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***Research paper***

# The Solution of the Intuitionistic Fuzzy Linear Fractional Partial Differential Equations Using the Homotopy Analysis Method

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**Abstract**

The present work is devoted to using an analytic approach, namely the homotopy analysis method, to solve the Fractional Partial Differential Equations with intuitionistic fuzzy initial data under generalized intuitionistic fuzzy Caputo derivative.

***Keywords:*** *Caputo fractional derivative, Generalized intuitionistic fuzzy derivative, Hukuhara difference, Intuitionistic Fuzzy fractional differential equation, The homotopy analysis method.*

## 1. Introduction

In this work , we will used the homotopy analysis method in order to solve the following problem

$$\begin{cases} {}^{C}_{gH}D^{\delta}_{\xi}\mathfrak{u}(\theta,\xi)+c\frac{\partial}{\partial\theta}\mathfrak{u}(\theta,\xi)=\mathscr{G}(\theta,\xi),\ 0\leq\theta,\xi<1; \\ \mathfrak{u}(\theta,0)=\psi(\theta); \\ \frac{\partial}{\partial\theta}\mathfrak{u}(\theta,0)=\Upsilon(\theta). \end{cases} \tag{1.1}$$

where $0<\delta\leq 1$, the operator ${}^{C}_{gH}D^{\delta}$ denote the Caputo fractional generalized derivative of order $\delta$, and $\psi,\Upsilon]0,1[\longrightarrow\mathbb{F}(\mathbb{R})$ and $\Upsilon(\theta)\longrightarrow_{|\theta|\longrightarrow\infty}0$.
The concept of intuitionistic fuzzy sets is intoduced by K. Atanassov [2]. The autors in [7] built the concept of intuitionistic fuzzy metric space and intuionistic fuzzy numbers. In [8] S. Melliani introduce the extension of Hukuhara difference in the intuitionistic fuzzy case. The autors in [9] introduce the concept of intuitionistic fuzzy Laplace's transform. Mehdi Dehghan, Jalil Manafian, and Abbas Saadatmandi in [4] solve the Linear Fractional Partial Differential Equations Using the Homotopy Analysis Method. From this end idea we introduce in this paper the concept of generalized intuitionistic fuzzy caputo derivative, and we give a solution of an intuitionistic fuzzy fractional equation by mean the homotompy analysis method.
This article is structured as follows. In Section 2, we review some concepts about intuitionistic fuzzy numbers. The concepts of generalized intuitionistic fuzzy derivation and generalized intuitionistic fuzzy Caputo derivation appear in Sections 3 and 4. In Section 5, we introduce the intuitionistic fuzzy Laplace transform. The method of homotopy analysis is presented in Section 6 . Finally, in Section 7, we present the solution to problem 1.1.

## 2. preliminaries

**Definition 1.** *[7] The set of intuitionistic fuzzy numbers is defined by:*

$$\mathbb{F}=\mathbb{F}(\mathbb{R})=\{<\mu,\nu>:\mathbb{R}\longrightarrow[0,1]^2\ ,\ 0\leq\mu+\nu\leq 1\},$$

*and it checks the following properties:*

*1- For all $<\mu,\nu>\in\mathbb{F}$ is normal, i.e :*
*There exists $x_0,x_1\in\mathbb{R}$ such that : $\mu(x_0)=1$ and $\nu(x_1)=1$.*

2- For all $<\mu,\nu>\in\mathbb{F}$ is intuitionistic convex, that's to say :
$\mu$ is fuzzy covex : $\mu(\lambda x+(1-\lambda)y)\geq\min\{\mu(x),\mu(y)\}$ , $\forall x,y\in\mathbb{R}$ , $\forall\lambda\in[0,1]$.
$\nu$ is fuzzy concave : $\nu(\lambda x+(1-\lambda)y)\geq\max\{\nu(x),\nu(y)\}$ , $\forall x,y\in\mathbb{R}$ , $\forall\lambda\in[0,1]$.
3- For all $<\mu,\nu>\in\mathbb{F}$, $\mu$ is lower continuous and $\nu$ is appear continuous .
4- $supp<\mu,\nu>=\overline{\{x\in\mathbb{R}\ ,\ \nu(x)<1\}}$ is bounded.

*And we define zero intuitionistic fuzzy by:*

$$\tilde{0}(x)=\begin{cases}(1,0)\ ;\ x=0,\\(0,1)\ ;\ x\neq 0\end{cases}$$

**Definition 2.** *[7] For $\alpha\in[0,1]$, we define the appear and lower $\alpha-$cut as following :*

$$[<\mu,\nu>]_{\alpha}=\{x\in\mathbb{R}\ ,\ \mu(x)\geq\alpha\}.$$

$$[<\mu,\nu>]^{\alpha}=\{x\in\mathbb{R}\ ,\ \nu(x)\leq 1-\alpha\}.$$

*And we can write:*

$$[<\mu,\nu>]_{\alpha}=\left[[<\mu,\nu>]_l^+(\alpha),[<\mu,\nu>]_r^+(\alpha)\right]$$

*and*

$$[<\mu,\nu>]^{\alpha}=\left[[<\mu,\nu>]_l^-(\alpha),[<\mu,\nu>]_r^-(\alpha)\right].$$

**Proposition 1.** *[7] Let $<\mu_1,\nu_1>,<\mu_2,\nu_2>\in\mathbb{F}$, we have :*

1) $<\mu_1,\nu_1>=<\mu_2,\nu_2>\Leftrightarrow[<\mu_1,\nu_1>]_{\alpha}=[<\mu_2,\nu_2>]_{\alpha}$ , $[<\mu_1,\nu_1>]^{\alpha}=[<\mu_2,\nu_2>]^{\alpha}$ , $\forall\alpha\in[0,1]$.
2) $<\mu_1,\nu_1>\oplus<\mu_2,\nu_2>=<\mu_1\vee\nu_1,\mu_2\wedge\nu_2>$,
*and according to the extension of Zadeh, we have:*

$$[<\mu_1,\nu_1>\oplus<\mu_2,\nu_2>]_{\alpha}=[<\mu_1,\nu_1>]_{\alpha}+[<\mu_2,\nu_2>]_{\alpha},$$

$$[<\mu_1,\nu_1>\oplus<\mu_2,\nu_2>]^{\alpha}=[<\mu_1,\nu_1>]^{\alpha}+[<\mu_2,\nu_2>]^{\alpha}.$$

3) $\lambda<\mu_1,\nu_1>=<\lambda\mu_1,\lambda\nu_1>$ , $\forall\lambda\in\mathbb{R}$,
*and according to the extension of Zadeh, we have:*

$$[\lambda<\mu_1,\nu_1>]_{\alpha}=\lambda\,[<\mu_1,\nu_1>]_{\alpha},$$

$$[\lambda<\mu_1,\nu_1>]^{\alpha}=\lambda\,[<\mu_1,\nu_1>]^{\alpha}.$$

*If $\lambda=0$ then: $\lambda<\mu_1,\nu_1>=\tilde{0}$.*

**Theorem 1.** *[7] Let $\mathscr{M}=\{M_{\alpha},M^{\alpha}\ ,\ \alpha\in[0,1]\}$ is the family of subsets of $\mathbb{R}$, verifies the following properties:*

