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Research paper

Controllability, Observability and Stability of Volterra Type Non-Linear Matrix Integro-Dynamic System on Time Scales

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Abstract

This paper investigates the controllability, observability and stability of the solution of Volterra type non linear matrix integro dynamic system on time scales.

Keywords: *Controllability; non-linear Volterra type matrix integro-dynamic system; observability; stability; time scales.*

1. Introduction

In many engineering problems, one may desire to have a system that follows a preassigned behaviour. In other words, necessary steps have to be taken to avoid unwanted behaviour in a system and to compel the system to follow a desired behaviour. The origin of control theory stems from determining these steps called controls. After R.E. Kalman introduced general control theory in 1960, many engineers and mathematicians got attracted by this theory [7, 9, 10, 11, 12, 13, 16]. The importance of control theory in mathematics and its applications in diverse areas such as adoptive controls [9], communication networks [10], switching systems [14], dynamic programs [15], are well established.
The theory of time scales, introduced by Hilger [3, 8] at the end of the twentieth century as a means to unify the difference and the differential calculus, is now a well-established subject.
In [7], J.M. Davis, Ian A. Gravagne, Billy J. Jackson, R.J. Marks discussed the controllability, observability realizability and stability of linear dynamic system on time scales.
On the other hand, the theory of Volterra integro-dynamic equation has drawn the attention of many mathematicians in the last decade [6]. In [1], Adivar derived principal matrix solution using variation of parameters formula for integro-dynamic equations on time scales. In [2], Becker investigated the solution using variation of parameters formula for a integro-dynamic equations and its adjoint. Burton and Mahfoud discussed the various stability properties of integro-dynamic equations in [4, 5, 6].
Controllability, observability and stability of Volterra integro dynamic system on time scales were studied Awais Yonus and Ghaus ur Rahman [16] . They considered linear integro dynamic system of the form

$x^{\Delta}(t) = A(t)x(t) + \int_{t_0}^{t} K(t,s)x(s)\Delta s + B(t)u(t).$

Anyhow, much of contribution on controllability, observability and stability of non-linear integro dynamic systems on time scales is not available in literature. With this motivation, in this paper, we establish some new results on controllability, observability and stability of non linear matrix integro dynamic system on time scales.

2. Preliminary Results

Throught this paper $\mathbb{T}$ denotes time scale(an arbitrary nonempty closed subset of the real numbers).

Definition 2.1: ([3]) The mappings σ and $\rho : \mathbb{T} \to \mathbb{R}$ where $\mathbb{T}$ is any closed subset of reals, are defined as $\sigma(t) = inf\{s \in \mathbb{T}: s > t\}$ and $\rho(t) = sup\{s \in \mathbb{T}: s < t\}$.

Definition 2.2:([3]) A non-maximal element t in $\mathbb{T}$ is called right dense if $\sigma(t) = t$; right scattered if $\sigma(t) > t$; left dense if $\rho(t) = t$ and left scattered if $\rho(t) < t$.

Definition 2.3:([3]) If $\mathbb{T}$ has a left scattered maximum M, then $\mathbb{T}^k = \mathbb{T} - \{M\}$, otherwise, $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right scattered minimum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$, otherwise, $\mathbb{T}^k = \mathbb{T}$.

Definition 2.4:([3]) The function $\mu^*: \mathbb{T}^k \to \mathbb{R}^+$ defined by $\mu^*(t) = \mu(\sigma(t), t)$ for $t \in \mathbb{T}$ is called graininess. If t is right dense, then $\mu^* = 0$ and if t is right scattered, then $\mu^* = \sigma(t) - t$.

Definition 2.5:([3]) A mapping $f: \mathbb{T} \to \mathbb{R}$ is said to be differentiable at $t \in \mathbb{T}^k$, if there exists an $\alpha \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists a neighbourhood Q of t satisfying $|f(\sigma(t)) -$

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$f(s) - (\sigma(t) - s)\alpha| \leq \varepsilon|\sigma(t) - s|$ for all $s \in Q$.

