCF-1) $M(x,y,t) \succ_{i_2} 0$.
(BCF-2) $M(x,y,t) = (1+i_2)(1+i_3)e^{i_1\theta}$ for all $t > 0$ iff $x = y$.
(BCF-3) $M(x,y,t) = M(y,x,t)$.
(BCF-4) $M(x,y,t) * M(y,z,s) \succeq_{i_2} M(x,z,t+s)$.
(BCF-5) $M(x,y,\cdot) : (0,\infty) \to r_s(1+i_2)(1+i_3)e^{i_1\theta}$ is continuous.
for all $x,y,z \in X, s,t > 0, r_s \in [0,1]$ and $\theta \in [0, \frac{\pi}{2}]$.
$(X,M,*)$ is called a tricomplex valued fuzzy metric spaces.

Example 3.3. *Let $X = \mathbb{R}$. We define $a * b = min\{a,b\}$, for all $a,b \in r_s(1+i_2)(1+i_3)e^{i_1\theta}$, where $r_s \in [0,1]$ and $\theta \in [0, \frac{\pi}{2}]$ and $g : \mathbb{R}^+ \to (0,\infty)$ be an increasing continuous function.*

$$M(x,y,t) = (1+i_2)(1+i_3)e^{i_1\theta} - \frac{d(x,y)}{g(t)}$$

for all $x,y \in X$ and $t \in (0,\infty)$. Then $(X,M,*)$ is a tricomplex valued fuzzy metric space.

Remark 3.4. If we take $g(t) = t$ as a identity function in Example 3.3, then we get

$$M(x,y,t) = (1+i_2)(1+i_3)e^{i_1\theta} - \frac{d(x,y)}{t}$$

for all $x,y \in X$ and $t \in (0,\infty)$. Then $(X,M,*)$ is a tricomplex valued fuzzy metric space.

Remark 3.5. If we take $g(t) = k > 0$ as a constant function in Example 3.3, then we get

$$M(x,y,t) = (1+i_2)(1+i_3)e^{i_1\theta} - \frac{d(x,y)}{k}$$

for all $x,y \in X$ and $t \in (0,\infty)$. Then $(X,M,*)$ is a tricomplex valued fuzzy metric space.

Example 3.6. *Let $X = \mathbb{R}^+$. We define $a * b = min\{a,b\}$, for all $a,b \in r_s(1+i_2)(1+i_3)e^{i_1\theta}$, where $r_s \in [0,1]$ and $\theta \in [0, \frac{\pi}{2}]$ and*

$$M(x,y,t) = (1+i_2)(1+i_3)e^{i_1\theta}\frac{t}{t+d(x,y)}$$

for all $x,y \in X$, $t \in (0,\infty)$ and $d(x,y) = |x-y|$. Then $(X,M,*)$ is a tricomplex valued fuzzy metric space.

Example 3.7. *Let $X = N$. We define $a*b = max(a+b-(1+i_2)(1+i_3)e^{i_1\theta}, 0)$, for all $a,b \in r_s(1+i_2)(1+i_3)e^{i_1\theta}$, where $r_s \in [0,1]$ and $\theta \in [0, \frac{\pi}{2}]$ and*

$$M(x,y,t) = \begin{cases} (1+i_2)(1+i_3)e^{i_1\theta} & if\, x = y \\ txy(1+i_2)(1+i_3)e^{i_1\theta} & if\, x \neq y \text{ and } t \leq 1 \\ xy(1+i_2)(1+i_3)e^{i_1\theta} & if\, x \neq y \text{ and } t > 1 \end{cases}$$

for all $x,y \in X$ and $t \in (0,\infty)$. Then $(X,M,*)$ is a tricomplex valued fuzzy metric space.

4. Topology Induced by a tricomplex valued fuzzy metric spaces

We introduce some new definitions Open ball, Interior point, Hausdorff space, Boundedness, Cauchy sequence, Limit point, Closure of the Set, Closed ball and Convergent as follows.

Definition 4.1. Let $(X,M,*)$ be a tricomplex valued fuzzy metric spaces. We define an open ball $B(x,r,t)$ with centre $x \in X$ and radius $r \in \mathbb{C}_2, 0 \prec r \prec (1+i_2)(1+i_3)e^{i_1\theta}, t > 0$ as

$$B(x,r,t) = \{y \in X : M(x,y,t) \succ (1+i_2)(1+i_3)e^{i_1\theta},$$

where $\theta \in [0, \frac{\pi}{2}]$.
A point $x \in X$ is called an interior point of set $A \subset X$, whenever there exists $r \in \mathbb{C}_2, 0 \prec r \prec (1+i_2)(1+i_3)e^{i_1\theta}$ such that

$$B(x,r,t) = \{y \in X : M(x,y,t) \succ (1+i_2)(1+i_3)e^{i_1\theta} - r\} \subset A,$$

where $\theta \in [0, \frac{\pi}{2}]$.
The subset A of X is called open whenever each element of A is an interior point of A.