1) $\alpha\leq\beta\Rightarrow M_{\beta}\subset M_{\alpha}$ and $M^{\beta}\subset M^{\alpha}$ for all $\alpha,\beta\in[0,1]$.
2) $M_{\alpha}$ and $M^{\alpha}$ are two non-empty compact convex subsets in $\mathbb{R}$ for all $\alpha\in[0,1]$.
3) For all nondecreansing sequence $\alpha_i\to\alpha$ on $[0,1]$, we have:

$$M_{\alpha}=\cap_i M_{\alpha_i}\ ,\ M^{\alpha}=\cap_i M^{\alpha_i}.$$

*Then, we define $\mu$ and $\nu$ by:*

$$\mu(x)=\begin{cases}0\ ,\ x\notin M_0,\\\sup_{\alpha\in[0,1]}M_{\alpha}\ ,\ x\in M_0\end{cases}$$

$$\nu(x)=\begin{cases}1\ ,\ x\notin M^0,\\1-\sup_{\alpha\in[0,1]}M^{\alpha}\ ,\ x\in M^0\end{cases}$$

*Therefore, $<\mu,\nu>\in\mathbb{F}$, $M_{\alpha}=[<\mu,\nu>]_{\alpha}$ and $M^{\alpha}=[<\mu,\nu>]^{\alpha}$.*

**Remerk 1.** *i- The family $\{[<\mu,\nu>]_{\alpha},[<\mu,\nu>]^{\alpha}\ ,\ \alpha\in[0,1]\}$ satisfies the previous properties of theorem 1.*
*ii- For all $\alpha\in[0,1]$ we have:*

$$[<\mu,\nu>]_{\alpha}\subset[<\mu,\nu>]^{\alpha}.$$

**Definition 3.** *[7] Let $<\mu,\nu>,<x,y>\in\mathbb{F}$, we define the following two distances on $\mathbb{F}$:*

$$\begin{aligned}d_{\infty}(<\mu,\nu>,<x,y>)=&\frac{1}{4}\sup_{\alpha\in(0,1]}|\,[<\mu,\nu>]_r^+(\alpha)-[<x,y>]_r^+(\alpha)\,|\\&+\frac{1}{4}\sup_{\alpha\in(0,1]}|\,[<\mu,\nu>]_l^+(\alpha)-[<x,y>]_l^+(\alpha)\,|\\&+\frac{1}{4}\sup_{\alpha\in(0,1]}|\,[<\mu,\nu>]_r^-(\alpha)-[<x,y>]_r^-(\alpha)\,|\\&+\frac{1}{4}\sup_{\alpha\in(0,1]}|\,[<\mu,\nu>]_l^-(\alpha)-[<x,y>]_l^-(\alpha)\,|,\end{aligned}$$

*and,*

$$d_p(<\mu,\nu>,<x,y>) = (\frac{1}{4}\int_0^1 |[<\mu,\nu>]_r^+(\alpha) - [<x,y>]_r^+(\alpha)|^p\, d\alpha$$
$$+\frac{1}{4}\int_0^1 |[<\mu,\nu>]_l^+(\alpha) - [<x,y>]_l^+(\alpha)|^p\, d\alpha$$
$$+\frac{1}{4}\int_0^1 |[<\mu,\nu>]_r^-(\alpha) - [<x,y>]_r^-(\alpha)|^p\, d\alpha$$
$$+\frac{1}{4}\int_0^1 |[<\mu,\nu>]_l^-(\alpha) - [<x,y>]_l^-(\alpha)|^p\, d\alpha)^{\frac{1}{p}},$$

*then, $(\mathbb{F}, d_p)$ is a complete metric space.*

## 3. The generalized Hukuhara derivative of an intuitionistic fuzzy-valued function

The concept of intuitionistic fuzzy Hukuhra difference is introduced by the autors in [8], in this paper we will give the definition of generalized Hukuhara difference betwen two intuitionistic fuzzy number.

**Definition 4.** *The generalized Hukuhara difference of two intuitionistic fuzzy number $<\mu,\nu>$, $<x,y>\in\mathbb{F}$ is defined as follows*

$$<\mu,\nu> \ominus_{gH} <x,y> = <z,w> \Leftrightarrow <\mu,\nu> = <x,y> \oplus <z,w>.$$

Note that the $(\alpha,\beta)$-level representation of fuzzy-valued function $h : [0,T] \longrightarrow \mathbb{F}$ expressed by $[h]_\alpha = \left[h_{\alpha,l}, h_{\alpha,r}\right]$ and $[h]^\alpha = \left[h^{\beta,l}, h^{\beta,r}\right]$

**Definition 5.** *The generalized Hukuhara derivative of a intuitionistic fuzzy-valued function $h : [0,T] \longrightarrow \mathbb{F}1$ at $t_0$ is defined as*

$$h'_{gH}(t_0) = \lim_{\xi\to\xi_0} \frac{h(\xi) -_{gH} h(\xi_0)}{\xi - \xi_0}$$

*if $h'_{gH}(\xi_0) \in \mathbb{F}$, we say that $h$ is generalized Hukuhara differentiable at $\xi_0$*

Also we say that $h$ is $[(i)-gH]$-differentiable at $\xi_0$ if

$$\begin{cases} \left(h'_{gH}\right)_\alpha = \left[(h_{\alpha,l})', (h_{\alpha,r})'\right] \\ \left(h'_{gH}\right)^\beta = \left[(h^{\beta,l})', (h^{\beta,r})'\right] \end{cases}$$

And that $h$ is $[(ii)-gH]$-differentiable at $\xi_0$ if

$$\begin{cases} \left(h'_{gH}\right)_\alpha = \left[(h_{\alpha,r})', (h_{\alpha,l})'\right] \\ \left(h'_{gH}\right)^\beta = \left[(h^{\beta,r})', (h^{\beta,l})'\right] \end{cases}$$

**Remark 2.** *We can defined the generalized derivative of higher order by*

$$\begin{cases} h^{(0)} = h \\ h^{(n)}_{gH} = \left(h^{(n-1)}\right)'_{gH}, \quad \forall n \in \mathbb{N}^* \end{cases} \tag{3.1}$$

**Theorem 2.** *Let $h(\xi)$ and $h'(\xi)$ are two differentiable intuitionistic fuzzy-valued functions. We set $[h(\xi)]_\alpha = [\underline{h}_\alpha(\xi), \overline{h}_\alpha(\xi)]$ and $[h(\xi)]^\beta = [\underline{h}^\beta(\xi), \overline{h}^\beta(\xi)]$, where $0 \le \alpha + \beta \le 1$*

- *Let $h(\xi)$ and $h'(\xi)$ be (i)-differentiable, or, let $h(\xi)$ and $h'(\xi)$ be (ii)-differentiable; then: $\underline{h}_\alpha(\xi)$,$\overline{h}_\alpha(\xi)$, $\underline{h}^\beta(\xi)$ and $\overline{h}^\beta(\xi)$ have first-order and second-order derivatives and*

  $$\begin{cases} [h''(\xi)]_\alpha = [\underline{h''}_\alpha(\xi), \overline{h''}_\alpha(\xi)] \\ [h''(\xi)]^\beta = [\underline{h''}^\beta(\xi), \overline{h''}^\beta(\xi)] \end{cases}$$