Definition 2.6:([3]) An $(n \times n)$ matrix valued function A on a time scale $\mathbb{T}$ is called regressive provided

$$I + \mu(t)A(t)$$

is invertible for all $t \in \mathbb{T}^k$.

Theorem 2.7:([3]) Assume $f: \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we have the following

(a) If f is differentiable at t, then f is continuous at t.

(b) If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(c) If t is right-dense, then f is differentiable at t iff the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

exists as a finite number. In this case

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(d) If f is differentiable at t, then

$$f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t).$$

Theorem 2.8:([3]) Suppose A, B and C are differentiable $(n \times n)$ matrix-valued functions. Then

(a) $(A + B)^\Delta = A^\Delta + B^\Delta$;

(b) $(\alpha A)^\Delta = \alpha A^\Delta$ if α is constant;

(c) $(AB)^\Delta = A^\Delta B^\sigma + AB^\Delta = A^\sigma B^\Delta + A^\Delta B$;

(d) $(ABC)^\Delta = A^\Delta BC + A^\sigma B^\Delta C^\sigma + A^\sigma BC^\Delta = A^\Delta BC + A^\sigma B^\Delta C + A^\sigma B^\sigma C^\Delta$.

Lemma 2.9: ([3]) Let $y \in C_{rd}(\mathbb{T}, \mathbb{R})$, $p \in \mathbb{R}^+$, $p \geq 0$ and $\alpha \in \mathbb{R}$. Then

$$y(t) \leq \alpha + \int_{t_0}^{t} y(s)p(s)\Delta s$$

for all $t \in \mathbb{T}$, implies

$$y(t) \leq \alpha e_p(t, t_0)$$

for all $t \in \mathbb{T}$.

Lemma 2.10: ([3]) Let $\alpha \in \mathbb{R}$ with $\alpha \in \mathbb{R}^+$. Then

$$e_\alpha(t, s) \geq 1 + \alpha(t - s) \text{ for all } t \geq s$$

3. Main Results

In this paper, a non linear integro dynamic system on time scale $\mathbb{T} = [0, \infty)$ of the form

$$\left.\begin{aligned} X^\Delta(t) &= A(t)X(t) + \left(\int_{t_0}^{t} K(t, s, X(s))\Delta s + B(t)\right) U(t), \\ Z(t) &= C(t)X(t), \\ X(t_0) &= X_0. \end{aligned}\right\} \quad (1)$$

is considered. Here $t_0 \in \mathbb{T}^k$, $X \in \mathbb{R}^{n \times n}$, $U \in \mathbb{R}^{n \times n}$. The matrix functions $A(t)$, $K(t, s, X(s))$, $B(t)$ and $C(t)$ are all $(n \times n)$ continuous and regressive with respect to their arguments on $\mathbb{T}$.

Theorem 3.1: Let $A(t)$, B(t) and K(t,s,X(s)) be continuous and regressive $(n \times n)$ matrices. Then the solution of (1) is given by

$$\begin{aligned} X(t) &= \Psi(t, t_0)X_0 \\ &\quad + \Psi(t, t_0) \int_{t_0}^{t} \Psi(t_0, \sigma(\tau)) \left(\int_{t_0}^{\tau} K(\tau, s, X(s))\Delta s + B(\tau)\right) \\ &\quad \times U(\tau)\Delta\tau \end{aligned} \quad (2)$$

where $\Psi(t, t_0)$ is state transition matrix and is given by $\Psi(t, t_0) = \Phi(t)\Phi^{-1}(t_0)$ and $\Phi(t)$ is a fundamental matrix of $X^\Delta(t) = A(t)X(t)$.