Definition 4.2. Let $(X,M,*)$ be a tricomplex valued fuzzy metric spaces. Then, $(X,M,*)$ is called a Hausdorff space if for any two distinct points $p,q \in X$, there exist two open balls $B(p,r_1,t_1) = B(p,(1+i_2)(1+i_3)e^{i_1\theta} - r_1, \frac{t}{2})$ and $B(q,r_2,t_2) = B(q,(1+i_2)(1+i_3)e^{i_1\theta} - r_2, \frac{t}{2})$ such that $B(p,r_1,t_1) \cap B(q,r_2,t_2) = \phi$.

Definition 4.3. Let $(X,M,*)$ be a tricomplex valued fuzzy metric spaces. A subset A of X is said to be bounded if and only if there exist $t > 0$ and $r \in \mathbb{C}_2, 0 \prec r \prec (1+i_2)(1+i_3)e^{i_1\theta}$ such that

$$M(x,y,t) \succ (1+i_2)(1+i_3)e^{i_1\theta} - r, \;\; for\ all\ x,y \in A.$$

Definition 4.4. A sequence x_n in a tricomplex valued fuzzy metric space $(X,M,*)$ is a Cauchy sequence if and only if

$$\lim_{n\to\infty} M(x_{n+p},x_n,t) = (1+i_2)(1+i_3)e^{i_1\theta}, p > 0, t > 0$$

or

$$\lim_{n\to\infty} |M(x_{n+p},x_n,t)| = 1, p > 0, t > 0.$$

Definition 4.5. A tricomplex valued fuzzy metric space in which every Cauchy sequence is convergent, is called tricomplex valued complete fuzzy metric spaces.

Definition 4.6. A point $x \in X$ is called limit point of a subset A of X whenever there exists $r \in \mathbb{C}_2, 0 \prec r \prec (1+i_2)(1+i_3)e^{i_1\theta}$, such that $B(x,r,t) \cap (A/X) \neq \phi$. A subset B of X is **closed** whenever each limit point of B belongs to B.

Definition 4.7. A tricomplex valued fuzzy metric space in which a subset $B(x,r,t)$ of X whenever there exists $r \in \mathbb{C}_2, 0 \prec r \prec (1+i_2)(1+i_3)e^{i_1\theta}$, such that $\overline{B[x,r,t]}$ the closure of $B(x,r,t)$, to be the set consisting of all the points of $B(x,r,t)$ together with all the limit points of $B(x,r,t)$.

Definition 4.8. Let $(X,M,*)$ be a tricomplex valued fuzzy metric space. We define a closed ball $B[x,r,t]$ with centre $x \in X$ and radius $r \in \mathbb{C}_2(0 \in r \in (1+i_2)(1+i_3)e^{i_1\theta}$ and for all $t > 0$ by

$$B[x,r,t] = \{y \in X : M(x,y,t) \in (1+i_2)(1+i_3)e^{i_1\theta} - r\}.$$

Definition 4.9. A tricomplex valued fuzzy metric space in which, a sequence $\{x_n\} \in X$ is convergent to $x_n \to x$ if and only if $M(x_n,x,t) \to (1+i_2)(1+i_3)e^{i_1\theta}$ as $n \to \infty$ or $|M(x_n,x,t)| \to 1$.

Proposition 4.10. *Every open ball is an open set in tricomplex valued fuzzy metric spaces.*

Proof. Consider an open ball $B(x,r,t)$. To show $B(x,r,t)$ to be open we show that at every point of $B(x,r,t)$, there exists an open ball contained in $B(x,r,t)$.
Let

$$y \in B(x,r,t) \Rightarrow M(x,y,t) \succ (1+i_2)(1+i_3)e^{i_1\theta} - r,$$

where $r \in \mathbb{C}_2$ and $0 \prec r \prec (1+i_2)(1+i_3)e^{i_1\theta}$, thus we can find a $t_0, 0 < t_0 < t$ such that $M(x,y,t_0) \succ (1+i_2)(1+i_3)e^{i_1\theta} - r$.
Let $r_0 = M(x,y,t_0) \succ (1+i_2)(1+i_3)e^{i_1\theta} - r$. Then we can find s, where $0 \prec s \prec (1+i_2)(1+i_3)e^{i_1\theta}$, such that

$$r_0 \succ (1+i_2)(1+i_3)e^{i_1\theta} - s \succ (1+i_2)(1+i_3)e^{i_1\theta} - r.$$

For given r_0 and s, where $r_0 \succ (1+i_2)(1+i_3)e^{i_1\theta} - s$, we can find $r_1, 0 \prec r_1 \prec (1+i_2)(1+i_3)e^{i_1\theta}$ such that $r_0 * r_1 \succeq (1+i_2)(1+i_3)e^{i_1\theta} - s$. Consider the ball $B(y,(1+i_2)(1+i_3)e^{i_1\theta} - r_1, t-t_0)$. We assert that $B(y,(1+i_2)(1+i_3)e^{i_1\theta} - r_1, t-t_0) \subset B(x,r,t)$. Let

$$\begin{aligned} z \in B(y,(1+i_2)(1+i_3)e^{i_1\theta} - r_1, t-t_0) \Rightarrow M(y,z,t-t_0) \\ &\succ (1+i_2)(1+i_3)e^{i_1\theta} - ((1+i_2)(1+i_3)e^{i_1\theta} - r_1) \\ &= r_1. \end{aligned}$$