- *Let $h(\xi)$ be (i)-differentiable and $h'(\xi)$ be (ii)-differentiable, or, let $h(\xi)$ be (ii)-differentiable and $h'(\xi)$ be (ii)-differentiable; then: $\underline{h}_\alpha(\xi)$,$\overline{h}_\alpha(\xi)$, $\underline{h}^\beta(\xi)$ and $\overline{h}^\beta(\xi)$ have first-order and second-order derivatives and*

  $$\begin{cases} [h''(\xi)]_\alpha = [\overline{h''}_\alpha(\xi), \underline{h''}_\alpha(\xi)] \\ [h''(\xi)]^\beta = [\overline{h''}^\beta(\xi), \underline{h''}^\beta(\xi)] \end{cases}$$

*Proof.* Just use the proof of theorem 3.1 in [1] for $[h(\xi)]_\alpha$ and $[h(\xi)]^\beta$. □

**Definition 6.** *Let $h : (0,T) \longrightarrow \mathbb{F}$. We say that $h$ of classe $\mathscr{C}^m$, $m \in \mathbb{N}$, if $h^{(m)}_{gh}$ exists and continuous, by respect to metric $d_\infty$.*

Now if the $(\alpha,\beta)$-levels of $h:(0,T)\longrightarrow\mathbb{F}$, are given by $[h]_\alpha=\left[h_{\alpha,l},h_{\alpha,r}\right]$ and $[h]^\beta=\left[h^{\beta,l},h^{\beta,r}\right]$ and $h_{\alpha,l}$, $h_{\alpha,r}$, $h^{\beta,l}$, $h^{\beta,r}$ are Riemann integrable on $[0,T]$. Since the family

$$\left\{\left[h_{\alpha,l},h_{\alpha,r}\right],\ \left[h^{\beta,l},h^{\beta,r}\right]\right\}$$

built an intuitionistic element and the integrale preserve the monotony then by the Theorem 1 the family

$$\left\{\left[\int_{[0,T]}h_{\alpha,l},\int_{[0,T]}h_{\alpha,r}\right],\left[\int_{[0,T]}h^{\beta,l},\int_{[0,T]}h^{\beta,r}\right]\right\}$$

define an intuitionistic fuzzy element, which is the integral of $h$ on $[0,T]$, we denote $\int_{[0,T]}h$

**Definition 7.** *Let $h:[0,T]\longrightarrow\mathbb{F}$ be a intuitionistic fuzzy-valued function, we say that $f$ is integrable on $[0,T]$ if $h_{\alpha,l}$, $h_{\alpha,r}$, $h^{\beta,l}$, $h^{\beta,r}$ defined in the previous are integrable on $[0,T]$*

# 4. Intuitionistic fuzzy generalized caputo-derivative

Let $h:[0,T]\longrightarrow\mathbb{F}$ be a intuitionistic fuzzy-valued integrable function on $[0,T]$, and $\delta\in(m-1,m]$ and $m\in\mathbb{N}^*$
it's $(\alpha,\beta)$-levels are defined by $[h]_\alpha=\left[h_{\alpha,l},h_{\alpha,r}\right]$ and $[h]^\beta=\left[h^{\beta,l},h^{\beta,r}\right]$
where $h_{\alpha,l}$, $h_{\alpha,r}$, $h^{\beta,l}$, $h^{\beta,r}\in\mathscr{C}^m([0,T])$.
We set

$$M_\alpha=\left[\frac{1}{\Gamma(m-\delta)}\int_0^\xi(\xi-s)^{m-\delta-1}\left(h_{\alpha,l}\right)^{(m)}(s),\frac{1}{\Gamma(m-\delta)}\int_0^\xi(\xi-s)^{m-\delta-1}\left(h_{\alpha,r}\right)^{(m)}(s)\right]$$

and

$$M^\beta=\left[\frac{1}{\Gamma(m-\delta)}\int_0^\xi(\xi-s)^{m-\delta-1}\left(h^{\beta,l}\right)^{(m)}(s),\frac{1}{\Gamma(m-\delta)}\int_0^\xi(\xi-s)^{m-\delta-1}\left(h^{\beta,r}\right)^{(m)}(s)\right]$$

**Proposition 2.** *The the family $\left\{M_\alpha,M^\beta,\ \alpha,\ \beta\in[0,1]\right\}$ defined an intuitionistic fuzzy element*

*Proof.* Just use the Theorem 1 □

**Definition 8.** *The intuitionistic fuzzy preceding item is called the generalized caputo derivative of h, we denote ${}^{C}_{gH}D^\alpha h$.*
*we say that h is ${}^{cf}[(i)-gH]$-differentiable at $\xi_0$ if*

$$\left[{}^{C}_{gH}D^\delta h\right]_\alpha=\left[D^\delta h_{\alpha,l},D^\delta h_{\alpha,r}\right]$$
$$\left[{}^{C}_{gH}D^\delta h\right]^\beta=\left[D^\delta h^{\beta,l},D^\delta h^{\beta,r}\right]$$

*and that h is ${}^{cf}[(ii)-gH]$-differentiable at $\xi_0$ if*

$$\left[{}^{C}_{gH}D^\delta h\right]_\alpha=\left[D^\delta h_{\alpha,r},D^\delta h_{\alpha,l}\right]$$
$$\left[{}^{C}_{gH}D^\delta h\right]^\beta=\left[D^\delta h^{\beta,r},D^\delta h^{\beta,l}\right]$$

As in the previuos definition we will give the difinition of intuitionistic fuzyy fractional Riemann-Liouville integral. If the $(\alpha,\beta)$-levels of $h:(0,T)\longrightarrow\mathbb{F}$, are given by $[h]_\alpha=\left[h_{\alpha,l},h_{\alpha,r}\right]$ and $[h]^\beta=\left[h^{\beta,l},h^{\beta,r}\right]$ and $h_{\alpha,l},h_{\alpha,r},h^{\beta,l},h^{\beta,r}$ are Riemann integrable on $(0,T]$. Since the family

$$\left\{\left[h_{\alpha,l},h_{\alpha,r}\right],\ \left[h^{\beta,l},h^{\beta,r}\right]\right\}$$

built an intuitionistic element and the integrale preserve the monotony then by Theorem 1 the family

$$\left\{\mathscr{A}_\alpha,\mathscr{A}^\beta\ :\ \alpha+\beta\in[0,1]\right\}$$

where

$$\mathscr{A}_\alpha=\left[\frac{1}{\Gamma(\delta)}\int_{(0,\xi)}(\xi-s)^{\delta-1}h_{\alpha,l}(s),\frac{1}{\Gamma(\delta)}\int_{(0,\xi)}(\xi-s)^{\delta-1}h_{\alpha,r}(s)\right]$$

and

$$\mathscr{A}^\beta=\left[\frac{1}{\Gamma(\delta)}\int_{(0,\xi)}(\xi-s)^{\delta-1}h^{\beta,l}(s),\frac{1}{\Gamma(\delta)}\int_{(0,\xi)}(\xi-s)^{\delta-1}h^{\beta,r}(s)\right]$$

define an intuitionistic fuzzy element, which is the Riemann-liouville fractional integral of $h$ on $(0,T)$, we denote $\frac{1}{\Gamma(\delta)}\int_{(0,\xi)}(\xi-s)^{\delta-1}h(s)ds$