Proof:
Let $\Phi(t)$ be the fundamental matrix of

$$X^\Delta(t) = A(t)X(t). \quad (3)$$

Then any solution of (3.3) is of the form

$$X(t) = \Phi(t)C.$$

where C is a constant matrix. Let the particular solution $\bar{X}(t)$ of (1) be of the form

$$\bar{X}(t) = \Phi(t)R(t).$$

Then we have

$$\begin{aligned} \Phi^\Delta(t)R(t) + \Phi(\sigma(t))R^\Delta(t) &= A(t)\Phi(t)R(t) \\ &\quad + \left(\int_{t_0}^{t} K(t, s, X(s))\Delta s + B(t)\right) U(t) \end{aligned}$$

$$R^\Delta(t) = \Phi^{-1}(\sigma(t)) \left(\int_{t_0}^{t} K(t, s, X(s))\Delta s + B(t)\right) U(t).$$

Integration on both sides between the limits t_0 to t, we have

$$R(t) = \int_{t_0}^{t} \Phi^{-1}(\sigma(\tau)) \left(\int_{t_0}^{\tau} K(\tau, s, X(s))\Delta s + B(\tau)\right) U(\tau)\Delta\tau.$$

and the general solution is given by

$$\begin{aligned} X(t) &= \Phi(t)C \\ &\quad + \Phi(t) \int_{t_0}^{t} \Phi^{-1}(\sigma(\tau)) \left(\int_{t_0}^{\tau} K(\tau, s, X(s))\Delta s + B(\tau)\right) U(\tau)\Delta\tau. \end{aligned}$$

Clearly, $C = \Phi^{-1}(t_0)X_0$

Hence the general solution of (1) is (2).

Definition 3.1:([7, 16]) The regressive non-linear system (1) is said to be completely controllable on $[t_0, t_f]$ if for any initial time t_0, initial state $X(t_0) = X_0$ and given final state X_f, there exists a finite time $t_f > t_0$ and a rd-continuous control $U(t)$ for $t_0 \leq t \leq t_f$ such that $X(t_f) = X_f$.

Theorem 3.2: The system (1) is completely controllable if and only if the $(n \times n)$ symmetric controllability Gramian matrix given by

$$G_c(t_0, t_f) = \int_{t_0}^{t_f} \Psi(t_0, \sigma(\tau))H(\tau, X(\tau))H^T(\tau, X(\tau))\Psi^T(t_0, \sigma(\tau))\Delta\tau$$

is invertible, where

$$H(\tau, X(\tau)) = \int_{t_0}^{\tau} K(\tau, s, X(s))\Delta s + B(\tau).$$

Proof:
Suppose that Gramian matrix $G_c(t_0, t_f)$ is invertible. Then for given X_0 and X_f, choose continuous matrix function $U(t)$ for $t \in [t_0, t_f)$ as

$$U(t) = -H^T(t, X(t))\Psi^T(t_0, \sigma(t))G_c^{-1}(t_0, t_f)[X_0 - \Psi(t_0, t_f)X_f]$$

and extend $U(t)$ continuously for all other values of t. The corresponding solution of the system (1) at $t = t_f$ can be written as

$$
\begin{aligned}
X(t_f) &= \Psi(t_f,t_0)X_0 \\
&\quad +\Psi(t_f,t_0)\int_{t_0}^{t_f} \Psi(t_0,\sigma(\tau))\left(\int_{t_0}^{\tau} K(\tau,s,X(s))\Delta s + B(\tau)\right) \\
&\quad \times U(\tau)\Delta\tau \\
&= \Psi(t_f,t_0)X_0 + \Psi(t_f,t_0)\int_{t_0}^{t_f} \Psi(t_0,\sigma(\tau))H(\tau,X(\tau)) \\
&\quad \{-H^T(\tau,X(\tau))\Psi^T(t_0,\sigma(\tau))G_c^{-1}(t_0,t_f) \\
&\quad \times [X_0 - \Psi(t_0,t_f)X_f]\}\Delta\tau \\
&= \Psi(t_f,t_0)X_0 - \Psi(t_f,t_0)G_c(t_0,t_f)G_c^{-1}(t_0,t_f) \\
&\quad \times [X_0 - \Psi(t_0,t_f)X_f] \\
&= \Psi(t_f,t_0)X_0 - \Psi(t_f,t_0)[X_0 - \Psi(t_0,t_f)X_f] \\
&= \Psi(t_f,t_0)X_0 - \Psi(t_f,t_0)X_0 + X_f \\
&= X_f.
\end{aligned}
$$