since $M(y,z,t-t_0) \succ r_1$.
Consider

$$\begin{aligned} M(x,z,t) &\succeq M(x,y,t_0) * M(y,z,t-t_0) \\ &\succeq r_0 * r_1 \\ &\succeq (1+i_2)(1+i_3)e^{i_1\theta} - s \\ &\succeq (1+i_2)(1+i_3)e^{i_1\theta} - r. \end{aligned}$$

which gives $M(x,z,t) \succ (1+i_2)(1+i_3)e^{i_1\theta} - r$. Which amounts to say that $z \in B(x,r,t)$. Hence $B(y,(1+i_2)(1+i_3)e^{i_1\theta} - r_1, t-t_0) \in B(x,r,t)$. This shows that $B(x,r,t)$ is an open set. □

Proposition 4.11. *Every tricomplex valued fuzzy metric space is Hausdorff .*

Proof. Let $(X,M,*)$ be a tricomplex valued fuzzy metric space. Let p,q be two distinct points of X. Then

$0 \prec M(x,y,t)(1+i_2)(1+i_3)e^{i_1\theta}$.

Let $M(x,y,t) = r$, for some $r \in \mathbb{C}_2$ then $0 \prec r \prec (1+i_2)(1+i_3)e^{i_1\theta}$. For each $r_0(r \prec r_0 \prec (1+i_2)(1+i_3)e^{i_1\theta})$, we can find a $r_1(r_1 \prec (1+i_2)(1+i_3)e^{i_1\theta})$ such that $r_1 * r_1 \succeq r_0$.
Now consider two open balls $B(p,(1+i_2)(1+i_3)e^{i_1\theta} - r_1, \frac{t}{2})$ and $B(q,(1+i_2)(1+i_3)e^{i_1\theta} - r_2, \frac{t}{2})$. Certainly

$$B(p,(1+i_2)(1+i_3)e^{i_1\theta} - r_1, \frac{t}{2}) \cap B(q,(1+i_2)(1+i_3)e^{i_1\theta} - r_2, \frac{t}{2}) = \phi.$$

If not then there exists

$$s \in B(p,(1+i_2)(1+i_3)e^{i_1\theta} - r_1, \frac{t}{2}) \cap B(q,(1+i_2)(1+i_3)e^{i_1\theta} - r_2, \frac{t}{2}).$$

Now consider

$$\begin{aligned} r &= M(x,y,t) \\ &\succeq M(p,s,\frac{t}{2}) * M(s,q,\frac{t}{2}) \\ &\succ r_1 * r_2 \succeq r_0. \end{aligned}$$

Which is a contradiction. Therefore $(X,M,*)$ is Hausdorff. □

Lemma 4.12. *Every closed ball is closed set in tricomplex valued fuzzy metric spaces.*

Proof. Let $(X,M,*)$ be a tricomplex valued fuzzy metric space and let $B[x,r,t]$ be closed ball in X. Let $z \in \overline{B[x,r,t]}$. Since X is first countable then there exists a sequence $\{z_n\}$ in $B[x,r,t]$ such that $z_n \to z$. Therefore $M(z_n,z,t) \to (1+i_2)(1+i_3)e^{i_1\theta}$ as $n \to \infty$ for all $t > 0$.
For a given $\varepsilon > 0$,

$$M(x,z,t+\varepsilon) \succeq M(x,z_n,t) * M(z_n,z,\varepsilon).$$

Hence

$$\begin{aligned} M(x,z,t+\varepsilon) &\succeq \lim_{n\to\infty} M(x,z_n,t) * \lim_{n\to\infty} M(z_n,z,\varepsilon) \\ &\succeq ((1+i_2)(1+i_3)e^{i_1\theta} - r) * (1+i_2)(1+i_3)e^{i_1\theta} \\ &= (1+i_2)(1+i_3)e^{i_1\theta} - r. \end{aligned}$$

If $M(x,z_n,t)$ is bounded then sequence $\{z_n\}$ has a sub-sequence, which can be again denoted by $\{z_n\}$ for which $\lim_{n\to\infty} M(x,z_n,t)$exists. In a particular case for $n \in \mathbb{N}$, taking $\varepsilon = \frac{1}{n}$. Then $M(x,z,t+\frac{1}{n}) \succeq (1+i_2)(1+i_3)e^{i_1\theta} - r$. Hence

$$\begin{aligned} M(x,z,t) &= \lim_{n\to\infty} M(x,z,t+\frac{1}{n}) \\ &\succeq (1+i_2)(1+i_3)e^{i_1\theta} - r. \end{aligned}$$

Which leads to $z \in B[x,r,t]$. This implies $\overline{B[x,r,t]} \subseteq B[x,r,t]$. But $B[x,r,t] \subseteq \overline{B[x,r,t]}$ always. Thus we have $\overline{B[x,r,t]} = B[x,r,t]$. Therefore $B[x,r,t]$ is a closed set. □

5. Main Result

We prove the Banach Contraction Theorem in the tricomplex valued complete fuzzy metric spaces as follows.