## 5. The intuitionistic fuzzy laplace transform

In this section $h:[a,\infty)\longrightarrow\mathbb{F}$ is an intuitionistic fuzzy-valued function where $a>0$. We set $[h(\xi)]_\alpha=[\underline{h}_\alpha(\xi),\overline{h}_\alpha(\xi)]$ and $[h(\xi)]^\beta=[\underline{h}^\beta(\xi),\overline{h}^\beta(\xi)]$, where $0\le\alpha+\beta\le 1$. assume that these four function are Riemann-integrable on $[a,b]$, and assume there are four positive function $\mathscr{M}_(\alpha)$, $\mathscr{M}_2(\alpha)$, $\mathscr{N}_1(\beta)$ and $\mathscr{N}_2(\beta)$ such that

$$\int_a^b \underline{h}_\alpha(\xi)d\xi\le\mathscr{M}_1(\alpha)$$

$$\int_a^b \underline{h}^\beta(\xi)d\xi\le\mathscr{N}_1(\beta)$$

$$\int_a^b \overline{h}_\alpha(\xi)d\xi\le\mathscr{M}_2(\alpha)$$

$$\int_a^b \overline{h}^\beta(\xi)d\xi\le\mathscr{N}_2(\beta)$$

For every $b\ge a$.
We define

$$\underline{A}_\alpha=\mathfrak{L}(\underline{h}_\alpha(\xi))=\int_0^\infty e^{-p\xi}\underline{h}_\alpha(\xi)d\xi$$

$$\underline{B}^\beta=\mathfrak{L}(\underline{h}^\beta(\xi))=\int_0^\infty e^{-p\xi}\underline{h}^\beta(\xi)d\xi$$

$$\overline{A}_\alpha=\mathfrak{L}(\overline{h}_\alpha(\xi))=\int_0^\infty e^{-p\xi}\overline{h}_\alpha(\xi)d\xi$$

$$\overline{B}^\beta=\mathfrak{L}(\overline{h}^\beta(\xi))=\int_0^\infty e^{-p\xi}\overline{h}^\beta(\xi)d\xi$$

**Definition 9.** *The intiutionistic fuzzy Laplace's transform is defined as follows*

$$\begin{aligned}\mathfrak{L}(h(\xi))&=\int_0^\infty h(\xi)\odot e^{-p\xi}d\xi\\ &=\left\{[\underline{A}_\alpha,\overline{A}_\alpha];[\underline{B}^\beta,\overline{B}^\beta]\right\}\end{aligned}$$

**Theorem 3.** *[9] Let $h,g:[a,b]\longrightarrow\mathbb{F}$ are continuous intuitionistic fuzzy valued function and $\lambda_1,\lambda_2$ are constants. Then*

$$\mathfrak{L}(\lambda_1\odot h(\xi)\oplus\lambda_2\odot g(\xi))=\lambda_1\odot\mathfrak{L}(h(\xi))\oplus\lambda_2\odot\mathfrak{L}(g(\xi))$$

**Theorem 4.** *[9] Let $h$ is continuous intutionistic fuzzy valued function on $[0,1)$ and*

*$\lambda\in\mathbb{R}$. Then $\mathfrak{L}\lambda\odot h(\xi)]=\lambda\odot\mathfrak{L}[h(\xi)]$*

**Theorem 5.** *[9] Let $h$ is continuous intutionistic fuzzy valued function and $\mathfrak{L}[h(\xi)]=\mathscr{H}(p)$. Then*

$$\mathfrak{L}[e^{a\xi}\odot\mathscr{H}(\xi)]=\mathscr{H}(p-a)$$

*where $e^{a\xi}$ is real valued function and $p-a>0$.*

**Theorem 6.** *[9] Let $h'(\xi)$ be an integrable fuzzy valued function, and $h(\xi)$ is the primitive of $h'(\xi)$ on $[0,\infty)$. Then $\mathfrak{L}[h'(\xi)]=p\odot\mathfrak{L}h(\xi)-_{gH}h(0)$ i.e. $\mathfrak{L}[h'(\xi)]=p\odot\mathfrak{L}h(\xi)-_{gH}h(0)$, when $h$ is $(i)$-gH differentiable and $\mathfrak{L}[h'(\xi)]=(-h(0))-_{gH}(-p\odot\mathfrak{L}[h(\xi)])$, when $h$ is $(ii)$-gH differentiable*

**Theorem 7.** *Let $h(\xi)$ and $h'(\xi)$ are two differentiable intuitionistic fuzzy-valued functions. We set $[h(\xi)]_\alpha=[\underline{h}_\alpha(\xi),\overline{h}_\alpha(\xi)]$ and $[h(\xi)]^\beta=[\underline{h}^\beta(\xi),\overline{h}^\beta(\xi)]$, where $0\le\alpha+\beta\le 1$*

- *If $h'$ and $D^\delta h$ are $(i)$-differentiable then*
  $$\mathfrak{L}(D^\delta h(s))=\left(s^{2-\delta}\mathfrak{L}(h(s))-_\varepsilon s^{1-\delta}h'(0)\right)-_\varepsilon s^{-\delta}h(0).$$
- *If $D^\delta h$ is $(i)$-differentiable and $h'$ is $(ii)$-differentiable then*
  $$\mathfrak{L}(D^\delta h(s))=\left(s^{1-\delta}h'(0)-_\varepsilon(-s^{2-\delta}\mathfrak{L}(h(s)))\right)-_\varepsilon s^{-\delta}h(0).$$
- *If $D^\delta h$ is $(ii)$-differentiable and $h'$ is $(i)$-differentiable then*
  $$\mathfrak{L}(D^\delta h(s))=s^{-\delta}h(0)-_\varepsilon\left(-\left(s^{2-\delta}\mathfrak{L}(h(s))-_\varepsilon s^{1-\delta}h'(0)\right)\right).$$
- *If $h'$ and $D^\delta h$ are $(ii)$-differentiable then*
  $$\mathfrak{L}(D^\delta h(s))=-s^{-\delta}h(0)-_\varepsilon\left(-s^{1-\delta}h'(0)-_\varepsilon(-s^{2-\delta}\mathfrak{L}(h(s)))\right).$$

*Proof.* Observe that $D^\delta=D^{\delta-1}D$ and use the theorem 6 □

## 6. The homotopy method

We apply the homotopy analysis method to the discussed problem. Let us consider the fractional differential equation,

$\mathscr{F}\mathscr{D}\mathfrak{u}(\theta,\xi)=0$

Based on the constructed zero-order deformation equation by Liao (2003), we give the following zero-order deformation equation in the similar way:

$(1-q)\mathfrak{L}(\mathfrak{u}(\theta,\xi;q))-\mathfrak{u}_0\theta,\xi)=q\varepsilon\mathscr{F}\mathscr{D}(\mathfrak{u}(\theta,\xi;q)),\ \ q\in[0,1],\ \varepsilon\neq 0$

$\mathfrak{L}$ is an auxiliary linear integer order operator and it possesses the property $\mathfrak{L}(C)=0$, $\mathfrak{u}$ is an unknown function.
Expanding $\mathfrak{u}$ in Taylor series with respect to $q$, one has

$$\mathfrak{u}(\theta,\xi;q)=\mathfrak{u}_0(\theta,\xi)+\sum_{m=1}^{\infty}\mathfrak{u}_m(\theta,\xi)q^m \tag{6.1}$$

where (6.1)

$$\mathfrak{u}_m(\theta,\xi)=\frac{\partial\mathfrak{u}(\theta,\xi;q)}{\partial q^m}|_{q=0}$$