Conversely, suppose that the system be controllable. If $G_c(t_0,t_f)$ is not invertiable then there exists a constant vector $Y \neq 0$ such that

$$
\begin{aligned}
0 &= Y^T G_c(t_0,t_f)Y \\
&= \int_{t_0}^{t_f} Y^T\Psi(t_0,\sigma(\tau))H(\tau,X(\tau))H^T(\tau,X(\tau))\Psi^T(t_0,\sigma(\tau))Y\Delta\tau \\
&= \int_{t_0}^{t_f} \| Y^T\Psi(t_0,\sigma(\tau))H(\tau,X(\tau)) \|^2 \,\Delta\tau
\end{aligned}
$$

and hence $Y^T\Psi(t_0,\sigma(\tau))H(\tau,X(\tau)) = 0$, for all $\tau \in [t_0,t_f]$. Since the system is controllable on $[t_0,t_f]$ and choose $X(t_0) = Y + \Psi(t_0,t_f)X_f$, then there exists a $U(\tau)$ such that

$$
\begin{aligned}
X_f &= \Psi(t_f,t_0)X_0 \\
&\quad +\Psi(t_f,t_0)\int_{t_0}^{t_f} \Psi(t_0,\sigma(\tau))\left(\int_{t_0}^{\tau} K(\tau,s,X(s))\Delta s + B(\tau)\right) \\
&\quad \times U(\tau)\Delta\tau.
\end{aligned}
$$

Or

$$
\begin{aligned}
\Psi(t_0,t_f)X_f &= X_0 + \int_{t_0}^{t_f} \Psi(t_0,\sigma(\tau))H(\tau,X(\tau))U(\tau)\Delta\tau \\
X_0 - Y &= X_0 + \int_{t_0}^{t_f} \Psi(t_0,\sigma(\tau))H(\tau,X(\tau))U(\tau)\Delta\tau \\
Y^TY &= -\int_{t_0}^{t_f} Y^T\Psi(t_0,\sigma(\tau))H(\tau,X(\tau))U(\tau)\Delta\tau \\
Y^TY &= 0 \\
\Rightarrow Y &= 0.
\end{aligned}
$$

Which is contradiction to $Y \neq 0$, and hence $G_c(t_0,t_f)$ is invertible.

Definition 3.2: ([7, 16]) The regressive non-linear system (1) is said to be completely observable on $[t_0,t_f]$ if any initial time t_0 and initial state $X(t_0) = X_0$, there exists a finite time $t_f > t_0$ such that the knowledge of the control $U(t)$ and output $Z(t)$ for $t_0 \leq t \leq t_f$ is sufficient to determine X_0 uniquely.

Theorem 3.3: The system (1) is completely observable if and only if the $(n \times n)$ symmetric observability Gramian matrix given by

$$
G_o(t_0,t_f) = \int_{t_0}^{t_f} \Psi^T(t,t_0)C^T(t)C(t)\Psi(t,t_0)\Delta t
$$

is invertible.

Proof:
Suppose that observability Gramian matrix $G_o(t_0,t_f)$ is invertible. Without loss of generality, assume that the control $U(t) = 0$ for $t \in [t_0,t_f]$. Then the solution of (1) is

$$X(t) = \Psi(t,t_0)X_0$$

and out put

$$Z(t) = C(t)\Psi(t,t_0)X_0.$$

Multiplying the above equation with $\Psi^T(t,t_0)C^T(t)$ and integration on both sides between the limits t_0 to t_f, we have