Theorem 5.1. *Let $(X,M,*)$ be a tricomplex valued complete fuzzy metric space such that*

$$\lim_{t\to\infty} M(x,y,t)=(1+i_2)(1+i_3)e^{i_1\theta},\ \ for\ all\ \ x,y\in X\ and\ t>0. \tag{5.1}$$

Let $T:X\to X$ be a mapping satisfying

$$\eta(\frac{1}{M(Tx,Ty,t)}-\frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}})\preceq_{i_3} k\eta(\frac{1}{M(x,y,t)}-\frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}), \tag{5.2}$$

for all $x,y\in X$ and $0<k<1$. Then T has a unique fixed point.

Proof. Suppose T satisfies condition (5.2). Let a_0 be an arbitrary point in X and We define a sequence $\{a_n\}$ in X by

$a_{n+1}=T_{a_n}, n=0,1,2,\ldots$

Applying condition (5.2) with $x=a_n$ and $y=a_{n+1}$, we have

$$\begin{aligned}\eta(\frac{1}{M(a_n,a_{n+1},t)}-\frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}) &= \eta(\frac{1}{M(Ta_{n-1},a_n,t)}-\frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}})\\ &\preceq_{i_3} k\eta(\frac{1}{M(a_{n-1},a_n,t)}-\frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}})\\ &\preceq_{i_3} k^2\eta(\frac{1}{M(a_{n-2},a_{n-1},t)}-\frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}})\\ &\vdots\\ &\preceq_{i_3} k^n\eta(\frac{1}{M(a_0,a_1,t)}-\frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}).\end{aligned}$$

Thus for any positive integer m and using $(TCF-4)$, we have

$$\begin{aligned}\eta(\frac{1}{M(a_n,a_{n+m},t)}-\frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}) &\preceq_{i_3} k\eta((\frac{1}{M(a_n,a_{n+1},t)}-\frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}})*\\ &\quad\cdots*(\frac{1}{M(a_{n+m-1},a_{n+m},t)}-\frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}))\\ &\preceq_{i_3} k^n\eta((\frac{1}{M(a_0,a_1,t)}-\frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}})*\\ &\quad\cdots*(\frac{1}{M(a_0,a_1,t)}-\frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}})).\end{aligned}$$

Which on letting $n\to\infty$, reduces to

$$\begin{aligned}\lim_{n\to\infty}(\eta(\frac{1}{M(a_n,a_{n+m},t)}-\frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}})) \preceq_{i_3}\preceq_{i_3} k^n\eta((\frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}-\frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}})*\\ \cdots*(\frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}-\frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}})).\end{aligned} \tag{5.3}$$

equation (5.3) gives rise to

$$\lim_{n\to\infty}(\eta(\frac{1}{M(a_n,a_{n+m},t)}-(1+i_2)(1+i_3)e^{i_1\theta}))\preceq_{i_3} 0$$

which implies that

$$\eta(\lim_{n\to\infty}(\frac{1}{M(a_n,a_{n+m},t)}-(1+i_2)(1+i_3)e^{i_1\theta}))\preceq_{i_3} 0,$$

since η is continuous. in view of Remark 3.1, we conclude that

$$\lim_{n\to\infty} M(a_n,a_{n+m},t)=(1+i_2)(1+i_3)e^{i_1\theta}.$$

We assert that $\{a_n\}$ is Cauchy sequence in X. Since X is complete, then essentially $a_n \to u$ as $n \to \infty$, where $u \in X$. Consequently

$$\begin{aligned}
\eta(\frac{1}{M(Tu,u,t)} - \frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}) \preceq_{i_3} & k\eta((\frac{1}{M(Tu,a_{n+1},\frac{t}{2})} - \frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}) \\
& *(\frac{1}{M(a_{n+1},u,\frac{t}{2})} - \frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}})) \\
& \preceq_{i_3} k\eta((\frac{1}{M(Tu,Ta_n,\frac{t}{2})} - \frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}) \\
& *(\frac{1}{M(a_{n+1},u,\frac{t}{2})} - \frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}})) \\
& \preceq_{i_3} k\eta((\frac{1}{M(u,a_n,\frac{t}{2k})} - \frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}) \\
& *(\frac{1}{M(a_{n+1},u,\frac{t}{2})} - \frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}))
\end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned}
\eta(\frac{1}{M(Tu,u,t)} - \frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}) \preceq_{i_3} & k\eta((\frac{1}{M(u,u,\frac{t}{2k})} - \frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}) \\
& *(\frac{1}{M(u,u,\frac{t}{2})} - \frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}))
\end{aligned}$$

Now by $(TCF-2)$, we have

$$\begin{aligned}
\eta(\frac{1}{M(Tu,u,t)} - \frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}) \preceq_{i_3} & k\eta((\frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}} - \frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}) \\
& *(\frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}} - \frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}))
\end{aligned}$$

Or

$M(Tu,u,t) = (1+i_2)(1+i_3)e^{i_1\theta}$.