Differentiating the equation $m$ times with respect to the embedding parameter $q$, then setting $q=0$, and finally dividing them by $m!$,we have the mth-order deformation equation

$$\mathfrak{L}(\theta,\xi)-\chi_m\mathfrak{u}_{m-1}(\theta,\xi)]=\varepsilon R_m(\vec{\mathfrak{u}}_{m-1}(\theta,\xi))$$

where

$$R_m(\vec{\mathfrak{u}}_{m-1}(\theta,\xi))=\frac{1}{(m-1)!}\frac{\partial^{m-1}\mathscr{F}\mathscr{D}(\mathfrak{u}(\theta,\xi;q))}{\partial q^{m-1}}|_{q=0}$$

and

$$\chi_m=\begin{cases}0,\ m\leq 1\\ 1,\ m>1\end{cases}$$

**Definition 10.** *[4] The mittag-leffler function is given by*

$$E_\alpha(\mathfrak{z})=\sum_{k=0}^{\infty}\frac{\mathfrak{z}^k}{\Gamma(1+\alpha k)}\quad \alpha\in\mathbb{C},\Re(\alpha)>0,\mathfrak{z}\in\mathbb{C} \tag{6.2}$$

*and its general form*

$$E_{\alpha,\beta}(\mathfrak{z})=\sum_{k=0}^{\infty}\frac{\mathfrak{z}^k}{\Gamma(\beta+\alpha k)}\quad \alpha,\beta\in\mathbb{C},\Re(\alpha)>0,\Re(\beta)>0,z\in\mathbb{C} \tag{6.3}$$

## 7. Solution of the problem (1.1)

we consider the following non-homogeneous fractional partial differential equation

$$\begin{cases}{}^{C}_{gH}D^{\delta}_{\xi}\mathfrak{u}(\theta,\xi)+{}_{gh}D_{\theta}\mathfrak{u}(\theta,\xi)=\mathscr{G}(\theta,\xi),\ 0\leq\theta,\xi<1\\ \mathfrak{u}(\theta,0)=\psi(\theta)\\ \frac{\partial}{\partial\theta}\mathfrak{u}(\theta,0)=\Upsilon(\theta)\end{cases} \tag{7.1}$$

where $0<\delta\leq 1$, the operator ${}^{C}_{gH}D^{\delta}$ denote the intuitionistic fuzzy Caputo fractional generalized derivative of order $\delta$, and $\psi,\Upsilon:[0,1[\longrightarrow\mathbb{F}$, $\mathscr{G}:[0,1[\times[0,1[\longrightarrow\mathbb{F}$ and $\Upsilon(\theta)\longrightarrow_{|\theta|\longrightarrow\infty}0$.
We choose the linear non-integer order operator

$$\mathfrak{L}[\mathfrak{v}(\theta,\xi;q)]={}^{C}_{gH}D^{\delta}_{\xi}\mathfrak{v}(\theta,\xi;q)$$

Let us consider the linear fractional differential equation

$$\mathscr{F}\mathscr{D}\mathfrak{v}(\theta,\xi;q)]=0.$$

where $\mathscr{F}\mathscr{D}\mathfrak{v}(\theta,\xi;q)]={}^{C}_{gH}D^{\delta}_{\xi}\mathfrak{v}(\theta,\xi;q)+\mathfrak{v}_{\theta}(\theta,\xi;q)-\mathscr{G}(\theta,\xi)$
Based on the constructed zero-order deformation equation by Liao (2003), we give the following zero-order deformation equation in the similar way:

$$(1-q)\mathfrak{L}[\mathfrak{v}(\theta,\xi;q)-\mathfrak{u}_0(\theta,\xi)]=q\varepsilon\mathscr{F}\mathscr{D}(\theta,\xi;q)].$$

where $q\in[0,1]$ and $\varepsilon\neq 0$
Clearly we have $\mathfrak{v}(\theta,\xi;0)=\mathfrak{u}_0(\theta,\xi)=\mathfrak{u}(\theta,0),\mathfrak{v}(\theta,\xi;1)=\mathfrak{v}(\theta,\xi)$.

the mth-order deformation equation is giving by :

$\mathfrak{L}[\mathfrak{u}_m(\theta,\xi)-\chi_m\mathfrak{u}_{m-1}(\theta,\xi)] = h\mathscr{F}\mathscr{R}\left(\mathfrak{u}_{(m-1)}(\theta,\xi)\right)$

where

$$\mathscr{F}\mathscr{R}(\mathfrak{u}_{(m-1)}(\theta,\xi)) = {}^{C}_{gH}D^{\delta}_{\xi}\mathfrak{u}_{(m-1)} + \mathfrak{u}_{(m-1)\theta} - (1-\chi_m)\mathscr{G}(\theta,\xi).$$

Now, the solution of (25), for m $\geq 1$ becomes

$$\mathfrak{u}_m(\theta,\xi) = \chi_m\mathfrak{u}_{(m-1)}(\theta,\xi) + \varepsilon\mathfrak{L}^{-1}\mathscr{F}\mathscr{R}(\mathfrak{u}_{(m-1)}(\theta,\xi)).$$

The solution is given by two cases.

- Case $i)$: $\mathfrak{u}$ is ${}^{cf}[(i)-gH]$-differentiable by rapport to $\xi$ and $\mathfrak{u}$ is $[(i)-gH]$-differentiable by raport to $\theta$ and $\xi$.

$$\begin{cases} D^{\delta}_{\xi}\underline{\mathfrak{u}_\alpha}(\theta,\xi) + D_\theta\underline{\mathfrak{u}_\alpha}(\theta,\xi) = \underline{\mathscr{G}_\alpha}(\theta,xi),\ 0\leq\theta,\xi<1 \\ \underline{\mathfrak{u}_\alpha}(\theta,0) = \underline{\psi_\alpha}(\theta); \quad 0\leq\alpha\leq 1 \\ \frac{\partial}{\partial\theta}\underline{\mathfrak{u}_\alpha}(\theta,0) = \underline{\Upsilon_\alpha}(\theta) \end{cases} \tag{7.2}$$

From (19), (24), and (27), we now successively obtain

$$\begin{aligned} \underline{\mathfrak{u}_{0\alpha}}(\theta,\xi) &= \underline{\mathfrak{u}_\alpha}(\theta,0) = \underline{\psi_\alpha}(\theta) \\ \underline{\mathfrak{u}_{1\alpha}}(\theta,\xi) &= \varepsilon D^{-\gamma}_{\xi}(D^{\delta}_{\xi}\underline{\mathfrak{u}_{0\alpha}} + \underline{\mathfrak{u}_{0\alpha}}_\theta - \underline{\mathscr{G}_\alpha}(\theta,\xi)) \\ &= \varepsilon D^{-\delta}_{\xi}(\underline{\psi_\alpha}_\theta - \underline{\mathscr{G}_\alpha}) \end{aligned}$$

$$\underline{\mathfrak{u}_{2\alpha}}(\theta,\xi) = \varepsilon(\varepsilon+1)D^{-\delta}_{\xi}(\underline{\psi_\alpha}_\theta - \underline{\mathscr{G}_\alpha}) + \varepsilon^2(D^{-\delta}_{\xi})^2(\underline{\psi_\alpha}_{\theta\theta} - \underline{\mathscr{G}_\alpha}_\theta)$$