$$
\int_{t_0}^{t_f} \Psi^T(t,t_0)C^T(t)C(t)\Psi(t,t_0)X_0\Delta t = \int_{t_0}^{t_f} \Psi^T(t,t_0)C^T(t)Z(t)\Delta t
$$

$$
G_o(t_0,t_f)X_0 = \int_{t_0}^{t_f} \Psi^T(t,t_0)C^T(t)Z(t)\Delta t
$$

or

$$
X_0 = G_o^{-1}(t_0,t_f)\int_{t_0}^{t_f} \Psi^T(t,t_0)C^T(t)Z(t)\Delta t.
$$

It follows that X_0 is uniquely determined with the knowledge of $Z(t)$ when $U(t) = 0$ for $t \in [t_0,t_f]$. Therefore the system (1) is observable.
Conversely, suppose that $G_o(t_0,t_f)$ is not invertible and the system (1) is observable. This implies that there exists a constant vector $Y \neq 0$ such that

$$
\begin{aligned}
0 &= Y^T G_o(t_0,t_f)Y \\
&= \int_{t_0}^{t_f} Y^T\Psi^T(t,t_0)C^T(t)C(t)\Psi(t_0,t)Y\Delta t \\
&= \int_{t_0}^{t_f} [C(t)\Psi(t,t_0)Y]^T[C(t)\Psi(t,t_0)Y]\Delta t \\
&= \int_{t_0}^{t_f} \| C(t)\Psi(t,t_0)Y \|^2 \,\Delta t
\end{aligned}
$$

and hence $C(t)\Psi(t,t_0)Y = 0$, $t \in [t_0,t_f]$. If $X_0 = Y$, then the out put becomes zero through out the time interval and hence X_0 can not be determined in this case from the knowledge of $Z(t)$. It is a contradiction to our assumption. Thus the observability Gramian matrix $G_o(t_0,t_f)$ is invertible.
Now, in the next two theorems, the stability properties of the system (1) are discussed.

Definition 3.3:([7, 16]) The system (1) is called stable if for any $\varepsilon > 0$ and $t_0 \geq 0$, there exists $\delta = \delta(t_0,\varepsilon) > 0$ such that for any solution $\bar{X}(t) = X(t,t_0,\bar{X}_0)$ of (1), $|\bar{X}_0 - X_0| \leq \delta$ implies $|\bar{X}(t) - X(t)| < \varepsilon$ for all $t \geq t_0 \geq 0$.

Definition 3.4:([7, 16]) The system (1) is called uniformly stable if δ in definition 3.3 is independent of the time t_0.

Definition 3.5:([7, 16]) The system (1) is called asymptotically stable if it is stable and if there exists $\delta > 0$ such that $|\bar{X}_0 - X_0| \leq \delta$ implies $|\bar{X}(t) - X(t)| \to 0$ as $t \to +\infty$.

Theorem 3.4: Assume that

(a) $|\Psi(t,t_0)| \leq S$,
(b) $|H(\tau,X(\tau))U(\tau)| \leq T|X(\tau)|$,

where $S > 0$ and $T > 0$. Then the solution of (1) is uniformly stable.

Proof :
For any $\varepsilon > 0$, let $\delta(\varepsilon) < \frac{\varepsilon}{Se_{ST}(t,t_0)}$ and $|X_0| < \delta(\varepsilon)$. Suppose that there exists $t_1 \geq t_0$ such that $|X(t_1)| = \varepsilon$ and $|X(t)| < \varepsilon$ on $[t_0, t_1)$
From (2), we have

$$\begin{aligned} X(t) &= \Psi(t,t_0)X_0 \\ &\quad +\Psi(t,t_0)\int_{t_0}^{t} \Psi(t_0,\sigma(\tau))\left(\int_{t_0}^{\tau} K(\tau,s,X(s))\Delta s + B(\tau)\right) \\ &\quad \times U(\tau)\Delta\tau, \text{on } [t_0,t_1] \\ |X(t)| &\leq |\Psi(t,t_0)||X_0| + \int_{t_0}^{t} |\Psi(t,\sigma(\tau))||H(\tau,X(\tau))||U(\tau)|\Delta\tau \\ &\leq S\delta(\varepsilon) + S\int_{t_0}^{t} T|X(\tau)|\Delta\tau \end{aligned}$$