Which implies that $Tu = u$. Thus u is a fixed point of T . According to the uniqueness of fixed point, assume $w \in X$ be another fixed point of T such that $w \neq u$. The inequality turns into

$$\begin{aligned}
\frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}} \preceq_{i_3} & \preceq_{i_3} \eta(\frac{1}{M(u,w,t)} - \frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}) \\
& = \eta(\frac{1}{M(Tu,Tw,t)} - \frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}) \\
& \succeq_{i_3} k\eta(\frac{1}{M(u,w,t)} - \frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}) \\
& \preceq_{i_3} k^2\eta(\frac{1}{M(u,w,t)} - \frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}) \\
& \quad \cdots \\
& \preceq_{i_3} k^n\eta(\frac{1}{M(u,w,t)} - \frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}),
\end{aligned}$$

which implies that

$$\frac{1}{2e^{\frac{i_1-i_2}{2}\theta}} \preceq_{i_3} k^n\eta(\frac{1}{M(u,w,t)} - \frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}).$$

thus we obtain

$(1+i_2)(1+i_3)e^{i_1\theta} \succeq_{i_3} M(u,w,t)$.

Since $k < 1$, then on making $n \to \infty$, we gets $u = w$. Thus, we conclude that T has a unique fixed point. □

Now, we furnish an example which shows the superiority of our result.

Example 5.2. *Let $X = \{0\} \cup \mathbb{N}$. We define $a * b = max(a+b-(1+i_2)(1+i_3)e^{i_1\theta}, 0)$, for all $a,b \in r_s(1+i_2)(1+i_3)e^{i_1\theta}$, where $r_s \in [0,1]$ and $\theta \in [0,\frac{\pi}{2}]$ and*

$$M(x,y,t) = \begin{cases} (1+i_2)(1+i_3)e^{i_1\theta} & if\, x = y \\ \frac{x}{y}(1+i_2)(1+i_3)e^{i_1\theta} & if\, x < y \\ \frac{y}{x}(1+i_2)(1+i_3)e^{i_1\theta} & if\, y < x \end{cases}$$

for all $x, y \in X$ *and* $t \in (0,\infty)$. *Then* $(X, M, *)$ *is a tricomplex valued fuzzy metric space. Certainly here*

$$\lim_{t\to\infty} M(x,y,t) = (1+i_2)(1+i_3)e^{i_1\theta},$$

for all $x, y \in X$ *and* $t \in (0,\infty)$. *Define* $T(x) = \frac{x}{5}$. *By a routine calculation, one can verify that* T *satisfies the condition*

$$M(Tx, Ty, kt) \succeq_{i_3} M(x,y,t), \quad \textit{for all } x, y \in X,$$

for $k = \frac{1}{4}$. *Thus all the conditions of Theorem* 5.1 *are satisfied and* $x = 0$ *is the unique fixed point of* T .

6. An $\alpha - \eta - \psi - \phi-$ Contraction Function

Definition 6.1. Let $(X, M, *)$ be a tricomplex valued fuzzy metric space. Let $T : X \to X$ be satisfies the following

$$\psi\Big(\frac{1}{M(Tx,Ty,t)} - \frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}\Big) \preceq_{i_3} \psi\Big(\frac{1}{M(x,y,t)} - \frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}\Big) - \phi\Big(\frac{1}{M(x,y,t)} - \frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}\Big),$$

for all $x, y \in X$ and $0 < k < 1$, such that $(1+i_2)(1+i_3)e^{i_1\theta}$
(i) ψ is continuous and decreasing with $\psi(t) = 0$ if and only if $t = 0$.
(ii) ϕ is continuous with $\phi(t) = 0$ if and only if $t = 0$.
f is called $\psi - \phi-$ contraction function.

Theorem 6.2. *Let* $(X, M, *)$ *be a tricomplex valued fuzzy metric space. Let* $T : X \to X$ *be satisfies the following*

$$\alpha(x, Tx, t)\alpha(y, Ty, t) \geq \eta(x, Tx, t)\eta(y, Ty, t)$$

this implies that

$$\psi\Big(\frac{1}{M(Tx,Ty,t)} - \frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}\Big) \preceq_{i_3} \psi\Big(\frac{1}{M(x,y,t)} - \frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}\Big) - \phi\Big(\frac{1}{M(x,y,t)} - \frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}\Big), \tag{6.1}$$

such that
(i) There exists $a_0 \in X$ *such that* $\alpha(a_0, T(x_0), t) \geq \eta(a_0, T(x_0), t)$ *for all* $t > 0$.

Then T *has a unique fixed point.*

Proof. Suppose T satisfies condition (6.1). Let a_0 be an arbitrary point in X and We define a sequence $\{a_n\}$ in X by

$$a_{n+1} = T_{a_n}, n = 0, 1, 2, \ldots$$

Since T is $\alpha-$ admissible with respect to η such that

$$\alpha(a_0, T(x_0)) \geq \eta(a_0, T(x_0), t)$$

Continuing this process we get

$$\alpha(a_n, Ta_{n+1}, t) \geq \eta(a_n, a_{n+1}, t)$$

clearly

$$\alpha(a_{n-1}, Ta_{n-1}, t)\alpha(a_n, Ta_n, t) \geq \eta(a_{n-1}, Ta_{n-1}, t)\eta(a_n, Ta_n, t).$$