$$\begin{aligned} \underline{\mathfrak{u}_{3\alpha}}(\theta,\xi) = {} & \varepsilon^3(D^{-\delta}_{\xi})^3(\underline{\psi_\alpha}_{\theta\theta\theta} - \underline{\mathscr{G}_\alpha}_{\theta\theta\theta}) + \varepsilon^2(\varepsilon+1)(D^{-\delta}_{\xi})^2(\underline{\psi_\alpha}_{\theta\theta} - \underline{\mathscr{G}_\alpha}_\theta) \\ & + \varepsilon(\varepsilon+1)^2D^{-\delta}_{\xi}(\underline{\psi_\alpha}_\theta - \underline{\mathscr{G}_\alpha}) + \varepsilon(\varepsilon+1)D^{-\delta}_{\xi}(\underline{\psi_\alpha}_{\theta\theta} - \underline{\mathscr{G}_\alpha}_\theta) \end{aligned}$$

$$\begin{aligned} \underline{\mathfrak{u}_{4\alpha}}(\theta,\xi) = {} & \varepsilon^4(D^{-\delta}_{\xi})^4(\underline{\psi_\alpha}_{\theta\theta\theta} - \underline{\mathscr{G}_\alpha}_{\theta\theta\theta}) + 2\varepsilon^3(\varepsilon+1)(D^{-\delta}_{\xi})^3(\underline{\psi_\alpha}_{\theta\theta\theta} - \underline{\mathscr{G}_\alpha}_{\theta\theta}) \\ & + 2\varepsilon^2(\varepsilon+1)^2(D^{-\delta}_{\xi})^2(\underline{\psi_\alpha}_{\theta\theta} - \underline{\mathscr{G}_\alpha}_\theta) + \varepsilon^2(\varepsilon+1)(D^{-\delta}_{\xi})^2(\underline{\psi_\alpha}_{\theta\theta\theta} - \underline{\mathscr{G}_\alpha}_{\theta\theta}) \\ & + \varepsilon(\varepsilon+1)^2D^{-\delta}_{\xi}(\underline{\psi_\alpha}_{\theta\theta} - \underline{\mathscr{G}_\alpha}_\theta) + \varepsilon(\varepsilon+1)^3D^{-\delta}_{\xi}(\underline{\psi_\alpha}_\theta - \underline{\mathscr{G}_\alpha}) \end{aligned}$$

and so on .if we substitute h=-1 in the above terms the dominant terms will be remaining and the rest terms vanish because they include the factor $\varepsilon^m(\varepsilon+1)^n, n,m\in\mathbb{N}$. Define $\underline{\mathscr{A}_\alpha}(\theta,\xi) = \underline{\psi_\alpha}_\theta(\theta) - \underline{\mathscr{G}_\alpha}(\theta,\xi)$ then we have:

$\underline{\mathfrak{u}_{0\alpha}}(\theta,\xi) = \underline{\psi_\alpha}(\theta)$

$\underline{\mathfrak{u}_{1\alpha}}(\theta,\xi) = -D^{-\delta}_{\xi}(\underline{\mathscr{A}_\alpha})$

$\underline{\mathfrak{u}_{2\alpha}}(\theta,\xi) = (D^{-\delta}_{\xi})^2(\underline{\mathscr{A}_\alpha}_\theta)$

$\underline{\mathfrak{u}_{3\alpha}}(\theta,\xi) = -(D^{-\delta}_{\xi})^3(\underline{\mathscr{A}_\alpha}_{\theta\theta})$

$\underline{\mathfrak{u}_{4\alpha}}(\theta,\xi) = (D^{-\xi}_{\xi})^4(\underline{\mathscr{A}_\alpha}_{\theta\theta\theta})$

$$\underline{\mathfrak{u}_\alpha}(\theta,\xi) = \underline{\psi_\alpha}(\theta) - \sum_{k=1}^{\infty}(-1)^k(D^{-\delta}_{\xi})^k(D^{k-1}_{\theta}\underline{\mathscr{A}_\alpha})$$

let consider the falowing equation

$$\begin{cases} D^{\delta}_{\xi}\overline{\mathfrak{u}}_\alpha(\theta,\xi) + D_\theta\overline{\mathfrak{u}}_\alpha(\theta,\xi) = \overline{\mathscr{G}}_\alpha(\theta,\xi),\ 0\leq\theta,\xi<1 \\ \overline{\mathfrak{u}}_\alpha(\theta,0) = \overline{\psi}_\alpha(\theta); \quad 0\leq\alpha\leq 1 \\ \frac{\partial}{\partial\theta}\overline{\mathfrak{u}}_\alpha(\theta,0) = \overline{\Upsilon}_\alpha(\theta) \end{cases} \tag{7.3}$$

then the solution is giving by

$$\overline{\mathfrak{u}_\alpha}(\theta,\xi) = \overline{\psi_\alpha}(\theta) - \sum_{k=1}^{\infty}(-1)^k(D^{-\delta}_{\xi})^k(D^{k-1}_{\theta}\overline{\mathscr{A}_\alpha})$$

where $\overline{\mathscr{A}_\alpha}(\theta,\xi) = \overline{\psi_\alpha}_\theta(\theta) - \overline{\mathscr{G}_\alpha}(\theta,\xi)$
let consider the equation

$$\begin{cases} D^{\delta}_{\xi}\underline{\mathfrak{u}}^\beta(\theta,\xi) + D_\theta\underline{\mathfrak{u}}^\beta(\theta,\xi) = \underline{\mathscr{G}}^\beta(\theta,\xi)\ 0\leq\theta,\xi<1 \\ \underline{\mathfrak{u}}^\beta(\theta,0) = \underline{\psi}^\beta(\theta); \quad 0\leq\beta\leq 1 \\ \frac{\partial}{\partial\theta}\underline{\mathfrak{u}}^\beta(\theta,0) = \underline{\Upsilon}^\beta(\theta) \end{cases} \tag{7.4}$$

we have

$$\underline{\mathfrak{u}^{\beta}}(\theta,\xi)=\underline{\psi^{\beta}}(\theta)-\sum_{k=1}^{\infty}(-1)^{k}(D_{\xi}^{-\delta})^{k}(D_{\theta}^{k-1}\underline{\mathscr{A}^{\beta}})$$

where $\underline{\mathscr{A}^{\beta}}(\theta,\xi)=\underline{\psi^{\beta}}_{\theta}(\theta)-\underline{\mathscr{G}^{\beta}}(\theta,\xi)$

$$\begin{cases} D_{\xi}^{\delta}\overline{\mathfrak{u}}^{\beta}(\theta,\xi)+D_{\theta}\overline{\mathfrak{u}}^{\beta}(\theta,\xi)=\overline{\mathscr{G}}^{\beta}(\theta,\xi),\ 0\leq\theta,\xi<1\\ \overline{\mathfrak{u}}^{\beta}(\theta,0)=\overline{\psi}^{\beta}(\theta);\quad 0\leq\beta\leq 1\\ \frac{\partial}{\partial\theta}\overline{\mathfrak{u}}^{\beta}(\theta,0)=\overline{\Upsilon}^{\beta}(\theta)\end{cases}\tag{7.5}$$

the solution of the equation is giving by

$$\overline{\mathfrak{u}^{\beta}}(\theta,\xi)=\overline{\psi^{\beta}}(\theta)-\sum_{k=1}^{\infty}(-1)^{k}(D_{\xi}^{-\delta})^{k}(D_{\theta}^{k-1}\overline{\mathscr{A}^{\beta}})$$

- Case *ii*): $\mathfrak{u}$ is ${}^{cf}[(ii)-gH]$-differentiable by rapport to $\xi$ and $\mathfrak{u}$ is $[(ii)-gH]$-differentiable by raport to $\theta$ and $\xi$.