By Lemma 2.9, we have

$$\begin{aligned} |X(t)| &\leq S\delta(\varepsilon)e_{ST}(t,t_0) \\ &< Se_{ST}(t,t_0)\frac{\varepsilon}{Se_{ST}(t,t_0)} \\ &= \varepsilon \end{aligned}$$

Therefore $|X(t_1)| < \varepsilon$, for $t \in [t_0, t_1]$ which is a contradiction. Thus the solution of (1) is uniformly stable.

Theorem 3.5: Assume that
(a) $|\Psi(t,t_0)||X_0|| \leq Se_P(t_0,t)$,
(b) $|\Psi(t,t_0)\Psi(t,\sigma(\tau))| \leq Te_P(0,t)$,
(c)

$$\begin{aligned} \sup_{t_0\leq s\leq \tau<\infty}\left(\int_{t_0}^{\tau} |K(\tau,s,X(\tau))|\Delta\tau + |B(\tau)|\right)|U(\tau)| \\ \leq Le_P(s,0)|X(s)|, \end{aligned}$$

where $S > 0$, $T > 0$, $L > 0$ and $(P \ominus LT) > 0$. Then every solution $X(t)$ of (1) tends to zero, as $t \to +\infty$.

Proof :
From (2), we have

$$\begin{aligned} X(t) &= \Psi(t,t_0)X_0 \\ &\quad +\Psi(t,t_0)\int_{t_0}^{t} \Psi(t_0,\sigma(\tau))\left(\int_{t_0}^{\tau} K(\tau,s,X(s))\Delta s + B(\tau)\right) \\ &\quad \times U(\tau)\Delta\tau, \end{aligned}$$

$$\begin{aligned} |X(t)| &\leq |\Psi(t,t_0)||X_0| \\ &\quad +|\Psi(t,t_0)|\int_{t_0}^{t} |\Psi(t_0,\sigma(\tau))| \\ &\quad \times\left(\int_{t_0}^{\tau} |K(\tau,s,X(s))|\Delta s + |B(\tau)|\right)|U(\tau)|\Delta\tau \\ &\leq Se_P(t_0,t) + LTe_P(0,t)\int_{t_0}^{t} e_P(s,0)|X(s)|\Delta s \\ &= Se_P(t_0,0)e_P(0,t) + LTe_P(0,t)\int_{t_0}^{t} e_P(s,0)|X(s)|\Delta s \end{aligned}$$

$$|X(t)|e_P(t,0) \leq Se_P(t_0,0) + LT\int_{t_0}^{t} e_P(s,0)|X(s)|\Delta s$$

By Lemma 2.9, we have

$$\begin{aligned} |X(t)|e_P(t,0) &\leq Se_P(t_0,0)e_{LT}(t,t_0) \\ &= Se_P(t_0,0)e_P(0,t)e_{LT}(t,t_0) \\ &= Se_P(t_0,0)e_P(0,t)e_{\ominus LT}(t_0,t) \\ &= Se_P(t_0,0)e_P(0,t)e_{\ominus LT}(t_0,0)e_{\ominus LT}(0,t) \\ &= Se_{P\ominus RT}(t_0,0)e_{P\ominus LT}(0,t). \end{aligned}$$

By Lemma 2.10, we have
$e_{P\ominus LT}(0,t) \leq \frac{1}{1+(P\ominus LT)t}$, so we obtain

$$|X(t)| \leq Se_{P\ominus LT}(t_0,0)\frac{1}{1+(P\ominus LT)t}$$

Since $(P \ominus LT) > 0$, we obtain the required result.