Applying condition (6.1) with $x = a_{n-1}$ and $y = a_n$, we have

$$\psi\Big(\frac{1}{M(Ta_{n-1},Ta_n,t)} - \frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}\Big) \preceq_{i_3} \psi\Big(\frac{1}{M(a_{n-1},a_n,t)} - \frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}\Big) - \phi\Big(\frac{1}{M(a_{n-1},a_n,t)} - \frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}\Big),$$

which implies that

$$\psi\Big(\frac{1}{M(a_n,a_{n+1},t)} - \frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}\Big) \preceq_{i_3} \psi\Big(\frac{1}{M(a_{n-1},a_n,t)} - \frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}\Big) - \phi\Big(\frac{1}{M(a_{n-1},a_n,t)} - \frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}\Big).$$

If $M(a_{n-1},a_n,t)=(1+i_2)(1+i_3)e^{i_1\theta}$, then $M(a_n,a_{n+1},t)=(1+i_2)(1+i_3)e^{i_1\theta}$. Otherwise, if $M(a_{n-1},a_n,t)<(1+i_2)(1+i_3)e^{i_1\theta}$, then

$$\psi(\frac{1}{M(a_n,a_{n+1},t)}-\frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}})\prec_{i_3}\psi(\frac{1}{M(a_{n-1},a_n,t)}-\frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}),$$

since ψ is non-decreasing. Thus for any positive integer m and using $(TCF-4)$, we have

$$\begin{aligned}M(a_n,a_{n+m},t)&\succeq_{i_3} M(a_n,a_{n+1},\frac{t}{k})*M(a_{n+1},a_{n+2},\frac{t}{k})*\\&\cdots*M(a_{n+p-1},a_{n+p},\frac{t}{k})\\&\succeq_{i_3}(1+i_2)(1+i_3)e^{i_1\theta}*\cdots*(1+i_2)(1+i_3)e^{i_1\theta}.\end{aligned}$$

We conclude that

$$\lim_{n\to\infty}M(a_n,a_{n+m},t)=(1+i_2)(1+i_3)e^{i_1\theta}.$$

We assert that $\{a_n\}$ is Cauchy sequence in X. Since X is complete, then essentially $a_n\to u$ as $n\to\infty$, where $u\in X$.

$$\begin{aligned}\psi(\frac{1}{M(Tu,u,t)}-\frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}})&\preceq_{i_3}\psi(\frac{1}{M(u,u,t)}-\frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}})\\&-\phi(\frac{1}{M(u,u,t)}-\frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}).\end{aligned}$$

If $M(Tu,u,t)=(1+i_2)(1+i_3)e^{i_1\theta}$, Otherwise, if $M(Tu,u,t)<(1+i_2)(1+i_3)e^{i_1\theta}$, then

$$\psi(\frac{1}{M(Tu,u,t)}-\frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}})\prec_{i_3}\psi(\frac{1}{M(u,u,t)}-\frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}),$$

Or

$$M(Tu,u,t)=(1+i_2)(1+i_3)e^{i_1\theta}.$$

Which implies that $Tu=u$. Thus u is a fixed point of T . According to the uniqueness of fixed point, assume $w\in X$ be another fixed point of T such that $w\neq u$. The inequality turns into

$$\psi(\frac{1}{M(Tu,w,t)}-\frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}})\prec_{i_3}\psi(\frac{1}{M(u,w,t)}-\frac{1}{(1+i_2)(1+i_3)e^{i_1\theta}}),$$

thus we obtain

$$(1+i_2)(1+i_3)e^{i_1\theta}\succeq_{i_3}M(u,w,t).$$

Since $k<1$, then on making $n\to\infty$, we gets $u=w$. Thus, we conclude that T has a unique fixed point. □

Theorem 6.3. *Let $(X,M,*)$ be a tricomplex valued complete fuzzy metric spaces such that*

$$\lim_{t\to\infty}M(x,y,t)=(1+i_2)(1+i_3)e^{i_1\theta}, \tag{6.2}$$

for all $x,y\in X$ and $t>0$. Let $T:X\to X$ be a mapping satisfying

$$M(Tx,Ty,kt)\succeq_{i_3} min\{M(x,y,t),M(x,Tx,t),M(y,Ty,t),\frac{M(x,Tx,t),M(y,Ty,t)}{M(x,y,t)}\}, \tag{6.3}$$

for all $x,y\in X$ and $0<k<1$. Then T has a unique fixed point.

Proof. Suppose T satisfies equation (6.3). Let a_0 be an arbitrary point in X and we define a sequence $\{a_n\}$ in X by

$$a_{n+1}=T_{a_n}, n=0,1,2,\ldots$$

we discuss two cases to get the desired fixed point.
Case-I When $x_n\neq x_{n+1}$. Applying condition (6.3) with $x=a_n$ and $y=a_{n+1}$, we have

$$\begin{aligned}M(a_n,a_{n+1},t)&=M(Ta_{n-1},Ta_n,t)\\&\succeq_{i_3} min\{M(a_n,a_{n+1},t),M(a_n,Ta_n,t),M(a_{n+1},Ta_{n+1},t)\}\\&\succeq_{i_3} min\{M(a_n,a_{n+1},t),M(a_n,a_{n+1},t),M(a_{n+1},a_{n+2},t)\}\\&\succeq_{i_3} min\{M(a_n,a_{n+1},t),M(a_{n+1},a_{n+2},t)\}\end{aligned} \tag{6.4}$$