$$\begin{cases} D_{\xi}^{\delta}\overline{\mathfrak{u}_{\alpha}}(\theta,\xi)+D_{\theta}\overline{\mathfrak{u}_{\alpha}}(\theta,\xi)=\underline{\mathscr{G}_{\alpha}}(\theta,\xi),\ 0\leq\theta,\xi<1\\ \overline{\mathfrak{u}_{\alpha}}(\theta,0)=\overline{\psi_{\alpha}}(\theta);\quad 0\leq\alpha\leq 1\\ \frac{\partial}{\partial\theta}\overline{\mathfrak{u}_{\alpha}}(\theta,0)=\underline{\Upsilon_{\alpha}}(\theta)\end{cases}\tag{7.6}$$

Define $\overline{\mathscr{A}_{\alpha}}(\theta,\xi)=\overline{\psi_{\alpha}}_{\theta}(\theta)-\underline{\mathscr{G}_{\alpha}}(\theta,\xi)$ then we have

$$\overline{\mathfrak{u}_{\alpha}}(\theta,\xi)=\overline{\psi_{\alpha}}(\theta)-\sum_{k=1}^{\infty}(-1)^{k}(D_{\xi}^{-\delta})^{k}(D^{k-1}\theta\overline{\mathscr{A}_{\alpha}})$$

let consider the falowing equation

$$\begin{cases} D_{\xi}^{\delta}\underline{\mathfrak{u}}_{\alpha}(\theta,\xi)+D_{\theta}\underline{\mathfrak{u}}_{\alpha}(\theta,\xi)=\overline{\mathscr{G}}_{\alpha}(\theta,\xi),\ 0\leq\theta,\xi<1\\ \underline{\mathfrak{u}}_{\alpha}(\theta,0)=\underline{\psi}_{\alpha}(\theta);\quad 0\leq\alpha\leq 1\\ \frac{\partial}{\partial\theta}\underline{\mathfrak{u}}_{\alpha}(\theta,0)=\overline{\Upsilon}_{\alpha}(\theta)\end{cases}\tag{7.7}$$

then the solution is giving by

$$\underline{\mathfrak{u}_{\alpha}}(\theta,\xi)=\underline{\psi_{\alpha}}(\theta)-\sum_{k=1}^{\infty}(-1)^{k}(D_{\xi}^{-\delta})^{k}(D_{\theta}^{k-1}\underline{\mathscr{A}_{\alpha}})$$

where $\underline{\mathscr{A}_{\alpha}}(\theta,\xi)=\underline{\psi_{\alpha}}_{\theta}(\theta)-\overline{\mathscr{G}_{\alpha}}(\theta,\xi)$
let consider the equation

$$\begin{cases} D_{\xi}^{\delta}\overline{\mathfrak{u}}^{\beta}(\theta,\xi)+D_{\theta}\overline{\mathfrak{u}}^{\beta}(\theta,\xi)=\underline{\mathscr{G}}^{\beta}(\theta,\xi),\ 0\leq\theta,\xi<1\\ \overline{\mathfrak{u}}^{\beta}(\theta,0)=\overline{\psi}^{\beta}(\theta);\quad 0\leq\beta\leq 1\\ \frac{\partial}{\partial\theta}\overline{\mathfrak{u}}^{\beta}(\theta,0)=\underline{\Upsilon}^{\beta}(\theta)\end{cases}\tag{7.8}$$

we have

$$\overline{\mathfrak{u}^{\beta}}(\theta,\xi)=\overline{\psi^{\beta}}(\theta)-\sum_{k=1}^{\infty}(-1)^{k}(D_{\xi}^{-\delta})^{k}(D_{\theta}^{k-1}\overline{\mathscr{A}^{\beta}})$$

where $\overline{\mathscr{A}^{\beta}}(\theta,\xi)=\overline{\psi^{\beta}}_{\theta}(\theta)-\underline{\mathscr{G}^{\beta}}(\theta,\xi)$

$$\begin{cases} D_{\xi}^{\delta}\underline{\mathfrak{u}}^{\beta}(\theta,\xi)+D_{\theta}\underline{\mathfrak{u}}^{\beta}(\theta,\xi)=\overline{\mathscr{G}}^{\beta}(\theta,\xi),\ 0\leq\theta,\xi<1\\ \underline{\mathfrak{u}}^{\beta}(\theta,0)=\underline{\psi}^{\beta}(\theta);\quad 0\leq\beta\leq 1\\ \frac{\partial}{\partial\theta}\underline{\mathfrak{u}}^{\beta}(\theta,0)=\overline{\Upsilon}^{\beta}(\theta)\end{cases}\tag{7.9}$$

the solution of the equation is giving by

$$\underline{\mathfrak{u}^{\beta}}(\theta,\xi)=\underline{\psi^{\beta}}(\theta)-\sum_{k=1}^{\infty}(-1)^{k}(D_{\xi}^{-\delta})^{k}(D_{\theta}^{k-1}\underline{\mathscr{A}^{\beta}})$$

where $\underline{\mathscr{A}^{\beta}}(\theta,\xi)=\underline{\psi^{\beta}}_{\theta}(\xi)-\underline{\mathscr{G}^{\beta}}(\theta,\xi)$

## 8. Numerical example

Now, we put $\mathscr{G}(\theta,\xi)=e^{-\theta-\xi}(1,2,3;0,2,4)$ and $\mathfrak{u}(\theta,0)=e^{-\theta}(1,2,3;0,2,4)$ then the problem yields

$$\begin{cases} {}^{C}_{gH}D^{\delta}_{\xi}\mathfrak{u}(\theta,\xi)+_{gh}D_{\theta}\mathfrak{u}(\theta,\xi)=e^{-\theta-\xi}(1,2,3;0,2,4),\ 0\le\theta,\xi<1 \\ \mathfrak{u}(\theta,0)=e^{-\theta}(1,2,3;0,2,4) \\ \frac{\partial}{\partial\theta}\mathfrak{u}(\theta,0)=\widetilde{0} \end{cases} \tag{8.1}$$

- $\mathfrak{u}$ is ${}^{cf}[(i)-gH]$-differentiable by rapport to $\xi$ and $\mathfrak{u}$ is $[(i)-gH]$-differentiable by raport to $\theta$ and $\xi$.