Corollary: Suppose $K(t,s,X(s)) = K(t,s)X(s)$, then the solution of (1) is

$$\begin{aligned} X(t) &= \Psi(t,t_0)X_0 \\ &\quad +\Psi(t,t_0)\int_{t_0}^{t} \Psi(t_0,\sigma(\tau))\left(\int_{t_0}^{\tau} K(\tau,s)X(s)\Delta s + B(\tau)\right) \\ &\quad \times U(\tau)\Delta\tau, \end{aligned}$$

4. Example

Example 4.1: Let us consider the following system on time scale

$$\left.\begin{aligned} X^{\Delta}(t) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}X(t) + \left(\int_{0}^{t} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}X(s)\Delta s + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)U(t), \\ Z(t) &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}X(t), \\ X(0) &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \end{aligned}\right\} \tag{4}$$

for $0 \leq s \leq t < \infty$.
Case(a): If $\mathbb{T} = \mathbb{R}^+$ then (4) is

$$\left.\begin{aligned} X'(t) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}X(t) + \left(\int_{0}^{t} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}X(s)ds + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)U(t), \\ Z(t) &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}X(t), \\ X(0) &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \end{aligned}\right\} \tag{5}$$

The controllable Gramian matrix is given by

$$G_c(t_0,t_f) = \begin{bmatrix} 1/2(-5+e^{2t_f}(5+2(-3+t_f)t_f)) & 1/2(-3+e^{2t_f}(3-2t_f)) \\ 1/2(-3+e^{2t_f}(3-2t_f)) & -1+e^{2t_f} \end{bmatrix}.$$

Clearly $G_c(t_0,t_f)$ is symmetric and invertible when $t_0 = 0$ and $t_f \neq 0$. By using Theorem 3.2, it implies that system (5) is controllable.
The observable Gramian matrix is given by

$$G_o(t_0,t_f) = \begin{bmatrix} 2t_f & 2t_f + t_f^2 \\ 2t_f + t_f^2 & 2t_f + 2t_f^2 + 2t_f^3/3 \end{bmatrix}.$$

Clearly $G_o(t_0,t_f)$ is symmetric and invertible when $t_0 = 0$ and $t_f \neq 0$. By using Theorem 3.3, it implies that system (5) is observable.
The solution of (5) is

$$X(t) = \begin{bmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{bmatrix}$$
$$= \begin{bmatrix} \frac{e^4(3-5t)+e^8(1+t)}{(1-18e^4+e^8)} & \frac{e^4(3-5t)+e^8(1+t)}{(1-18e^4+e^8)} \\ +\frac{3e^{4+2t}(-7+3t)+e^{2t}(1+3t)}{(1-18e^4+e^8)} & +\frac{3e^{4+2t}(-7+3t)+e^{2t}(1+3t)}{(1-18e^4+e^8)} \\ 1+\frac{-1+13e^4}{(1-18e^4+e^8)} & 1+\frac{-1+13e^4}{(1-18e^4+e^8)} \\ +\frac{e^{2t}(1+2t+e^4(-13+6t))}{(1-18e^4+e^8)} & +\frac{e^{2t}(1+2t+e^4(-13+6t))}{(1-18e^4+e^8)} \end{bmatrix}.$$

It can be observed that $x_{11}(t) = x_{12}(t)$ and $x_{21}(t) = x_{22}(t)$ and which are plotted in the interval $[0,2]$ and are shown in figure 1.

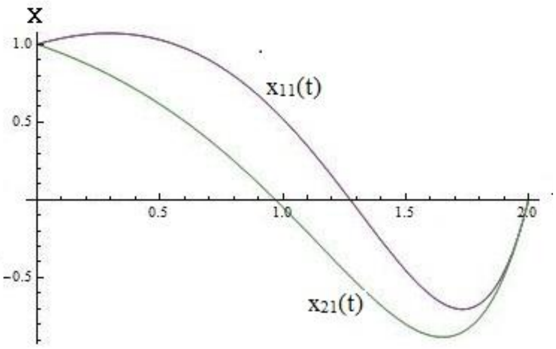

Figure 1:: Solution of (5)

From the figure 1, it can be observe that for initial time $t_0 = 0$, initial state $X(t_0) = 1$ and final state $X_f = 0$, then there exists a finite final time $t_f = 2$ such that $X(t_f) = X_f$. Hence the system (5) is controllable and observable from the definitions 3.1 and 3.2.