Now suppose

$$min\{M(a_n,a_{n+1},t),M(a_{n+1},a_{n+2},t)\}=M(a_n,a_{n+1},t)$$

we deduce that $M(a_n,a_{n+1},t) \succeq_{i_3} M(a_n,a_{n+1},t)$. This leads to a contradiction, therefore by (6.3), we must have

$$\begin{aligned} M(a_n,a_{n+1},t) &\succeq_{i_2} M(a_{n-1},a_n,\frac{t}{k}) \\ &\succeq_{i_3} M(a_{n-2},a_{n-1},\frac{t}{k^2}) \\ &\succeq_{i_3} M(a_{n-3},a_{n-2},\frac{t}{k^3}) \\ &\quad\vdots \\ &\succeq_{i_3} M(a_0,a_1,\frac{t}{k^n}). \end{aligned} \tag{6.5}$$

In general

$$\begin{aligned} M(a_n,a_{n+m},kt) &\succeq_{i_3} M(a_n,a_{n+1},\frac{t}{k}) * \cdots * M(a_{n+m-1},a_{n+m},\frac{t}{k}) \\ &\succeq_{i_3} M(a_0,a_1,\frac{t}{k^n}) * \cdots * M(a_0,a_1,\frac{t}{k^{n+m-1}}). \end{aligned}$$

Which on letting $n \to \infty$, reduces to

$$\lim_{n\to\infty} M(a_n,a_{n+m},t) \succeq_{i_3} (1+i_2)(1+i_3)e^{i_1\theta} * (1+i_2)(1+i_3)e^{i_1\theta} * \cdots * (1+i_2)(1+i_3)e^{i_1\theta}.$$

Since $k<1$ and $\lim_{n\to\infty} M(x,y,t) = (1+i_2)(1+i_3)e^{i_1\theta}$. Which implies that

$$\lim_{n\to\infty} M(a_n,a_{n+m},t) \succeq_{i_3} (1+i_2)(1+i_3)e^{i_1\theta}.$$

We observe that $\{a_n\}$ is Cauchy sequence in X. Since X is complete, there exists some $u \in X$ such that $x_n \to u$ as $n \to \infty$. Implying thereby the convergence of $\{x_n\}$ and $\{x_{n+1}\}$ being sub-sequences of the convergent sequence $\{x_n\}$. Then $x_n \to u$ and $x_{n+1} \to u$ as $n \to \infty$. Now we shall show that u is a fixed point of T . Setting $x = u$ and $y = x_{n+1}$ in inequality (6.2), one yields

$$\begin{aligned} M(Tu,u,kt) &\succeq_{i_3} M(Tu,a_{n+2},\frac{t}{2}) * M(a_{n+2},u,\frac{t}{2}) \\ &\succeq_{i_3} min\{M(u,u,\frac{t}{2k}),M(u,Tu,\frac{t}{2k}),M(u,u,\frac{t}{2k})\} * M(u,u,\frac{t}{2k}) \\ &= min\{M(u,Tu,\frac{t}{2k}),(1+i_2)(1+i_3)e^{i_1\theta}\} \\ &= M(u,Tu,\frac{t}{2k}) \end{aligned} \tag{6.6}$$

Since $k \in (0,\frac{1}{2})$, we get $Tu = u$. Thus u is a fixed point of mapping T. Hence u is a common fixed point of mappings T. To investigate the uniqueness of common fixed point, let $w \in X$ be another common fixed point of mappings T such that $w \neq u$, then

$$\begin{aligned} M(u,w,kt) &= M(Tu,Tw,kt) \\ &\succeq_{i_3} min\{M(w,Tw,t),M(u,Tu,t),M(u,w,t\} \\ &\succeq_{i_3} min\{M(w,w,t),M(u,u,t),M(u,w,t)\} \\ &= min\{(1+i_2)(1+i_3)e^{i_1\theta},(1+i_2)(1+i_3)e^{i_1\theta},M(u,w,t)\} \end{aligned} \tag{6.7}$$

Since $M(u,w,t) \in (1+i_2)(1+i_3)e^{i_1\theta}$, $r_s \in [0,1]$ and $\theta \in [0,\frac{\pi}{2}]$, also $M(u,w,t) \preceq (1+i_2)(1+i_3)e^{i_1\theta}$, then we certainly have

$$min\{(1+i_2)(1+i_3)e^{i_1\theta},(1+i_2)(1+i_3)e^{i_1\theta},M(u,w,t)\} = M(u,w,t).$$

Therefore we get $M(u,w,kt) \succeq M(u,w,t)$, which yields $u = w$, Thus u is the unique common fixed point of T.
Case II When $x_n = x_{n+1}$, observe that $\{x_n\}$ is a constant sequence and so convergent. This concludes the proof. □

We present example validates the aforesaid theorem.