$$\begin{cases} D^{\delta}_{\xi}\underline{\mathfrak{u}_{\alpha}}(\theta,\xi)+D_{\xi}\underline{\mathfrak{u}_{\alpha}}(\theta,\xi)=e^{-\theta-\xi}(\alpha+1),\ 0\le\theta,\xi<1 \\ \underline{\mathfrak{u}_{\alpha}}(\theta,0)=e^{-\theta}(\alpha+1);\quad 0\le\alpha\le1 \\ \frac{\partial}{\partial\theta}\underline{\mathfrak{u}_{\alpha}}(\theta,0)=0 \end{cases} \tag{8.2}$$

we have
$\underline{\mathscr{A}_{\alpha}}=-(\alpha+1)(e^{-\theta}+e^{-\theta-\xi})$,
$\underline{\mathscr{A}_{\alpha}}_{\theta}=(\alpha+1)(e^{-\theta}+e^{-\theta-\xi})$,
$\underline{\mathscr{A}_{\alpha}}_{\theta\theta}=\underline{\mathscr{A}_{\alpha}}=(\alpha+1)(e^{-\theta}+e^{-\theta-\xi})$,

$$\begin{aligned} D^{-\delta}(\underline{\mathscr{A}_{\alpha}}) &= D^{-\delta}_{\xi}(-(\alpha+1)(e^{-\theta}+e^{-\theta-\xi})) \\ &= (\alpha+1)(\frac{e^{-\theta}}{\Gamma(\delta+1)}\xi^{\delta}+e^{-\theta}D^{-\delta}_{\xi}e^{-\xi}), \end{aligned}$$

$$\begin{aligned} (D^{-\delta}_{\xi})^{2}(\underline{\mathscr{A}_{\alpha}}_{\theta}) &= D^{-2\delta}_{\xi}((\alpha+1)(e^{-\theta}+e^{-\theta-\xi})) \\ &= (\alpha+1)(\frac{e^{-\theta}}{\Gamma(2\delta+1)}\xi^{2\delta}+e^{-\theta}D^{-2\delta}_{\xi}e^{-\xi}), \end{aligned}$$

$$\begin{aligned} (D^{-\delta}_{\xi})^{3}(\underline{\mathscr{A}_{\alpha}}_{\theta\theta}) &= D^{-3\delta}_{\xi}(-(\alpha+1)(e^{-\theta}+e^{-\theta-\xi})) \\ &= (\alpha+1)(\frac{e^{-\theta}}{\Gamma(3\delta+1)}\xi^{3\delta}-e^{-\theta}D^{3\delta}_{\xi}e^{-\xi}), \end{aligned}$$

To solve $D^{-\delta}_{\xi}(e^{-\xi})$, the Laplace transform can be used:

$$\mathfrak{L}(D^{-\delta}_{\xi}(e^{-\xi}))=\frac{1}{s^{\delta(s+1)}},$$

and with the use of the inverse Laplace transform we have,

$$\begin{aligned} D^{-\delta}_{\xi}(e^{-\xi}) &= \mathfrak{L}^{-1}(\frac{1}{s^{\delta+1}}(1-\frac{1}{s+1})) \\ &= \frac{\xi^{\delta}}{\Gamma(\delta+1)}-E(\xi,\delta+1,-1), \end{aligned}$$

$$\begin{aligned} D^{-2\delta}_{\xi}(e^{-\xi}) &= \mathfrak{L}^{-1}(\frac{1}{s^{2\delta+1}}(1-\frac{1}{s+1})) \\ &= \frac{\xi^{2\delta}}{\Gamma(2\delta+1)}-E(\xi,2\delta+1,-1), \end{aligned}$$

$$\begin{aligned} D^{-3\delta}_{\xi}(e^{-\xi}) &= \mathfrak{L}^{-1}(\frac{1}{s^{3\delta+1}}(1-\frac{1}{s+1})) \\ &= \frac{\xi^{3\delta}}{\Gamma(3\delta+1)}-E(\xi,3\delta+1,-1), \end{aligned}$$

where $E(\xi,\delta,a)=\frac{1}{\Gamma(\delta)}\int_{0}^{\xi}\tau^{\delta-1}e^{a(\xi-\tau)}d\tau$
With the use of the above formula the solution $\mathfrak{u}(\theta,\xi)$ can be obtained; hence, we have:

$$\begin{aligned} \underline{\mathfrak{u}_{\alpha}}(\theta,\xi)) &= (\alpha+1)(e^{\theta}+2e^{-\theta}\{\frac{\xi^{\delta}}{\Gamma(\delta+1)} \\ &+\frac{\xi^{2\delta}}{\Gamma(2\delta+1)}+...\}-e^{-\theta}\{E(\xi,\delta+1,-1)+E(\xi,2\delta+1,-1) \\ &+E(\xi,3\delta+1,-1)+...\}) \end{aligned}$$

thus, we get:

$$\underline{u_\alpha}(\theta,\xi) = (\alpha+1)(e^{-\theta} + 2e^{-\theta}\sum_{k=1}^{\infty}\frac{\xi^{k\delta}}{\Gamma(k\delta+1)} - e^{-\theta}\sum_{k=1}^{\infty}E(\xi,k\delta+1,-1))$$

where for $\delta = 1$ we have:

$$\begin{aligned} E(\xi,k+1,-1) &= \frac{1}{\Gamma(k+1)}\int_0^{\xi}\tau^k e^{-(\xi-\tau)}d\tau \\ &= \frac{e^{-\xi}}{\Gamma(k+1)}\int_0^{\xi}\tau^k e^{\tau}d\tau \end{aligned}$$

Finaly the exacte solution will be,

$$\underline{u_\alpha}(\theta,\xi) = (\alpha+1)e^{-\theta}(e^{\xi} + sinh(\xi))$$

let consider the equation

$$\begin{cases} D_\xi^\delta\overline{u_\alpha}(\theta,\xi) + D_\theta\overline{u_\alpha}(\theta,\xi) = e^{-\theta-\xi}(3-\alpha),\ 0 \le \theta,\xi < 1 \\ \overline{u_\alpha}(\theta,0) = e^{-\theta}(3-\alpha+1); \quad 0 \le \alpha \le 1 \\ \frac{\partial}{\partial\theta}\overline{u_\alpha}(\theta,0) = 0 \end{cases} \tag{8.3}$$

So the solution is giving by,

$$\overline{u_\alpha}(\theta,\xi) = (3-\alpha)e^{-\theta}(e^{\xi} + sinh(\xi))$$

and the solution of the problem

$$\begin{cases} D_\xi^\delta\underline{u^\beta}(\theta,\xi) + D_\theta\underline{u^\beta}(\theta,\xi) = 2e^{-\theta-_xi}(1-\beta),\ 0 \le \theta,\xi < 1 \\ \underline{u^\beta}(\theta,0) = 2e^{-\theta}(1-\beta); \quad 0 \le \beta \le 1 \\ \frac{\partial}{\partial\theta}\underline{u^\beta}(\theta,0) = 0 \end{cases} \tag{8.4}$$

is giving by,

$$\underline{u^\beta}(\theta,\xi) = 2(1-\beta)e^{-\theta}(e^{\xi} + sinh(\xi))$$

$$\begin{cases} D_\xi^\delta\overline{u^\beta}(\theta,\xi) + D_\theta\overline{u^\beta}(\theta,\xi) = 2e^{-\theta-\xi}(1+\beta)),\ 0 \le \theta,\xi < 1 \\ \overline{u^\beta}(\theta,0) = 2e^{-\theta}(1+\beta); \quad 0 \le \beta \le 1 \\ \frac{\partial}{\partial\theta}\overline{u^\beta}(\theta,0) = 0 \end{cases} \tag{8.5}$$

is giving by,

$$\overline{u^\beta}(\theta,\xi) = 2(1+\beta)e^{-\theta}(e^{\xi} + sinh(\xi))$$

## 9. Conclusion

In this paper, the homotopy analysis method has been proposed and implemented to derive new approximate analytical solutions for intuitionistic fuzzy fractional partial differential equations. The efficiency and simplicity of the homotopy analysis transform method are highlighted. This work contributes to the expanding field of fractional calculus and intuitionistic fuzzy mathematics, offering a new perspective on solving fractional partial differential equations with intuitionistic fuzzy initial data. The results presented in this paper not only provide valuable insights into the application of the homotopy analysis method in the context of intuitionistic fuzzy systems but also pave the way for further exploration and practical applications in this interdisciplinary research area.

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