Case(b): If $\mathbb{T} = \mathbb{N}$ then (4) is

$$\left.\begin{aligned} \Delta X(n) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} X(n) \\ &\quad + \left(\sum_{s=0}^{n-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} X(s) + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right) U(n) \\ Z(n) &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} X(n), \\ X(0) &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \end{aligned}\right\} \tag{6}$$

The controllable Gramian matrix is given by

$$G_c(n_0, n_f) = \begin{bmatrix} \frac{2}{27}[-20+5(2^{2+2n_f}) & -\frac{2}{9}[4-2^{2+2n_f} \\ -3(2^{3+2n_f}n_f)+9(2^{2n_f}n_f^2)] & -\frac{2}{9}[4-2^{2+2n_f}+3(2^{2n_f}n_f)] \\ -\frac{2}{9}[4-2^{2+2n_f}+3(2^{2n_f}n_f)] & \frac{2}{3}[-1+2^{2n_f}] \end{bmatrix}.$$

Clearly $G_c(n_0, n_f)$ is symmetric and invertible when $n_0 = 0$ and $n_f \neq 0$. By using Theorem 3.2, it implies that system (6) is controllable.

The observable Gramian matrix is given by

$$G_o(n_0, n_f) = \begin{bmatrix} 2n_f & (1+n_f)n_f \\ n_f(1+n_f) & \frac{1}{3}n_f(1+n_f)(1+2n_f) \end{bmatrix}.$$

Clearly $G_o(n_0, n_f)$ is symmetric and invertible when $n_0 = 0$ and $n_f \neq 0$. By using Theorem 3.3, it implies that system (6) is observable.

The solution of (6) is

$$X(n) = \begin{bmatrix} x_{11}(n) & x_{12}(n) \\ x_{21}(n) & x_{22}(n) \end{bmatrix}$$
$$= \begin{bmatrix} 1/9[32-23(4^n) & 1/9[32-23(4^n) \\ +3(8+3(4^n))n] & +3(8+3(4^n))n] \\ 8/3-5(4^n)/3 & 8/3-5(4^n)/3 \\ +3(4^{-1+n})n & +3(4^{-1+n})n \end{bmatrix}.$$

It can be observed that $x_{11}(n) = x_{12}(n)$ and $x_{21}(n) = x_{22}(n)$ are plotted in the interval $[0,2]$ and are shown in figure 2.

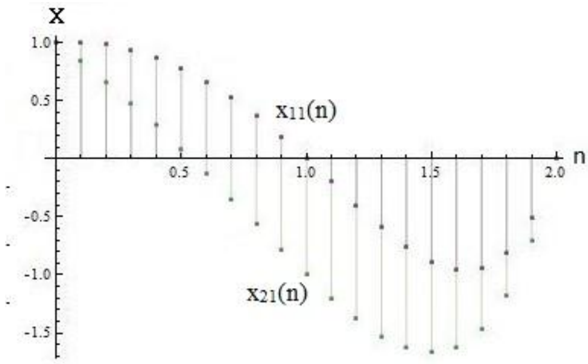

Figure 2: Solution of (6)

From the figure 2, it can be observe that for initial time $n_0 = 0$, initial state $X(n_0) = 1$ and final state $X_f = 0$, then there exists a finite final time $n_f = 2$ such that $X(n_f) = X_f$. Hence the system (6) is controllable and observable from the definitions 3.1 and 3.2.

5. Conclusion

In this paper, the necessary and sufficient conditions for the controllability and observability of the solution of the system (1) are established. The results are demonstrated using examples. Graphically, the idea of controllability and observability for the given problem is illustrated in figures 1 and 2.

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