Example 6.4. *Let $X = \{0\} \cup \mathbb{N}$. We define $a*b = min\{a,b\}$, for all $a,b \in r_s(1+i_2)e^{i_1\theta}$, where $r_s \in [0,1]$ and $\theta \in [0,\frac{\pi}{2}]$, $d(x,y) = |x-y|$ and*

$$M(x,y,t) = (1+i_2)(1+i_3)e^{i_1\theta}e^{\frac{-d(x,y)}{t}}$$

for all $x,y \in X$ and $t \in (0,\infty)$. Then $(X,M,*)$ is a tricomplex valued fuzzy metric space. Certainly here

$$\lim_{t\to\infty} M(x,y,t) = (1+i_2)(1+i_3)e^{i_1\theta},$$

for all $x,y \in X$ and $t \in (0,\infty)$. Define $T(x) = \frac{x}{5}$. By a routine calculation, one can verify that T satisfies the condition

$$M(Tx,Ty,kt) \succeq_{i_3} min\{M(x,y,t),M(x,Tx,t),M(y,Ty,t),\},$$

for all $x,y \in X$ and $k = \frac{1}{3}$. Thus all the conditions of Theorem 6.3 are satisfied and $x = 0$ is the unique fixed point of T .

7. Application

We also provide an application to substantiate the utility of our established result to find the unique solution of the higher degree polynomial equations. Since polynomials are used to describe curves of varied types, people exploitage them in the real world to graph curves. For the example, an engineer is designing a roller coaster would have used polynomials to model the curves, for instance, the roller coaster designers may use polynomials to explain the curves in their rides, while a civil engineer would use polynomials to texture and style of roads, buildings and other structures.
An application on higher degree polynomial equations is as follows having a degree greater or equal to 3 given by [14] and We have elucidate this higher degree polynomial equations in tricomplex valued fuzzy metric spaces. For any natural number $\alpha \geq 3$ and real $|\beta| \leq 1$, the following equation

$$\beta^{\alpha}+1=(\alpha^4-1)\beta^{\alpha+1}+\alpha^4\beta \tag{7.1}$$

has a unique real solution.

Proof. It is not difficult to see that if $|\alpha|>1$, equation (7.1) does not have a solution. So, let $X=\mathbb{C}_3([0,1],\mathbb{R})$. and for all $\alpha, r \in X$, let

$$M(\beta,r,t)=(1+i_2)(1+i_3)e^{i_1\theta}\frac{t}{t+|\beta-r|}.$$

Hence $(X,M,*)$ is a complete tricomplex valued fuzzy metric spaces. Now, let

$$T\beta=\frac{\beta^{\alpha}+1}{(\alpha^6-1)\beta^{\alpha}+\alpha^6} \tag{7.2}$$

Notice that, since $\alpha \geq 2$, we can deduce that $\alpha^4 \geq 6$, for all $t>0$ and $k\in(0,1)$. Thus,

$$\begin{aligned}
M(T\beta,Tr,kt) &= (1+i_2)(1+i_3)e^{i_1\theta}\frac{kt}{kt+|\frac{\beta^{\alpha}+1}{(\alpha^4-1)\beta^{\alpha}+\alpha^4}-\frac{r^{\alpha}+1}{(\alpha^4-1)r^{\alpha}+\alpha^4}|}\\
&= (1+i_2)(1+i_3)e^{i_1\theta}\frac{kt}{kt+|\frac{\beta^{\alpha}-r^{\alpha}}{((\alpha^4-1)\beta^{\alpha}+\alpha^4)((\alpha^4-1)r^{\alpha}+\alpha^4)}|}\\
&\succeq_{i_3} (1+i_2)(1+i_3)e^{i_1\theta}\frac{kt}{kt+|\frac{\beta-r}{\alpha^4}|}\\
&\succeq_{i_3} (1+i_2)(1+i_3)e^{i_1\theta}\frac{t}{t+|\frac{\beta-r}{\alpha^4}|}\\
&\succeq_{i_3} (1+i_2)(1+i_3)e^{i_1\theta}\frac{t}{t+|\frac{\beta-r}{\alpha}|}\\
&\succeq_{i_3} (1+i_2)(1+i_3)e^{i_1\theta}\frac{t}{t+|\beta-r|}\\
&\succeq_{i_3} M(\beta,r,t).
\end{aligned}$$

Therefore all the conditions of Theorem 5.1 are satisfied. Thus, T possesses a unique fixed point in X, and equation (7.1) has a unique real solution. □

Conclusions. In this article, motivated and inspired by the work of Ramot et al.[17], Ismat Beg et al.[10] and Choi et al. [7]. We extend and improve some existing generalize the complex valued fuzzy metric spaces by Singh et al.[20]. In this research endeavor, we extend the topological aspects pertinent to tricomplex valued fuzzy metric spaces. Our thorough investigations yield compelling results, bolstered by the inclusion of pertinent examples that substantiate and reinforce our findings. Furthermore, we demonstrate the practical relevance of our established results by applying them to ascertain the unique solutions of polynomial equations of higher degrees.
This work not only broadens the understanding of topological aspects within the realm of tricomplex valued fuzzy metric spaces but also offers a valuable avenue for practical applications. By paving the way for new insights and methodologies, our research contributes to the ongoing discourse in this specialized field, providing a solid foundation for future exploration and advancement.

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